Solution to Exercise 14.15
The parameters for the Generalized Leapfrog Theorem

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1 The constraints

We are given real numbers t_1, t_2, such that

$$1 < 1 - \phi + \phi t_2 < t_1 < t_2, \tag{1}$$

where

$$\phi = 233/303. \tag{2}$$

We also have the real constant $0 < \alpha < 1$, which is given in the Sparse Ball Theorem. Consider real numbers θ, β, δ, μ, a, and h, and define

$$w := \frac{1}{2}(\cos\theta - \sin\theta) - \frac{t_2 - t_1 + \delta}{2\delta t_2}, \tag{3}$$

$$h' := h - \frac{2\beta(2+h)}{1-2\beta}, \tag{4}$$

$$a' := a - \frac{2\beta(a+\sin\theta)}{1-2\beta}, \tag{5}$$

$$\beta' := \frac{\beta(1+h)}{h+\cos\theta}, \tag{6}$$

$$\delta' := \frac{\delta(h+\cos\theta)}{1+h}, \tag{7}$$

and

$$\xi := \frac{12(1+h)(3a+\sin\theta)}{h' - 6a - 2\sin\theta - 8\beta(1+h) - 48\beta(1+h)^2}. \tag{8}$$

The parameters must be chosen, subject to the following constraints:

$$0 < \theta < \pi/4, \tag{9}$$

$$\frac{t_2 - t_1}{t_2 - 1} < \delta < 1, \tag{10}$$

$$\cos\theta - \sin\theta > \frac{t_2 - t_1 + \delta}{\delta t_2}, \tag{11}$$

$$0 < \delta < 1, \tag{12}$$

$$\mu \geq \max\left(w, (1 + a + h/2 + \sin\theta)\left(1 + \frac{1}{\delta(1 - 2\beta)}\right)\right), \tag{13}$$

$$0 < \beta < \min\left(\delta, \cos\theta, \frac{h}{4(1+h)}, \frac{a}{4a + 2\sin\theta}\right), \tag{14}$$

$$0 < a < h < \frac{\cos\theta - \beta}{1 + \beta}, \tag{15}$$

$$h' > 6a + 2\sin\theta + 8\beta(1+h), \tag{16}$$

$$\beta(1+h)^2 < \delta(h + \cos\theta)^2, \tag{17}$$

$$\frac{\delta'(1-\beta')}{2(1-\beta') + \delta'(3-\beta')} \geq 1/6, \tag{18}$$

$$h' > 6a + 2\sin\theta + 8\beta(1+h) + 48\beta(1+h)^2, \tag{19}$$

$$\mu \geq (a + 1 + h/2)(1 + 1/\delta) + (\xi + \sin\theta)/\delta, \tag{20}$$

$$\xi < \cos\theta - h, \tag{21}$$

$$\cos\theta > 2(a + \sin\theta) + 1/t_2, \tag{22}$$

$$\begin{aligned} \xi \quad \leq \quad & \frac{3t_1(1+h) + (t_2\cos\theta - 1 - 2t_2(a+\sin\theta))(\cos\theta - h)}{9t_2(1+h) + t_2\cos\theta - 1 - 2t_2(a+\sin\theta)} \\ & - \frac{3t_2(1+h)(1 + 2a + \sin\theta + 2\beta(a+\sin\theta))}{9t_2(1+h) + t_2\cos\theta - 1 - 2t_2(a+\sin\theta)}, \end{aligned} \tag{23}$$

$$\alpha a' - \sin\theta - \frac{6a\beta}{1-\beta} \geq \frac{1}{2}\alpha a, \tag{24}$$

and

$$\beta < \frac{a'}{2a + a'}. \tag{25}$$

2 Choice for the parameters

We define

$$\beta := \min\left(10^{-4}, \alpha/49\right), \tag{26}$$

$$\delta := 99/100, \tag{27}$$

$$h := 1/100, \tag{28}$$

$$\mu := 5, \tag{29}$$

$$a := \min\left(10^{-4}, \frac{t_2 - 1}{10^6 t_2}, \frac{t_1 - (1-\phi) - \phi t_2}{10^5 t_2}\right), \tag{30}$$

and choose θ, such that

$$0 < \theta < \pi/4, \tag{31}$$

$$\cos\theta > 91/100, \tag{32}$$

$$\sin\theta \leq \frac{t_2 - 1}{200 t_2}, \tag{33}$$

$$\sin\theta \leq \alpha a/8, \tag{34}$$

$$\cos\theta > \frac{2}{10^6} + \frac{1}{100} + \left(1 - \frac{2}{10^6} - \frac{1}{100}\right)\frac{1}{t_2}, \tag{35}$$

$$\cos\theta - \sin\theta > \frac{t_2 - t_1 + \delta}{\delta t_2}, \tag{36}$$

and

$$\cos\theta \geq \frac{1375a}{9/1000 - 18a} + \frac{2}{9} + \frac{7}{9}\frac{1}{t_2} + \frac{101}{30}\frac{t_2 - t_1}{t_2}. \tag{37}$$

3 Verification of the constraints

3.1 Choosing θ as in (35) is possible

In order for (35) to be possible, we need

$$0 < \frac{2}{10^6} + \frac{1}{100} + \left(1 - \frac{2}{10^6} - \frac{1}{100}\right)\frac{1}{t_2} < 1.$$

It is clear that this quantity is strictly positive. It is strictly less than one, if and only if $1/t_2 < 1$, which is true by (1).

3.2 Choosing θ as in (36) is possible

In order for (36) to be possible, we need

$$0 < \frac{t_2 - t_1 + \delta}{\delta t_2} + \sin\theta < 1.$$

Since $t_2 > t_1$ by (1), this quantity is strictly positive. Using (30) and (34), and the fact that $0 < \alpha < 1$, we have

$$\frac{t_2 - t_1 + \delta}{\delta t_2} + \sin\theta < \frac{t_2 - t_1 + \delta}{\delta t_2} + \frac{t_1 - (1-\phi) - \phi t_2}{10^5 t_2}.$$

By (1), we have $t_1 - (1-\phi) - \phi t_2 > 0$. Also, by (27), we have $1/10^5 < 1/\delta$. Hence,

$$\frac{t_2 - t_1 + \delta}{\delta t_2} + \sin\theta < \frac{t_2 - t_1 + \delta}{\delta t_2} + \frac{t_1 - (1-\phi) - \phi t_2}{\delta t_2} = \frac{(1-\phi)(t_2-1)}{\delta t_2} + \frac{1}{t_2}.$$

Thus, it suffices to show that

$$\frac{(1-\phi)(t_2-1)}{\delta t_2} + \frac{1}{t_2} < 1,$$

which is equivalent to

$$\frac{(1-\phi)(t_2-1)}{\delta t_2} < \frac{t_2-1}{t_2},$$

which is equivalent to

$$1 - \phi < \delta,$$

which is true by (2) and (27).

3.3 Choosing θ as in (37) is possible

In order for (37) to be possible, we need

$$0 < \frac{1375a}{9/1000 - 18a} + \frac{2}{9} + \frac{7}{9}\frac{1}{t_2} + \frac{101}{30}\frac{t_2 - t_1}{t_2} < 1.$$

Since $a \leq 10^{-4}$ (by (30)), and since $t_2 > t_1$ (by (1)), this quantity is strictly positive. It is strictly less than one, if and only if

$$\frac{1375a}{9/1000 - 18a} < \frac{7}{9}\frac{t_2 - 1}{t_2} - \frac{101}{30}\frac{t_2 - t_1}{t_2},$$

which is equivalent to

$$1375a < \frac{7}{9}(9/1000 - 18a)\frac{t_2 - 1}{t_2} - \frac{101}{30}(9/1000 - 18a)\frac{t_2 - t_1}{t_2},$$

which is equivalent to

$$\left(1375 + 14\frac{t_2 - 1}{t_2} - \frac{303}{5}\frac{t_2 - t_1}{t_2}\right) a < \frac{7}{1000}\frac{t_2 - 1}{t_2} - \frac{303}{10^4}\frac{t_2 - t_1}{t_2},$$

which is equivalent to

$$\left(1375 + 14 - \frac{303}{5} - \frac{14}{t_2} + \frac{303}{5}\frac{t_1}{t_2}\right) a < \frac{7}{1000} - \frac{303}{10^4} - \frac{7}{1000}\frac{1}{t_2} + \frac{303}{10^4}\frac{t_1}{t_2}.$$

Since $1375 + 14 - 303/5 < 1329$ and $7/1000 - 303/10^4 = -233/10^4$, it suffices to show that

$$\left(1329 - \frac{14}{t_2} + \frac{303}{5}\frac{t_1}{t_2}\right) a < \frac{303}{10^4}\frac{t_1}{t_2} - \frac{7}{1000}\frac{1}{t_2} - \frac{233}{10^4}.$$

Since $t_1 < t_2$ (by (1)), we have

$$1329 - \frac{14}{t_2} + \frac{303}{5}\frac{t_1}{t_2} < 1329 + \frac{303}{5} < 1400.$$

Thus, it suffices to show that

$$1400a < \frac{303}{10^4}\frac{t_1}{t_2} - \frac{7}{1000}\frac{1}{t_2} - \frac{233}{10^4}.$$

Using (2), the right-hand side is equal to

$$\frac{303}{10^4}\frac{t_1 - (1-\phi) - \phi t_2}{t_2}.$$

Thus, it suffices to show that

$$1400a < \frac{303}{10^4}\frac{t_1 - (1-\phi) - \phi t_2}{t_2}.$$

But this inequality is satisfied, because of (30) and the fact that $1/10^5 < \frac{303}{1400 \cdot 10^4}$.

3.4 Verification of (9)

Constraint (9) holds, because of (31).

3.5 Verification of (10)

It follows from (27) that $\delta < 1$. Again using (27), the requirement

$$\frac{t_2 - t_1}{t_2 - 1} < \delta$$

is equivalent to

$$\frac{t_2 - t_1}{t_2 - 1} < \frac{99}{100},$$

which can be written as

$$t_2 < 100t_1 - 99.$$

By (1), we have

$$t_2 < \frac{t_1 - (1-\phi)}{\phi}.$$

Thus, it suffices to show that

$$\frac{t_1 - (1-\phi)}{\phi} < 100t_1 - 99,$$

which can be written as

$$(100 - 1/\phi)t_1 > (100 - 1/\phi),$$

which is true, because, by (1), $t_1 > 1$.

3.6 Verification of (11)

Constraint (11) holds, because of (36).

3.7 Verification of (12)

Constraint (12) follows from our choice of δ in (27).

3.8 Verification of (13)

Observe that $w > 0$, because of (11). Using (3), (29), and the fact that $t_2 > t_1$, we have

$$w = \frac{1}{2}(\cos\theta - \sin\theta) - \frac{t_2 - t_1 + \delta}{2\delta t_2} \le 1/2 \le 5 = \mu.$$

Using (26), (27), (28), and (30), we have

$$\begin{aligned}
(1 + a + h/2 + \sin\theta)\left(1 + \frac{1}{\delta(1-2\beta)}\right) &\le \left(1 + \frac{1}{10^4} + \frac{1}{200} + 1\right)\left(1 + \frac{1}{\frac{99}{100}(1 - 2\cdot 10^{-4})}\right) \\
&\le 5 \\
&= \mu.
\end{aligned}$$

3.9 Verification of (14)

Using (26) and (27), we have $0 < \beta < \delta$. Using (26) and (32), we have

$$\cos\theta > 91/100 > 10^{-4} \geq \beta.$$

Using (26) and (28), we have

$$\frac{h}{4(1+h)} = 1/404 > 10^{-4} \geq \beta.$$

The inequality

$$\beta < \frac{a}{4a + 2\sin\theta}$$

is equivalent to

$$(4a + 2\sin\theta)\beta < a.$$

Using (34) and the fact that $0 < \alpha < 1$, we have

$$(4a + 2\sin\theta)\beta \leq (4a + \alpha a/4)\beta < 5a\beta,$$

which is less than a, because, by (26), $\beta < 1/5$.

3.10 Verification of (15)

Using (28) and (30), we have $0 < a \leq 10^{-4} < h$. The inequality

$$h < \frac{\cos\theta - \beta}{1+\beta}$$

is equivalent to

$$\cos\theta > \beta + h(1+\beta).$$

Using (26), (28), and (32), we have

$$\beta + h(1+\beta) \leq 10^{-4} + \frac{1}{100}\left(1 + 10^{-4}\right) < 91/100 < \cos\theta.$$

3.11 Verification of (16)

The constraint (16) is implied by (19), which will be verified later.

3.12 Verification of (17)

Using (26) and (28), we have

$$\beta(1+h)^2 \leq \frac{1}{10^4}\left(1+\frac{1}{100}\right)^2 = \frac{101^2}{10^8}.$$

Thus, it suffices to show that

$$\frac{101^2}{10^8} < \delta(h+\cos\theta)^2.$$

Using (27) and (28), this is equivalent to

$$\frac{101^2}{10^8} < \frac{99}{100}\left(\frac{1}{100}+\cos\theta\right)^2,$$

which is equivalent to

$$\left(\frac{1}{100}+\cos\theta\right)^2 > \frac{101^2}{99\cdot 10^6}.$$

Using (32), we have

$$\left(\frac{1}{100}+\cos\theta\right)^2 > \left(\frac{92}{100}\right)^2 > \frac{101^2}{99\cdot 10^6}.$$

3.13 Verification of (18)

We first show that

$$\frac{9}{10}\delta < \delta' < \delta. \tag{38}$$

The inequality $\delta' < \delta$ follows from the definition of δ' in (7). The inequality $\frac{9}{10}\delta < \delta'$ is equivalent to

$$\frac{9}{10} < \frac{h+\cos\theta}{1+h},$$

which is equivalent to

$$10\cos\theta + h > 9.$$

Using (32), we have

$$10\cos\theta + h > 10\cos\theta > \frac{91}{10} > 9.$$

This proves that (38) holds.

In exactly the same way, using the definition of β' in (6), we get

$$\frac{9}{10}\beta' < \beta < \beta'. \tag{39}$$

Now we can show that constraint (18) is satisfied. Using (38) and (39), we get

$$\frac{\delta'(1-\beta')}{2(1-\beta')+\delta'(3-\beta')} \geq \frac{\frac{9}{10}\delta\left(1-\frac{10}{9}\beta\right)}{2+3\delta},$$

which, using (26) and (27), is at least

$$\frac{\frac{9}{10}\frac{99}{100}\left(1-\frac{10}{9}\cdot 10^{-4}\right)}{2+\frac{3\cdot 99}{100}} \geq 1/6.$$

3.14 Verification of (19)

Using the definition of h' in (4), the constraint (19) can be written as

$$h > \frac{2\beta(2+h)}{1-2\beta} + 6a + 2\sin\theta + 8\beta(1+h) + 48\beta(1+h)^2.$$

Using (26), (28), (30), and (34), and the fact that $0 < \alpha < 1$, we get

$$\begin{aligned} &\frac{2\beta(2+h)}{1-2\beta} + 6a + 2\sin\theta + 8\beta(1+h) + 48\beta(1+h)^2 \\ &< \frac{\frac{2}{10^4}(2+1/100)}{1-\frac{2}{10^4}} + \frac{6}{10^4} + \frac{1}{4\cdot 10^4} + \frac{8}{10^4}\left(1+\frac{1}{100}\right) + \frac{48}{10^4}\left(1+\frac{1}{100}\right)^2, \end{aligned}$$

which is less than $1/100 = h$.

3.15 Some inequalities for ξ

The value of ξ is defined in (8). We first prove that

$$\xi \leq \frac{25a}{\frac{2}{1000} - 4a}. \tag{40}$$

Using the definitions of h' and ξ in (8) and (4), respectively, we have

$$\xi = \frac{12(1+h)(3a+\sin\theta)}{h - \frac{2\beta(2+h)}{1-2\beta} - 6a - 2\sin\theta - 8\beta(1+h) - 48\beta(1+h)^2}.$$

Using (26), (28), (30), and (34), and the fact that $0 < \alpha < 1$, we have

$$\begin{aligned} \xi &\leq \frac{12\cdot\frac{101}{100}(3a+a/8)}{\frac{1}{100} - \frac{2\cdot 10^{-4}.\frac{201}{100}}{1-2\cdot 10^{-4}} - 6a - \frac{1}{4\cdot 10^4} - \frac{8}{10^4}\cdot\frac{101}{100} - \frac{48}{10^4}\left(\frac{101}{100}\right)^2} \\ &= \frac{\frac{75}{2}a}{\frac{1}{101} - \frac{1}{10^4}\left(\frac{402}{101(1-2\cdot 10^{-4})} + \frac{100}{404} + 8 + \frac{48\cdot 101}{100}\right) - \frac{600}{101}a}. \end{aligned}$$

Since

$$\frac{1}{101} - \frac{1}{10^4}\left(\frac{402}{101\left(1-2\cdot 10^{-4}\right)} + \frac{100}{404} + 8 + \frac{48\cdot 101}{100}\right) > 3/1000,$$

and since

$$\frac{600}{101}a \leq 6a,$$

we have

$$\xi \leq \frac{\frac{75}{2}a}{\frac{3}{1000} - 6a} = \frac{25a}{\frac{2}{1000} - 4a},$$

completing the proof of (40).

We next prove that

$$\xi \leq \frac{9}{110}\left(\cos\theta - 1/t_2\right) - \frac{1}{55}\frac{t_2 - 1}{t_2} - \frac{3(1+h)}{11}\frac{t_2 - t_1}{t_2}. \tag{41}$$

To prove this inequality, we observe that, using (40), it suffices to show that

$$\frac{25a}{\frac{2}{1000} - 4a} \leq \frac{9}{110}\left(\cos\theta - 1/t_2\right) - \frac{1}{55}\frac{t_2 - 1}{t_2} - \frac{3(1+h)}{11}\frac{t_2 - t_1}{t_2},$$

which is equivalent to (where we use the fact that $h = 1/100$, see (28))

$$\frac{25a}{\frac{2}{1000} - 4a} \leq \frac{9}{110}\cos\theta - \frac{1}{55} - \left(\frac{9}{110} - \frac{1}{55}\right)\frac{1}{t_2} - \frac{303}{1100}\frac{t_2 - t_1}{t_2},$$

which is equivalent to

$$\frac{1375a}{\frac{9}{1000} - 18a} \leq \cos\theta - \frac{2}{9} - \frac{7}{9}\frac{1}{t_2} - \frac{101}{30}\frac{t_2 - t_1}{t_2},$$

which is true, because of (37).

3.16 Verification of (20)

Using (30) and (40), we get

$$\xi \leq \frac{25a}{\frac{2}{1000} - 4a} \leq \frac{25 \cdot 10^{-4}}{\frac{2}{1000} - 4 \cdot 10^{-4}} = \frac{25}{16}.$$

Thus, using (27), (28), and (30), we get

$$\begin{aligned}
(a + 1 + h/2)(1 + 1/\delta) + (\xi + \sin\theta)/\delta &\leq \left(\frac{1}{10^4} + 1 + \frac{1}{200}\right)\left(1 + \frac{100}{99}\right) + \frac{\frac{25}{16} + 1}{99/100} \\
&= \left(\frac{1}{10^4} + 1 + \frac{1}{200}\right)\frac{199}{99} + \frac{100 \cdot 41}{99 \cdot 16} \\
&< 5 \\
&= \mu.
\end{aligned}$$

3.17 Verification of (21)

It follows from (41) that

$$\xi \leq \frac{9}{110}.$$

Using (28) and (32), we get

$$\xi + h \leq \frac{9}{110} + \frac{1}{100} < \frac{91}{100} < \cos\theta.$$

3.18 Verification of (22)

Using (30) and (33), we get

$$\begin{aligned} 2(a+\sin\theta) + \frac{1}{t_2} &\leq \frac{2}{10^6}\frac{t_2-1}{t_2} + \frac{1}{100}\frac{t_2-1}{t_2} + \frac{1}{t_2} \\ &= \frac{2}{10^6} + \frac{1}{100} + \left(1 - \frac{2}{10^6} - \frac{1}{100}\right)\frac{1}{t_2}, \end{aligned}$$

which, by (35), is less than $\cos\theta$.

3.19 Verification of (23)

The constraint (23) is equivalent to

$$\begin{aligned} \xi\left(9t_2(1+h) + t_2\cos\theta - 1 - 2t_2(a+\sin\theta)\right) + 3t_2(1+h)(1+2a+\sin\theta+2\beta(a+\sin\theta)) \\ \leq 3t_1(1+h) + \left(t_2\cos\theta - 1 - 2t_2(a+\sin\theta)\right)(\cos\theta - h). \quad (42) \end{aligned}$$

Observe that

$$t_2\cos\theta - 1 - 2t_2(a+\sin\theta) \leq t_2.$$

Using (26), (28), (30), and (33), we get

$$\begin{aligned} 2a + \sin\theta + 2\beta(a+\sin\theta) &\leq \frac{2}{10^6}\frac{t_2-1}{t_2} + \frac{1}{200}\frac{t_2-1}{t_2} + \frac{2}{10^4}\left(\frac{1}{10^6}\frac{t_2-1}{t_2} + \frac{1}{200}\frac{t_2-1}{t_2}\right) \\ &\leq \frac{1}{30(1+h)}\frac{t_2-1}{t_2}. \end{aligned}$$

Using (28) and (32), we get

$$\cos\theta - h \geq \frac{91}{100} - \frac{1}{100} = \frac{9}{10}.$$

Hence, (42) will follow from the claim that

$$\xi\left(9t_2(1+h) + t_2\right) + 3t_2(1+h) + \frac{1}{10}(t_2 - 1) \leq 3t_1(1+h) + \frac{9}{10}\left(t_2\cos\theta - 1 - 2t_2(a+\sin\theta)\right).$$

Since, using (28), $9t_2(1+h)+t_2 \le 11t_2$, it suffices to show that

$$11\xi t_2 + 3t_2(1+h) + \frac{1}{10}(t_2-1) \le 3t_1(1+h) + \frac{9}{10}\left(t_2\cos\theta - 1 - 2t_2(a+\sin\theta)\right),$$

which is equivalent to

$$11\xi t_2 + 3(t_2-t_1)(1+h) + \frac{1}{10}(t_2-1) + \frac{18}{10}t_2(a+\sin\theta) \le \frac{9}{10}\left(t_2\cos\theta - 1\right).$$

Using (30) and (33), we get

$$\frac{18}{10}t_2(a+\sin\theta) \le \frac{18}{10}t_2\left(\frac{1}{10^6}\frac{t_2-1}{t_2} + \frac{1}{200}\frac{t_2-1}{t_2}\right) = \left(\frac{18}{10^7}+\frac{9}{1000}\right)(t_2-1) \le \frac{1}{10}(t_2-1).$$

Thus, it suffices to show that

$$11\xi t_2 + 3(t_2-t_1)(1+h) + \frac{1}{5}(t_2-1) \le \frac{9}{10}\left(t_2\cos\theta - 1\right),$$

which is equivalent to

$$\xi \le \frac{9}{110}\left(\cos\theta - 1/t_2\right) - \frac{1}{55}\frac{t_2-1}{t_2} - \frac{3(1+h)}{11}\frac{t_2-t_1}{t_2},$$

which we have shown to hold in (41).

3.20 Verification of (24)

We have to show that

$$\frac{1}{2}\alpha a + \sin\theta + \frac{6a\beta}{1-\beta} \le \alpha a'.$$

We know from (34) that

$$\sin\theta \le \alpha a/8.$$

Using (26) and the fact that $0<\alpha<1$, we get

$$48\beta + \alpha\beta \le 49\beta \le \alpha,$$

which can be rewritten as $48\beta \le \alpha(1-\beta)$, which can be rewritten as

$$\frac{\beta}{1-\beta} \le \alpha/48.$$

Thus, it suffices to show that

$$\frac{1}{2}\alpha a + \frac{1}{8}\alpha a + \frac{1}{8}\alpha a \le \alpha a',$$

which is equivalent to

$$\frac{3}{4}a \leq a'. \tag{43}$$

Using the definition of a' in (5), this is equivalent to

$$a - \frac{2\beta(a + \sin\theta)}{1 - 2\beta} \geq \frac{3}{4}a,$$

which is equivalent to

$$\frac{2\beta(a + \sin\theta)}{1 - 2\beta} \leq \frac{1}{4}a.$$

Using (34) and the fact that $0 < \alpha < 1$, we have

$$\sin\theta \leq \alpha a/8 \leq a/8.$$

Thus, it suffices to show that

$$\frac{2\beta(a + a/8)}{1 - 2\beta} \leq \frac{1}{4}a,$$

which is equivalent to

$$\frac{\frac{9}{4}\beta}{1 - 2\beta} \leq \frac{1}{4},$$

which is equivalent to

$$9\beta \leq 1 - 2\beta,$$

which is equivalent to

$$\beta \leq 1/11,$$

which is true, because of (26).

3.21 Verification of (25)

We have to show that

$$\beta < \frac{a'}{2a + a'},$$

which is equivalent to

$$(2a + a')\beta < a'.$$

We have seen in (43) that $3a/4 \leq a'$. Also, the definition of a' in (5) implies that $a' < a$. Therefore, it suffices to show that

$$(2a + a)\beta < \frac{3}{4}a,$$

which is equivalent to

$$\beta < 1/4,$$

which is true, because of (26).