A Proofs and Intermediate Results

Proof of lemma 1:

1. If \( M \models p(\bar{a}) : t \), then \( M \models p(\bar{a}) : t_c \) and \( M \models p(\bar{a}) : f_c \). Thus, \( M \not\models T(\text{IC}) \), a contradiction.

2. We know that \( M \models p(\bar{a}) : t_c \lor p(\bar{a}) : f_c \) and \( M \models p(\bar{a}) : t_d \lor p(\bar{a}) : f_d \) (since \( p(\bar{a}) : t_d \in T(\text{DB}, IC') \) or \( p(\bar{a}) : f_d \in T(\text{DB}, IC') \)). Thus, one of the following cases must be true: (1) \( M \models p(\bar{a}) : t_c \) and \( M \models p(\bar{a}) : t_d \), and therefore \( M \models p(\bar{a}) : t \), (2) \( M \models p(\bar{a}) : t_c \) and \( M \models p(\bar{a}) : f_d \), and therefore \( M \models p(\bar{a}) : t_a \), (3) \( M \models p(\bar{a}) : f_c \) and \( M \models p(\bar{a}) : t_d \), and therefore \( M \models p(\bar{a}) : f_a \), (4) \( M \models p(\bar{a}) : f_c \) and \( M \models p(\bar{a}) : f_d \), and therefore \( M \models p(\bar{a}) : f \).

\[ \square \]

Proof of lemma 2: We have to prove that \( M(\text{DB}, \text{DB}') \models T(\text{DB}) \) and \( M(\text{DB}, \text{DB}') \models T(\text{IC}) \).

1. Let us consider \( p(\bar{a}) : \alpha \in T(\text{DB}) \). If \( \alpha = t_d \), then \( p(\bar{a}) \in \text{DB} \), and then by considering (1) we obtain that \( I_F(p(\bar{a})) = t \) or \( I_F(p(\bar{a})) = f \), and therefore \( M(\text{DB}, \text{DB}') \models p(\bar{a}) : \alpha \). If \( \alpha = f_d \), then \( p(\bar{a}) \notin \text{DB} \), and then by considering (1) we obtain that \( I_F(p(\bar{a})) = f \) or \( I_F(p(\bar{a})) = t \), and therefore \( M(\text{DB}, \text{DB}') \models p(\bar{a}) : \alpha \).

2. (a) Let us suppose that \( p_1 (\bar{T}_1) : t_c \lor \cdots \lor p_n (\bar{T}_n) : t_c \lor q_1 (\bar{b}_1) : f_c \lor \cdots \lor q_m (\bar{b}_m) : f_c \in T(\text{IC}) \), and let us assume that \( p_1 (\bar{a}_1) : t_c \lor \cdots \lor p_n (\bar{a}_n) : t_c \lor q_1 (\bar{b}_1) : f_c \lor \cdots \lor q_m (\bar{b}_m) : f_c \) was obtained from this constraint by instantiating in the domain of the database. In this case we have that \( p_1 (\bar{T}_1) \lor \cdots \lor p_n (\bar{T}_n) \lor \neg q_1 (\bar{T}_1) \lor \cdots \lor \neg q_m (\bar{T}_m) \) is an element of \( \text{IC} \), and therefore we have that \( \text{DB} \models p_1 (\bar{a}_1) \lor \cdots \lor p_n (\bar{a}_n) \lor \neg q_1 (\bar{b}_1) \lor \cdots \lor \neg q_m (\bar{b}_m) \).

Firstly, we are going to consider what happens if \( \text{DB}' \models_\text{DB} p_i (\bar{a}_i) \) \((1 \leq i \leq n) \). If \( p_i \) is a built-in predicate, then \( I_R(p_i (\bar{a}_i)) = t \), since \( M(\text{DB}, \text{DB}') \) gives to the built-in predicates in the database the appropriate truth values, and therefore \( M(\text{DB}, \text{DB}') \models p_i (\bar{a}_i) : t_c \). If \( p_i \) is not a built-in predicate, then \( I_F(p_i (\bar{a}_i)) = t \) or \( I_F(p_i (\bar{a}_i)) = f \), and therefore \( M(\text{DB}, \text{DB}') \models p_i (\bar{a}_i) : t_c \).

Secondly, we are going to consider what happens if \( \text{DB}' \models_\text{DB} \neg q_i (\bar{b}_i) \) \((1 \leq i \leq m) \). If \( q_i \) is a built-in predicate, then \( I_R(q_i (\bar{b}_i)) = f \), since \( M(\text{DB}, \text{DB}') \) gives to the built-in predicates in the database the appropriate truth values, and therefore \( M(\text{DB}, \text{DB}') \models q_i (\bar{b}_i) : f_c \). If \( q_i \) is not a built-in predicate, then \( I_F(q_i (\bar{b}_i)) = f \) or \( I_F(q_i (\bar{b}_i)) = f \), and therefore \( M(\text{DB}, \text{DB}') \models q_i (\bar{b}_i) : f_c \).

(b) Let us consider a predicate \( p \in P \). By considering (1) we know that for every tuple \( \bar{a} \) (of appropriate arity) \( I_F(p(\bar{a})) = t \), \( I_F(p(\bar{a})) = f \), \( I_F(p(\bar{a})) = t_a \) or \( I_F(p(\bar{a})) = f_a \), and therefore \( M(\text{DB}, \text{DB}') \models p(\bar{a}) : t_c \lor p(\bar{a}) : f_c \). Thus, we conclude that \( M(\text{DB}, \text{DB}') \models \forall x(p(x)) : t_c \lor p(x) : f_c \).
t_c \lor p(x) : f_c). Additionally, if \( I_P(p(\overline{a})) = t \) or \( I_P(p(\overline{a})) = t_a \), then 
\( M(DB, DB') \not\models p(\overline{a}) : t_c \), and if \( I_P(p(\overline{a})) = f \) or \( I_P(p(\overline{a})) = f_a \), then 
\( M(DB, DB') \not\models p(\overline{a}) : t_c \). Thus, we also conclude that 
\( M(DB, DB') \models \forall \overline{x} (\neg p(x) : t_c \lor \neg p(x) : f_c) \).

\[ \square \]

**Proof of lemma 3:** We are going to prove that \( DB_M \models_{DB} IC \). Let us suppose that \( p_1(T_1) \lor \cdots \lor p_n(T_n) \lor \neg q_1(T_1) \lor \cdots \lor \neg q_m(T_m) \) is an integrity constraint in IC, and let us assume that \( p_1(\overline{a_1}) \lor \cdots \lor p_n(\overline{a_n}) \lor \neg q_1(\overline{b_1}) \lor \cdots \lor \neg q_m(\overline{b_m}) \) was obtained from it by instantiated in the domain of the database. In this case we have that \( p_1(\overline{a_1}) : t_c \lor \cdots \lor p_n(\overline{a_n}) : t_c \lor q_1(\overline{b_1}) : f_c \lor \cdots \lor q_m(\overline{b_m}) : f_c \). Could be obtained by instantiated an integrity constraint in \( T(\text{IC}) \). Thus, we have that 
\( M \models p_1(\overline{a_1}) : t_c \lor \cdots \lor p_n(\overline{a_n}) : t_c \lor q_1(\overline{b_1}) : f_c \lor \cdots \lor q_m(\overline{b_m}) : f_c \).

Firstly, we are going to consider what happens if \( M \models p_i(\overline{a_i}) : t_c \) (1 \( \leq i \leq n \)). If \( p_i \) is a built-in predicate, then \( I_R(p_i(\overline{a_i})) = t \), since \( M \) gives to the built-in predicates in the database the value \( t \) or \( f \), and if in this case we suppose that \( I_R(p_i(\overline{a_i})) = f \) then \( M \not\models p_i(\overline{a_i}) : t_a \), a contradiction. Therefore 
\( DB_M \models_{DB} p_i(\overline{a_i}) \). If \( p_i \) is not a built-in predicate, then \( p_i(\overline{a_i}) : t_d \in T(DB) \) or \( p_i(\overline{a_i}) : f_d \in T(DB) \). In the first case we have that \( M \models p_i(\overline{a_i}) : t \), and therefore \( p_i(\overline{a_i}) \in DB_M \). In the second case \( M \models p_i(\overline{a_i}) : t_n \), and therefore \( p_i(\overline{a_i}) \in DB_M \).

Secondly, we are going to consider what happens if \( M \models q_i(\overline{b_i}) : f_c \) (1 \( \leq i \leq m \)). If \( q_i \) is a built-in predicate, then \( I_R(q_i(\overline{b_i})) = f \), since \( M \) gives to the built-in predicates in the database the value \( t \) or \( f \), and if in this case we suppose that \( I_R(q_i(\overline{b_i})) = t \) then \( M \not\models q_i(\overline{b_i}) : f_c \), a contradiction. Therefore 
\( DB_M \models_{DB} \neg q_i(\overline{b_i}) \). If \( q_i \) is not a built-in predicate, then \( q_i(\overline{b_i}) : t_d \in T(DB) \) or \( q_i(\overline{b_i}) : f_d \in T(DB) \). In the first case we have that \( M \models q_i(\overline{b_i}) : f_a \), and therefore \( q_i(\overline{b_i}) \not\in DB_M \). In the second case \( M \models q_i(\overline{b_i}) : f \), and therefore \( q_i(\overline{b_i}) \not\in DB_M \).

\[ \square \]

**Proof of proposition 1:**

1. By Lemma 3, we conclude that \( DB_M \models_{DB} IC \).
2. Now, we need to prove that \( DB_M \) is minimal. Let us suppose this is not true. Then, there is a database instance \( DB^* \) such that \( DB^* \models_{DB} IC \) and 
\( \Delta(DB, DB^*) \subseteq \Delta(DB, DB_M) \).
   (a) From Lemma 2, we conclude that \( M((DB, DB^*) \models T(DB, IC)) \).
   (b) Now, we are going to prove that \( M(DB, DB^*) \preceq M \).
   If \( M(DB, DB^*) \models p(\overline{a}) : t_a \), then by considering (1) we can conclude that 
\( p(\overline{a}) \not\in DB \) and \( p(\overline{a}) \in DB^* \), and therefore \( p(\overline{a}) \in \Delta(DB, DB^*) \).
   But \( \Delta(DB, DB^*) \subseteq \Delta(DB, DB_M) \), and therefore \( p(\overline{a}) \in DB_M \). Thus, 
we can conclude that \( M \models p(\overline{a}) : t \lor p(\overline{a}) : f_a \). If we suppose that 
\( M \models p(\overline{a}) : t \), then \( M \not\models p(\overline{a}) : f_a \), but we know that 
\( M \models T(DB, IC) \) and \( p(\overline{a}) : t_d \in T(DB, IC) \), since \( p(\overline{a}) \not\in DB \), a contradiction. Therefore, 
\( M \models p(\overline{a}) : t_a \).
   If \( M(DB, DB^*) \models p(\overline{a}) : f_a \), then by considering (1) we can conclude that 
\( p(\overline{a}) \in DB \) and \( p(\overline{a}) \not\in DB^* \), and therefore \( p(\overline{a}) \in \Delta(DB, DB^*) \).
But $\Delta(\text{DB}, \text{DB}') \triangleq \Delta(\text{DB}, \text{DB}_M)$, and therefore $p(\bar{a}) \notin \text{DB}_M$. Thus, we can conclude that $\mathcal{M} \models p(\bar{a}) : f \lor p(\bar{a}) : f_a$. If we suppose that $\mathcal{M} \models p(\bar{a}) : f$, then $\mathcal{M} \not\models p(\bar{a}) : t_d$, but we know that $\mathcal{M} \models T(\text{DB}, \text{IC})$ and $p(\bar{a}) : t_d \in T(\text{DB}, \text{IC})$, since $p(\bar{a}) \in \text{DB}$, a contradiction. Therefore, $\mathcal{M} \models p(\bar{a}) : f_a$. Thus, we can deduce that $\mathcal{M}(\text{DB}, \text{DB}') \leq_{\Delta} \mathcal{M}$. Finally, we know that there exists $p(\bar{a})$ such that it is not in $\Delta(\text{DB}, \text{DB}')$ and it is in $\Delta(\text{DB}, \text{DB}_M)$. Thus, $p(\bar{a}) \in \text{DB}$ and $p(\bar{a}) \notin \text{DB}'$, and therefore $\mathcal{M}(\text{DB}, \text{DB}') \models p(\bar{a}) : t$, or $p(\bar{a}) \notin \text{DB}$ and $p(\bar{a}) \notin \text{DB}'$, and therefore $\mathcal{M}(\text{DB}, \text{DB}') \models p(\bar{a}) : f$. Then, we have that $\mathcal{M}(\text{DB}, \text{DB}') \not\models p(\bar{a}) : t_a$ and $\mathcal{M}(\text{DB}, \text{DB}') \not\models p(\bar{a}) : f_a$. Additionally, since $p(\bar{a}) \in \Delta(\text{DB}, \text{DB}_M)$, we can conclude that $p(\bar{a}) \in \text{DB}$ and $p(\bar{a}) \notin \text{DB}_M$, or $p(\bar{a}) \notin \text{DB}$ and $p(\bar{a}) \notin \text{DB}_M$. In the first case we can conclude that $\mathcal{M} \models p(\bar{a}) : f_a$, since $\mathcal{M}$ must be satisfied $p(\bar{a}) : f \lor p(\bar{a}) : f_a$, and if we suppose that $\mathcal{M} \models p(\bar{a}) : f$, then $\mathcal{M} \not\models p(\bar{a}) : t_d$, but $p(\bar{a}) : t_d \in T(\text{DB}, \text{IC})$ in this case, a contradiction. In the second case we can conclude that $\mathcal{M} \models p(\bar{a}) : t_a$, since $\mathcal{M}$ must be satisfied $p(\bar{a}) : t \lor p(\bar{a}) : t_a$, and if we suppose that $\mathcal{M} \models p(\bar{a}) : t$, then $\mathcal{M} \not\models p(\bar{a}) : f_a$, but $p(\bar{a}) : f_a \in T(\text{DB}, \text{IC})$ in this case, a contradiction. Thus, we can conclude that $\mathcal{M} \models p(\bar{a}) : t_a \lor p(\bar{a}) : f_a$. Therefore we can deduce that $\mathcal{M} \not\models \mathcal{M}(\text{DB}, \text{DB}')$.

Finally, we deduce that $\mathcal{M}$ is not e-consistent maximal in the class of the models of $T(\text{DB}, \text{IC})$, with respect to $\Delta$, a contradiction.

\hspace{1cm} \Box

Proof of proposition 2:

1. By Lemma 2, we conclude that $\mathcal{M}(\text{DB}, \text{DB}') \models T(\text{DB}, \text{IC})$.

2. Let us suppose that $\mathcal{M}(\text{DB}, \text{DB}')$ is not e-consistent maximal in the class of models of $T(\text{DB}, \text{IC})$ with respect to $\Delta$. Then, there exists $\mathcal{M} \models T(\text{DB}, \text{IC})$, such that $\mathcal{M} <_{\Delta} \mathcal{M}(\text{DB}, \text{DB}')$. By using this it is possible to prove that $\Delta(\text{DB}, \text{DB}_M) \triangleq \Delta(\text{DB}, \text{DB}')$.

(a) Let us suppose that $p(\bar{a}) \in \Delta(\text{DB}, \text{DB}_M)$. Then $p(\bar{a}) \in \text{DB}$ and $p(\bar{a}) \notin \text{DB}_M$, or $p(\bar{a}) \notin \text{DB}$ and $p(\bar{a}) \notin \text{DB}_M$. In the first case we can conclude that $p(\bar{a}) : t_d \in T(\text{DB}, \text{IC})$ and $\mathcal{M} \models p(\bar{a}) : f \lor p(\bar{a}) : f_a$. If we suppose that $\mathcal{M} \models p(\bar{a}) : f$, then $\mathcal{M} \not\models p(\bar{a}) : t_d$, a contradiction. Thus, we have that $\mathcal{M} \models p(\bar{a}) : f_a$. But $\mathcal{M} <_{\Delta} \mathcal{M}(\text{DB}, \text{DB}')$, and therefore $\mathcal{M}(\text{DB}, \text{DB}') \models p(\bar{a}) : f_a$. Then, by considering (1) we conclude that $p(\bar{a}) \notin \text{DB}'$, and therefore in this case it is possible to conclude that $p(\bar{a}) \notin \Delta(\text{DB}, \text{DB}')$.

In the second case we can conclude that $p(\bar{a}) : f_d \in T(\text{DB}, \text{IC})$ and $\mathcal{M} \models p(\bar{a}) : t \lor p(\bar{a}) : t_a$. If we suppose that $\mathcal{M} \models p(\bar{a}) : t$, then $\mathcal{M} \not\models p(\bar{a}) : f_a$, a contradiction. Thus, we have that $\mathcal{M} \models p(\bar{a}) : t_a$. But $\mathcal{M} <_{\Delta} \mathcal{M}(\text{DB}, \text{DB}')$, and therefore $\mathcal{M}(\text{DB}, \text{DB}') \models p(\bar{a}) : t_a$. Then, by considering (1) we conclude that $p(\bar{a}) \in \text{DB}'$, and therefore in this case it is possible to conclude that $p(\bar{a}) \notin \Delta(\text{DB}, \text{DB}')$. Thus, we can conclude that $\Delta(\text{DB}, \text{DB}_M) \triangleq \Delta(\text{DB}, \text{DB}')$. 
(b) Since $\mathcal{M}(\text{DB}, \text{DB}') \notin \Delta \mathcal{M}$, there exists $p(\bar{a})$ such that $\mathcal{M}(\text{DB}, \text{DB}') \models p(\bar{a}) : t_a \lor p(\bar{a}) : f_a$ and $\mathcal{M} \models p(\bar{a}) : t \lor p(\bar{a}) : f$. By using (1) and the first fact it is possible to conclude that $p(\bar{a}) \in \Delta(\text{DB}, \text{DB}')$. If we suppose that $p(\bar{a}) \in \text{DB}$, then $p(\bar{a}) : t_a \in \mathcal{T}(\text{DB}, \text{IC})$, and therefore by considering the second fact it is possible to deduce that $\mathcal{M}$ must satisfy $p(\bar{a}) : t$. Thus, we can conclude that in this case $p(\bar{a}) \in \text{DB}_M$, and therefore $p(\bar{a}) \notin \Delta(\text{DB}, \text{DB}_M)$. By the other hand, if we suppose that $p(\bar{a}) \notin \text{DB}$, then $p(\bar{a}) : f_a \in \mathcal{T}(\text{DB}, \text{IC})$, and therefore by considering the second fact it is possible to deduce that $\mathcal{M}$ must satisfy $p(\bar{a}) : f$. Thus, we can conclude that in this case $p(\bar{a}) \notin \text{DB}_M$, and therefore $p(\bar{a}) \notin \Delta(\text{DB}, \text{DB}_M)$. Finally, we conclude that $\Delta(\text{DB}, \text{DB}_M) \subseteq \Delta(\text{DB}, \text{DB}')$.

We know that $\text{DB}'$ is a database instance, and therefore $\Delta(\text{DB}, \text{DB}')$ must be a finite set. Thus, we can conclude that $\Delta(\text{DB}, \text{DB}_M)$ is a finite set, and therefore $\text{DB}_M$ is a database instance. With the help of Lemma 3, we deduce that $\text{DB}_M \models \text{IC}$. But this is a contradiction, since $\text{DB}'$ is a repair of $\text{DB}$ with respect to $\text{IC}$ and $\Delta(\text{DB}, \text{DB}_M) \subseteq \Delta(\text{DB}, \text{DB}')$.

\[\square\]

Proof of Lemma 4: Let us suppose that

$$\mathcal{T}(\text{DB}, \text{IC}) \models r_1(\bar{a}_1) : t_a \lor \cdots \lor r_k(\bar{a}_k) : t_a.$$  

(5)

Because of the form of the clauses in $\mathcal{T}(\text{DB}, \text{IC})$, the above $a$-clause can be obtained by applying a series of reduction and resolution rules to the clauses in $\mathcal{T}(\text{DB}) \cup \mathcal{T}(\mathcal{B})$ (the database part of $\mathcal{T}(\text{DB}, \text{IC})$ plus builtins) and a clause of the form

$$r_1(\bar{t}_1) : f_c \lor \cdots \lor r_j(\bar{t}_j) : f_c \lor r_{j+1}(\bar{t}_{j+1}) : t_c \lor \cdots \lor r_k(\bar{t}_k) : t_c,$$

(6)

where the latter is a clause obtained from $\mathcal{T}(\text{IC})$ (the constraint part of $\mathcal{T}(\text{DB}, \text{IC})$) by resolution (and factorization) alone.

Furthermore, it is easy to show that resolution applied to a pair of range-restricted constraints yields a range-restricted constraint. Thus, (6) is range restricted.

Since (5) is obtained from (6) by resolution and reduction with the clauses in $\mathcal{T}(\text{DB})$, there must be clauses $r_i(\bar{c}_i) : T \in \mathcal{T}(\text{DB})$, $1 \leq i \leq j$ (which are resolved with (6)), and clauses $r_{i'}(\bar{e}_{i'}) : T \in \mathcal{T}(\text{DB})$, $j < i' \leq k$ (which are reduced with (6)), such that there is a substitution $\theta$ for which $\bar{t}_i - \bar{a}_i$ (1 $\leq i \leq h$).

Therefore, due to the range-restrictedness of (6), every constant in $\bar{e}_{i'}$ ($j < i' \leq k$) occurs in some $\bar{c}_i$ (1 $\leq i \leq j$). Since every constant in $\bar{c}_i$ is in the active domain of $\text{DB}$, we conclude that every constant mentioned in (5) belongs to the active domain of $\text{DB}$.  

\[\square\]
Proof of corollary 1: By Lemma 4, the clauses in $T^o(DB, IC)$ can mention only the constants that occur in the active domain of $DB$, which is a finite set.

Proof of theorem 3: At the end of section 6 we showed that the decision problem is equivalent to the problem of deciding, given a finite collection of sets, and a subset of the union of the family, whether the subset can be extended to a minimal hitting set of the family. In the following lemmas we prove that this is $NP$-complete.

Lemma 5. Given a finite collection of sets $S$ and a hitting set of it $H$, $H$ is a minimal hitting set of $S$ if and only if for each $h \in H$ there exists an $A \in S$ such that $A \cap H = \{h\}$.

Proof

($\Rightarrow$) Let us suppose that the lemma is not true. Then there exists $h \in H$ such that for every $A \in S, A \cap H \neq \{h\}$. We are going to prove $H' = H - \{h\}$ is also a hitting set. Let us consider $A \in S$. If $h \in A$, then there exists another $h' \in H$ such that $h' \in A$, since $A \cap H \neq \{h\}$, and therefore $A \cap H' \neq \emptyset$. If $h \notin A$, then there exist $h' \neq h$ such that $h' \in A \cap H$, and therefore $A \cap H' \neq \emptyset$. Thus, we obtain a contradiction.

($\Leftarrow$) If $H' \subseteq H$, then there exists $h \in H$ such that $h \notin H'$. But we know that there is a set $A \in S$ such that $A \uparrow H = \{h\}$, and therefore $A \uparrow H' = \emptyset$. Thus, $H'$ is not a hitting set of $S$.

Lemma 6. Given a finite collection of sets $S$ and a set $H \subseteq \cup S$, the problem of deciding if there exists a minimal hitting set $H'$ of $S$ such that $H \subseteq H'$ is $NP$.

Proof We are going to reduce our problem to SAT. For each $x \in \cup S$ we introduce a propositional letter $x$, and we define:

$$f(S, H) = (\bigwedge_{h \in H} \bigvee_{\{A \in S \mid h \in A\}} \bigwedge_{\{a \in A \mid a \neq h\}} \neg a) \wedge \left(\bigwedge_{h \in H} \left(\bigwedge_{\{A \in S \mid A \cap H = \emptyset\}} \bigvee_{a \in A} a\right)\right).$$

There exists a minimal hitting set $H'$ of $S$ which contains $H$ if and only if $f(H, S)$ is a satisfied formula.

($\Rightarrow$) For every proposition letter $x$ in $f(H, S)$ we define $\sigma(x) = 1$ if and only if $x \in H'$.

1. If $h \in H$, then $h \notin H'$, and therefore by lemma 5 we conclude that there exists $A \in S$ such that $A \cap H' = \{h\}$. Thus, for every $a \in A - \{h\}$ we have that $a \notin H'$ and then $\sigma(a) = 0$. We conclude that $\sigma(\bigvee_{\{A \in S \mid h \in A\}} \bigwedge_{\{a \in A \mid a \neq h\}} \neg a) = 1$.

2. $\sigma(\bigwedge_{h \notin H} h) = 1$, since $H \subseteq H'$. 
3. If \( A \subseteq S \) and \( A \cap H = \emptyset \), then \( A \cap (H' - H) \neq \emptyset \), since \( H' \) is a hitting set of \( S \). Thus, there exists \( a \in H' \) such that \( a \in A \), and therefore \( \sigma(a) = 1 \). We conclude that \( \sigma(\bigvee_{a \in A} a) = 1 \).

\( (\Leftarrow) \) Let \( \sigma \) such that \( \sigma(f(H, S)) = 1 \). We construct \( H'' = \{ x \mid \sigma(x) = 1 \} \). \( H \subseteq H'' \), since \( \sigma(\bigwedge_{h \in H} h) = 1 \). \( H'' \) is a hitting set of \( S \). Let us consider \( A \subseteq S \). If \( A \cap H \neq \emptyset \), then \( A \cap H'' \neq \emptyset \). If \( A \cap H = \emptyset \), then \( \sigma(\bigvee_{a \in A} a) = 1 \), and therefore \( A \cap (H'' - H) \neq \emptyset \).

\( H'' \) is a finite set. Then there exists a minimal hitting set of \( S \) such that \( H' \subseteq H'' \). We are going to prove that \( H \subseteq H' \). By contradiction, let us suppose that there exists \( h \in H \) such that \( h \not\in H' \). We know that \( \sigma(\bigvee_{A \subseteq S} A) \bigwedge_{a \in A \bigwedge a \neq h} \neg a = 1 \). Then there exists \( A \subseteq S \) such that \( \sigma(\bigvee_{A \subseteq S} A) \bigwedge_{a \in A \bigwedge a \neq h} \neg a = 1 \), and therefore \( A \cap H'' = \emptyset \), by definition of \( H' \) and given that \( h \not\in H' \). Thus, we conclude a contradiction.

**Lemma 7.** Given a finite collection of sets \( S \) and a set \( H \subseteq \bigcup S \), the problem of deciding if there exists a minimal hitting set \( H' \) of \( S \) such that \( H \subseteq H' \) is NP-hard

**Proof.** We are going to reduce SAT(3) to our problem. Given a formula \( \varphi = C_1 \land \cdots \land C_k \), where every \( C_i \) is a clause, we define \( PL(\varphi) \) as the set of propositional letters mentioned in it. Additionally, for each clause \( C_i \), of the form \( p_1 \lor \cdots \lor p_n \lor \neg q_1 \lor \cdots \lor \neg q_m \), we define

\[
CH(C_i) = \{ p_1, \ldots, p_n, q_1, \ldots, q_m, 0 \}.
\]

After that, we define \( f(\varphi) = (S, H) \), where

\[
S = \{ \{ v.p, p.0 \} \mid p \in PL(\varphi) \} \cup \{ v.p, p.1 \} \mid p \in PL(\varphi) \} \cup \{ CH(C_i) \mid 1 \leq i \leq k \}
\]

\[
H = \{ v.p \mid p \in PL(\varphi) \}
\]

We are going to prove that \( \varphi \) is consistent if and only if there exists a minimal hitting set \( H' \) of \( S \) such that \( H \subseteq H' \).

\( (\Rightarrow) \) Let \( \sigma \) that satisfies \( \varphi \). We define

\[
H'' = H \cup \{ p.0 \mid p \in PL(\varphi) \} \cup \{ p.1 \mid p \in PL(\varphi) \} \cup \{ \sigma(p) = 1 \}
\]

\( H'' \) is a hitting set of \( S \), and therefore there exists \( H' \) minimal hitting set of \( S \) such that \( H' \subseteq H'' \), since \( H'' \) is a finite set. If we suppose that there is \( v.p \in H \) such that \( v.p \not\in H' \), then \( H' \cap \{ v.p, p.0 \} = \emptyset \) or \( H' \cap \{ v.p, p.1 \} = \emptyset \), given that \( \sigma(p) = 1 \) or \( \sigma(p) = 0 \). Thus, we conclude a contradiction.

\( (\Leftarrow) \) Let us suppose that there exists \( H' \) minimal hitting set of \( S \) such that \( H \subseteq H' \). Notice that for every \( p \in PL(\varphi) \) we have that \( p.0 \not\in H' \) or \( p.1 \not\in H' \), since if both elements would be in \( H' \), then \( H' - \{ v.p \} \) will be a hitting set, a contradiction given that \( H' \) is minimal. Thus, we can define a function \( \sigma : PL(\varphi) \rightarrow \{ 0, 1 \} \) by means of the rule \( \sigma(p) = 1 \) if and only if \( p.1 \in H' \). We have that \( \sigma(\varphi) = 1 \), given that for every clause \( C_i \), \( H' \cap CH(C_i) \neq \emptyset \). \( \square \)