Self-Organized Criticality

Asad B. Sayeed

asayeed@mbl.ca

Carleton University
The Issue

Physical sciences seek simplicity and predictability:

- Elementary particles
- Physical laws for simple, measurable events.

How do complex phenomena arise from these?
Complex Phenomena

Examples of systems that display some form of emergent organization:

- Living organisms
- Economic systems
- Ecological systems

Less obvious systems:

- Works of fiction
- Geological systems over time
Complex Phenomena

We need to:

- Identify fundamental patterns in these systems
- Find the common characteristics of these patterns
- Make generalizations that are applicable to complex systems found in nature.
Background

- Per Bak, Chao Tang, Kurt Wiesenfeld, 1987 at Brookhaven: first discussion of SOC, sandpile model.

- This presentation largely based on Bak’s book:
Agenda

- Underlying concepts: contingency, complexity
- Characteristics and examples of complex phenomena
- Self-organized criticality and the Sandpile Model
- Other examples and speculation
Contingency

What is science?

- Most “hard” sciences depend on isolated, repeatable experiments (physics, chemistry, etc).
- Attempt to isolate variables.
- Take phenomena out of the random environment in which they often exist.
- But complex systems contain lots of random interactions—cannot be taken out of context!
Contingency

Historical sciences:

- Examples: evolutionary biology, economics, etc
- Variables can’t be isolated without simplifying the system.
- Repeatability often unattainable because specific instances of behaviour may be unique (would human beings evolve a second time? Why not intelligent cephalopods?).
Contingency

- Study focuses on *contingency*: what chains of causality are produced by what events?
- Contingency produces narratives that are readily explainable.
- Contingency involves freak accidents (asteroid hitting Earth, Mr. Caveman getting squashed by mammoth...).
Contingency

- Explanation only possible after the fact:
  - Theoretically possible to explain how cephalopods evolved.
  - Infeasible to predict how they will evolve.
  - Infeasible to predict how they *would have* evolved...
- Only possible to make tiny predictions with massive computational effort. Otherwise, speculation!
So we can’t have a theory that predicts cephalopods, only explains them.

- Can explain why there is variation.
- Abstract: should explain complexity without reference to lower levels.
- Statistical: predict probabilities, not details.
Example: evolution

- Theory should describe all possible scenarios.
- Independent of environment (should be valid on Mars).
- Independent of medium (ie, without reference to DNA, proteins, etc).
Complexity: Abstractness

Example: earthquakes

- Need to construct model independent of details of earth.
- Just like evolution, should be valid every place where earthquake phenomena are possible.

In other words, look at process, not details!
Complexity: Statistical

Example: evolution

• Evidence for theories usually presented as anecdotes.
• Cannot make predictions from anecdotes.
• Karl Popper: prediction distinguishes science from pseudoscience.

We predict statistical patterns rather than specific instances
Complexity: Phenomena

Ubiquitous phenomena in complex systems:

- Large catastrophic events
- Fractals
- $1/f$ noise
- Zipf’s law
Catastrophe

Complex systems can exhibit catastrophic behaviour.

- One part of system affects many others. Domino effect.
- Small perturbations can grow larger and larger.
- Earthquakes
  - Cracks propagate to produce bursts of energy.
  - Pattern emerges—Richter scale is logarithmic.

- Measure fluctuation in cotton prices over 30 months.
- Plot price fluctuation vs. month
- Log plot months vs fluctuation size.
Commodity price fluctuation: cotton (Mandelbrot, 1963). From Bak:

Figure 3. (a) Monthly variations of cotton prices (Mandelbrot, 1963) during a period of 30 months. (b) The curve shows the number of months where the relative variation exceeded a given fraction. Note the smooth transition from small variations to large variations. The straight line indicates a power law. Other commodities follow a similar pattern.

- Larger fluctuations are exponentially fewer than smaller ones. Same for earthquake sizes.
- Large catastrophes follow same pattern as small perturbations.
- Economists, biologists, etc. tend to avoid or “explain away” exceptional events. Perhaps they shouldn’t!
Fractals

Geometrical structures with features of all length scales.

- Term coined by Mandelbrot.
- Ubiquitous in nature
- Geographical features: fjords of Norway
  - Hierarchical structure.
  - Fjords within fjords within fjords. “Scale Free.”
Fractals

From Bak:

Figure 6. (a) The coast of Norway. Note the "fractal," hierarchical geometry, with fjords, and fjords within fjords, and so on. Mandelbrot has pointed out that landscapes often are fractals. (From Feder, 1988.)
Fractals

From Bak:

![Graph showing log L(km) vs log δ(km)]

\[ L(\delta) = a \delta^{1-D} \]

\[ D = 1.52 \pm 0.01 \]

**Figure 6.** Continued (b) The length \( L \) of the coast measured by covering the coast with boxes, like the ones shown in (a), of various lengths \( \delta \). The straight line indicates that the coast is fractal. The slope of the line yields the “fractal dimension” of the coast of Norway, \( D = 1.52 \).
Fractals

- Smaller boxes allow the measuring of more coastline detail.
- Exponentially smaller boxes mean exponentially longer coastline.
- Norwegian fjords are natural fractals.
1/f Noise

Example: quasar signals

- Strength of signal inversely proportional to frequency \((1/f)\).
- Seemingly random fluctuations over short periods of time.
- Fluctuations in fluctuation sizes over longer periods of time—like a fractal.
1/f Noise

Quasar signal intensity over 80 years (from Bak)
1/f Noise

Contrast with white noise (from bak)
\(1/f\) Noise

- Fluctuations of all sizes present for quasar.
- Fluctuations observed at all scales: scale free!
- Not so for white noise.
- \(1/f\) noise is ubiquitous (global temperature, musical compositions, ...)
- No general understanding of this complex phenomenon.
Zipf’s Law

- Regularities in systems of human origin (city sizes, etc).
- Literature: James Joyce’s *Ulysses*.
- Words ranked by frequency.
- Plot log frequency vs. log word rank.
- Joyce is minimizing his effort (Zipf). Straight line is emergent.
Zipf’s Law

From Bak:

Figure 8. Continued (b) Ranking of words in the English language. The curve shows how many words appear with more than a given frequency.
The Power Law

- We keep seeing straight lines on double log graphs
- x-axis: some quantity $s$
- y-axis: some function $N(s)$, often occurances of $s$.
- $N(s) = s^{-\tau}$
  $\Rightarrow \log N(s) = -\tau \log s$
- This gives us our straight line.
The Power Law

- This relationship ubiquitous for events in complex systems, as we have seen.
- Scale invariant: straight line everywhere, no special features for certain values.
- **But** breaks down at high and low values (fjords not infinitely large or small).
(Non)Equilibrium

- Systems at equilibrium: gases, sand on a beach
- Small disturbances don’t propagate much—none of the previous phenomena exhibited.
- Thus, no room for contingency.
- Equilibrium systems are not complex!
(Non)Equilibrium

- Natural systems are only stable over human lifespans—illusory.
- Evolution and other changes happen episodically (Gould and Eldridge: punctuated equilibrium).
- No general theory of intermittent bursts until now.
Chaos vs. Complexity

Chaotic systems

- Simple systems based on simple equations
- Example: predator/prey (Feigenbaum)
  - $x_n$, number of individuals alive in year $n$
  - $x_{n+1} = \lambda x_n (1 - x_n)$ for some parameter $\lambda$.
  - Low values of $\lambda$ produce cycle.
  - Higher value of lambda eventually produce white noise.
- Chaos depends on tuning a parameter to get white noise
Chaos vs. Complexity

Complex systems

- Follow power laws, produce $1/f$ signals, etc.
- Do so at all scales
- Not parametric—no “blind watchmaker”
- In other words, behaviour is self-organized!
Self-Organized Criticality

- Simple, local rules
- Two phases: transient and critical
- System evolves through transient phase (a period of stasis) to critical phase.
- At critical phase, a small fluctuation can cause a massive change—an “avalanche.”
- Exploration of such system can unify Zipf’s law, catastrophic behaviour, fractal behaviour, etc—scale free, log plots.
- Let’s look at an example...
The Sandpile Model

- An \( m \times n \) matrix \( Z \).
- Drop single grain of sand onto (randomly chosen) \((x, y)\). Update: \( Z(x, y) \leftarrow Z(x, y) + 1 \).
- If \( Z(x, y) \) now exceeds 3, avalanche.
  - \( Z(x, y) \leftarrow Z(x, y) - 4 \) (should make it 0).
  - \( Z(x \pm 1, y) \leftarrow Z(x \pm 1, y) + 1 \),
    \( Z(x, y \pm 1) \leftarrow Z(x, y \pm 1) + 1 \).
- Propagate avalanches from \( Z(x \pm 1, y \pm 1) \) as needed.
- Fall off edge into sand heaven (gone forever).
The Sandpile Model

Before avalanche:
The Sandpile Model

After avalanche:
Go directly to NetLogo
(Do not pass Go, do not collect $200)
The Sandpile Model

From Bak:

Figure 11. Size distribution of avalanches in systems of coupled pendulums or, equivalently, in the sandpile model. The figure shows how many avalanches there are of each size on a logarithmic plot. The distribution is a power law with exponent 1.1. This is our very first plot. By performing longer simulations on bigger systems one can extend the range of the power law.
The Sandpile Model

- Transient/stable phase, then reaches critical phase.
- Contingency matters at critical phase, less at transient.
- Avalanches within avalanches within avalanches—leave behind fractal formations.
- Log plot of count of avalanche sizes vs. avalanche sizes is straight line. (like Zipf’s Law)
- Large number of avalanche sizes represented—more time, more sizes. ($1/f$ noise).
The Sandpile Model

- Non-equilibrium: open dissipative system
- Parameter-independent (can set max height to 8, change avalanche fall size, etc). No external organizer, ie, self-organized.
- Sandpile moves from state to state via avalanche—no way to extrapolate from single grain of sand.
- Possible to claim that grain of sand caused avalanche. Not possible to claim that avalanche would not have happened if grain not there.
Mathematics of Sandpiles

- Want to predict avalanches without actually simulating them.
- Need to compute power law exponent, $\tau$ (from $N(s) = s^{-\tau}$).
- Impossible even though each state is deduced from prior state.
- Power law impossible to prove.
- However, possible to check if sandpile state is critical without dropping grain.
Speculation: Mass Extinction

- Dinosaurs: killed by meteor?
- Weaknesses in theory: dinosaurs seem to have started dying out earlier (2M years) than meteor.
- Perhaps environment at critical state.
- Perhaps avalanche of events leading to extinction of dinosaurs.
- Bad luck rather than bad genes—not necessarily even external events, or maybe only indirectly
Speculation: Language change

- My own speculation.
- Great Vowel Shift of English, circa 1400-1600. Massive sudden change in pronunciation. Avalanche?
- Portuguese hasn’t changed in centuries.
- Small changes (slang, accents, etc) seem to occur.
- Perhaps a case of SOC/punctuated equilibria?
Other Applications

- Economics (Bak says some pretty controversial things)
- Brain development
- Real sandpiles
- Celestial phenomena
- And so on and so forth
Summary

- Many patterns in nature observe power laws, over time, space, etc.
- Nonequilibrium systems exhibit complexity.
- Complex systems evolve to critical state independently of parameters (self-organized).
- Avalanches follow power law distribution.
- Sandpile model of general case. Many real-world examples.
Al Gore, *Earth in the Balance*:

- “The sand pile theory—self-organized criticality—is irresistible as a metaphor... The formation of identity is akin to the formation of the sand pile... One reason I am drawn to this theory is that it has helped me understand change in my own life.”

Per Bak:

- “Maybe Gore is stretching the point too far.”
Self-organized criticality affected AI’s life. Just think! What can SOC do for you?
The End

- Questions?
- Comments?
- Anybdy care for a mint? (Just checking.)