Epipolar Geometry

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Problem Definition

- Simple stereo configuration
  - Corresponding points are on same horizontal line
  - This makes correspondence search a 1D search
  - Need only look for matches on same horizontal line
- if two cameras are in an arbitrary location is there a similar constraint to make search 1D?
  - Yes, called epipolar constraint
  - Based on epipolar geometry
  - We will derive this constraint
  - Consider two cameras that can see a single point P
  - They are in an arbitrary positions and orientation
    - One camera is rotated and translated relative to the other camera

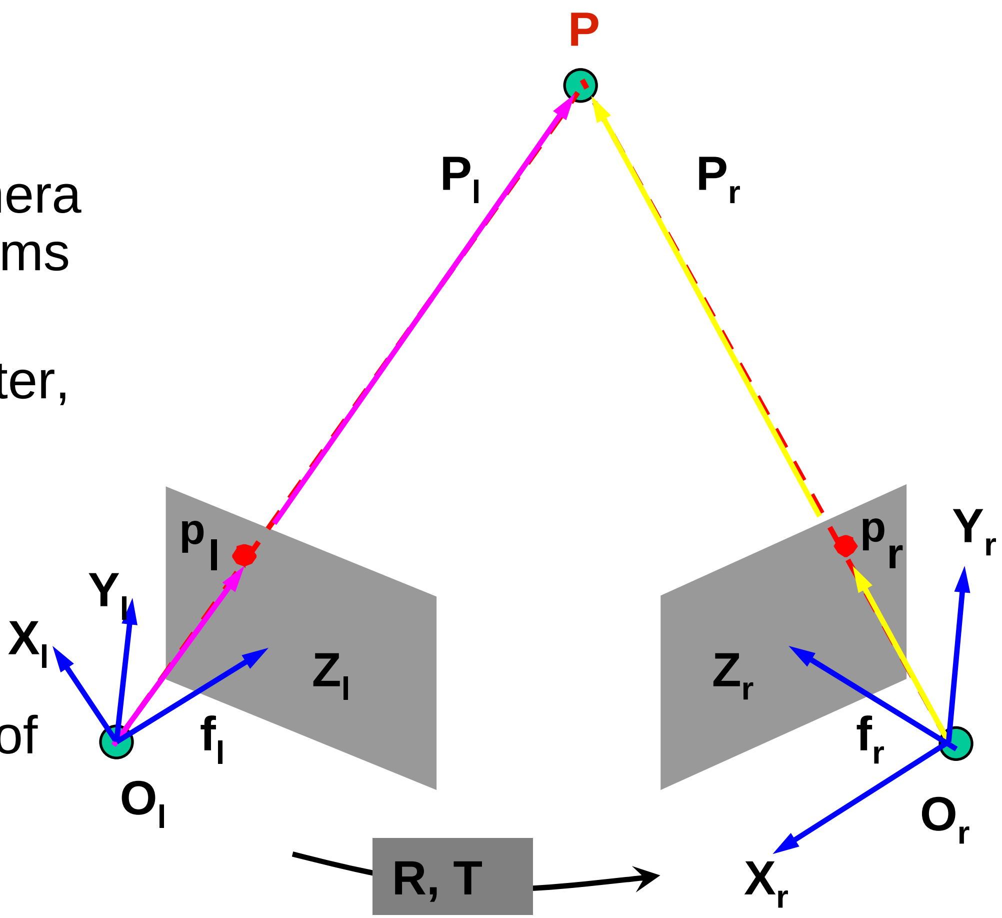
Parameters of a Stereo System

Intrinsic Parameters

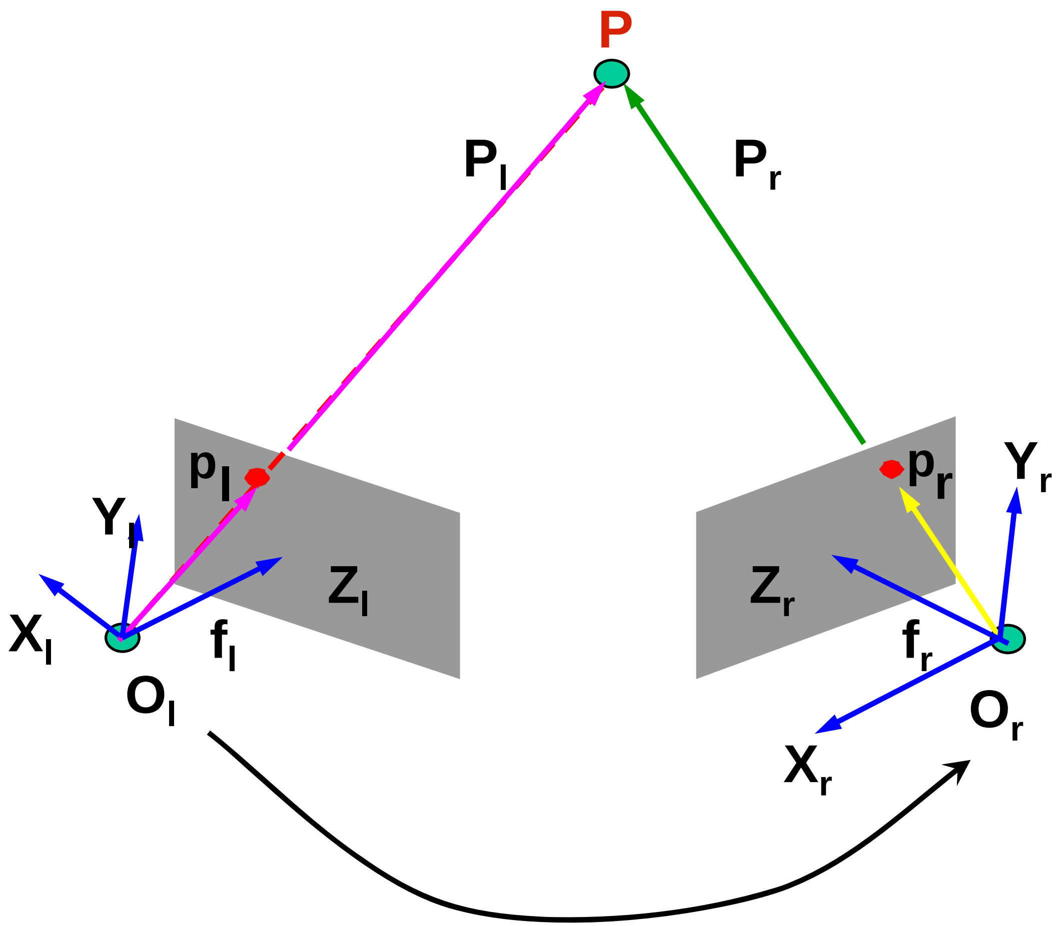
- Characterize the transformation from camera to pixel coordinate systems of each camera
- Focal length, image center, aspect ratio

Extrinsic parameters

- Describe the relative position and orientation of the two cameras
- Rotation matrix R and translation vector T

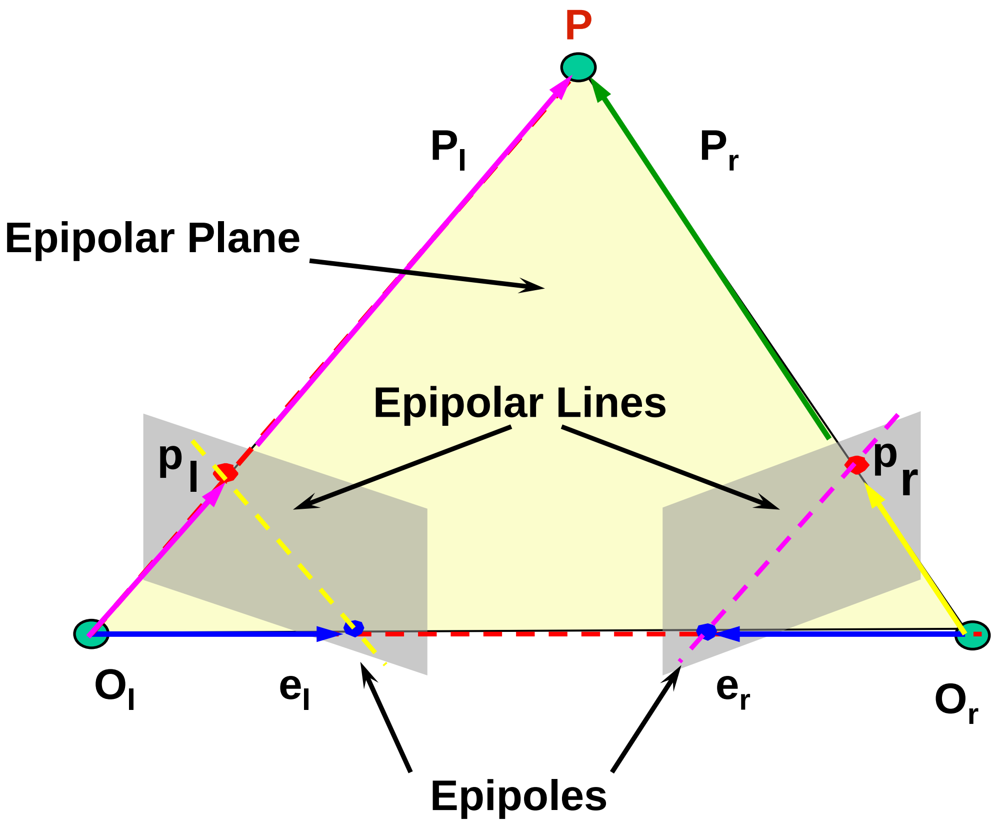


Epipolar Geometry



rotation R translation T

Epipolar Geometry

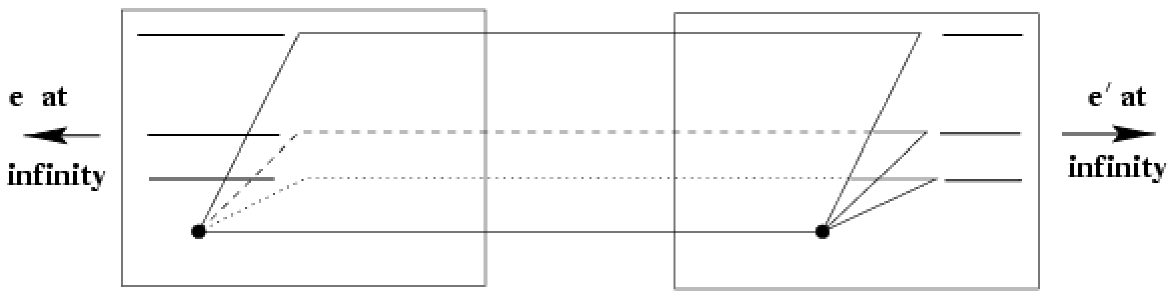


$$P_r = R(P_l - T)$$

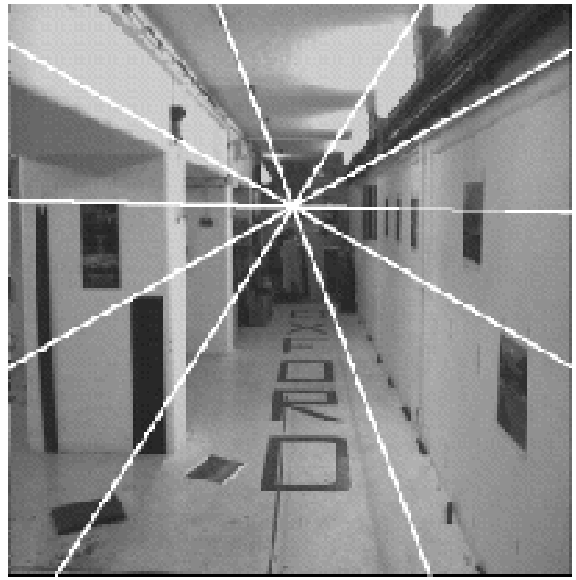
Shape of epipolar lines

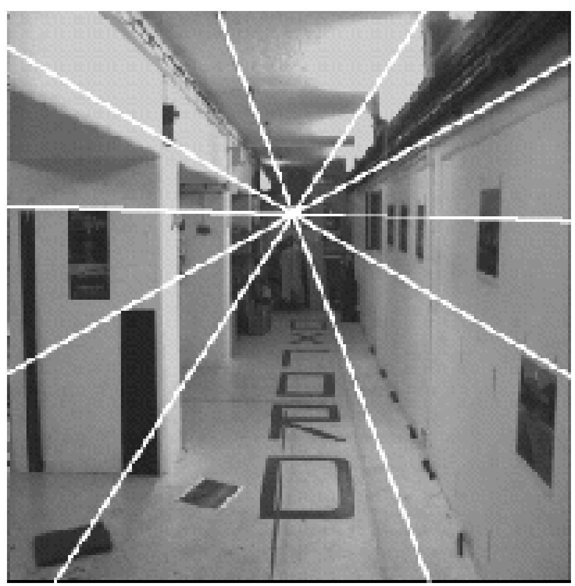
- Translating the cameras without rotation then the epipolar lines are parallel
- Translating the cameras in the direction of the camera y axis (horizontal) you get the simple stereo configuration of horizontal epipolar lines
- Translating the cameras in the z axis produces epipolar lines that emanate from the epipole (sometimes called focus of projection)

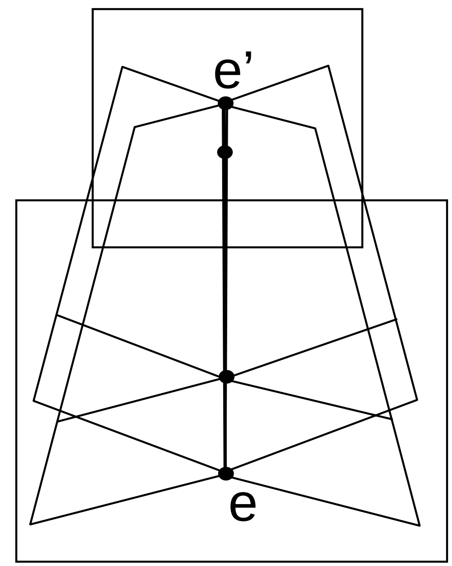
Example: motion parallel with image plane



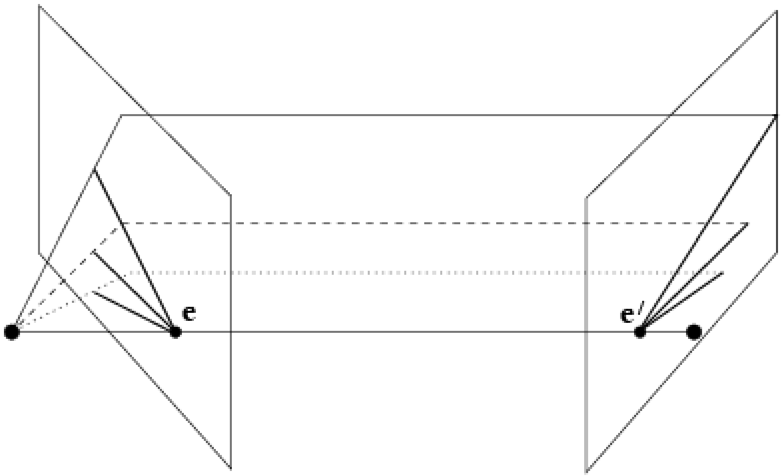
Example: forward motion

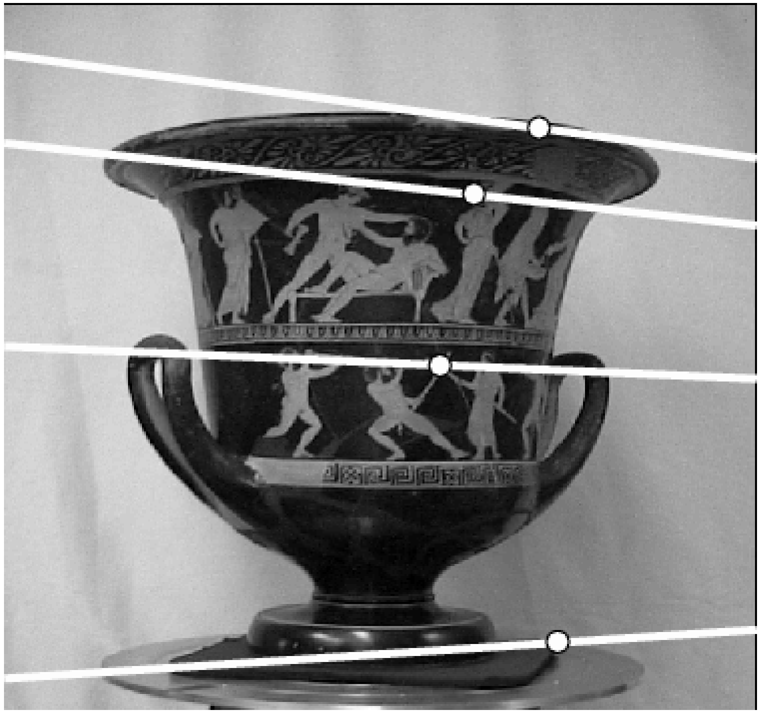


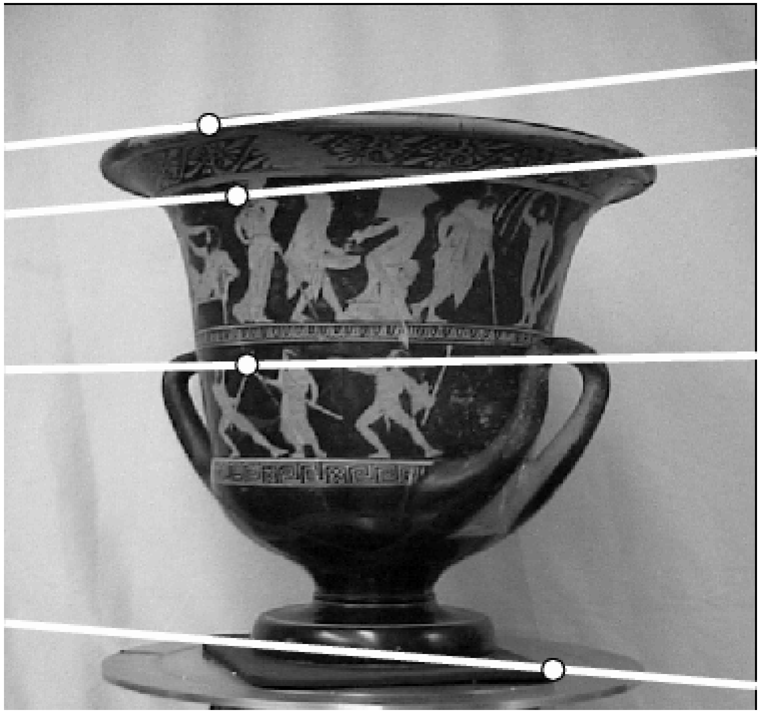




Example: converging cameras







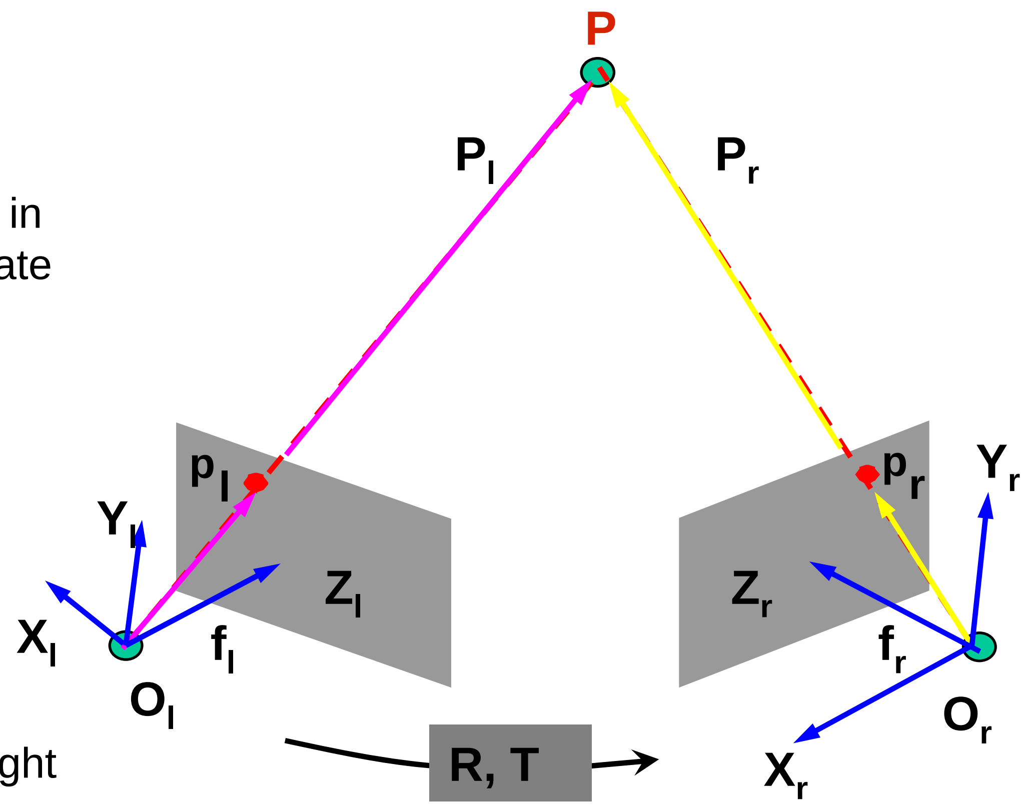
Epipolar Geometry

Notations

- $P_l = (X_l, Y_l, Z_l)$, $P_r = (X_r, Y_r, Z_r)$
  - Vectors of the same 3-D point P, in the left and right camera coordinate systems respectively
- Extrinsic Parameters
  - Translation Vector $T = (O_r - O_l)$
  - Rotation Matrix R

$$\mathbf{P_r} = \mathbf{R}(\mathbf{P_l} - \mathbf{T})$$

- $p_l = (x_l, y_l, z_l)$, $p_r = (x_r, y_r, z_r)$
  - Projections of P on the left and right image plane respectively
  - For all image points, we have $z_l = f_l$, $z_r = f_r$



$$\mathbf{p}_l = \frac{f_l}{Z_l}\mathbf{P_l} \qquad \mathbf{p}_r = \frac{f_r}{Z_r}\mathbf{P}_r$$

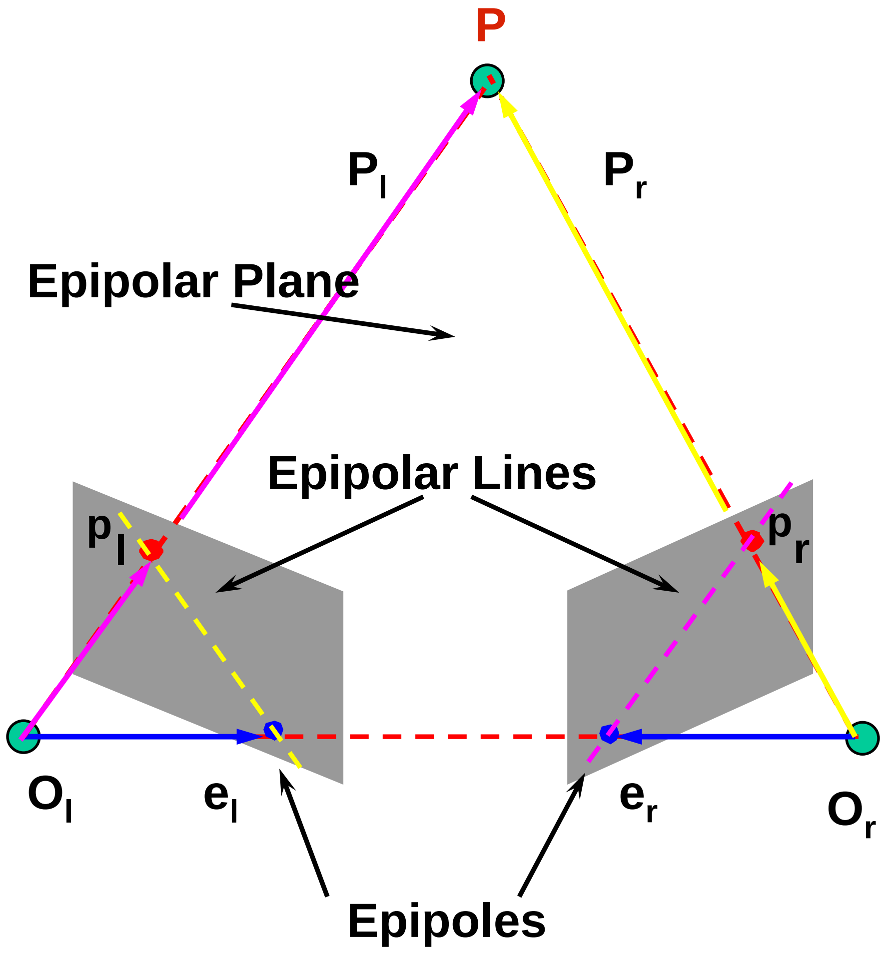
Epipolar Geometry

Motivation: where to search correspondences?

- Epipolar Plane
  - A plane going through point P and the centers of projection (COPs) of the two cameras
- Epipolar Lines
  - Lines where epipolar plane intersects the image planes
- Epipoles
  - The image in one camera of the COP of the other

Epipolar Constraint

- Corresponding points must lie on epipolar lines
- True for EVERY camera configuration!



Epipolar Geometry

Epipolar plane: plane going through point P and the centers of projection (COPs) of the two cameras

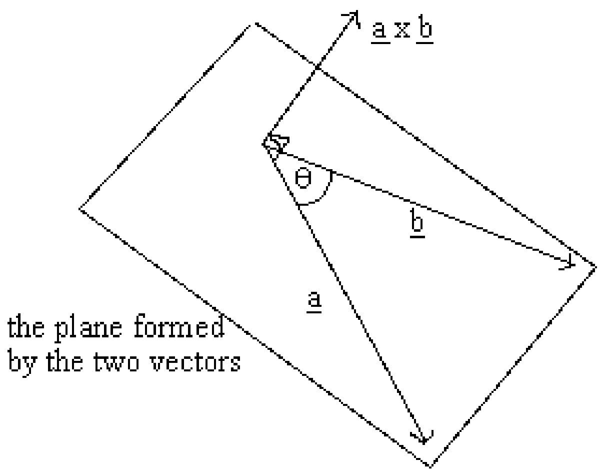
Epipolar lines: where this epipolar plane intersects the two image planes

Epipoles: The image in one camera of the COP of the other

Epipolar Constraint: Corresponding points between the two images must lie on epipolar lines

Cross product

- Consider two vectors in 3D space
  - (a1,a2,a3) and (b1,b2,b3)
- **a** x **b** = **n** a b sin q
- Cross product is at 90 degrees to both vectors
  - Normal to the plane defined by the two vectors



Cross product

- Two possible normal directions
  - We use the right hand rule to compute direction
- $\underline{\mathbf{a}} \times \underline{\mathbf{b}} =$

  $(a1\,\underline{\mathbf{i}} + a2\,\underline{\mathbf{j}} + a3\,\underline{\mathbf{k}}) \times (b1\,\underline{\mathbf{i}} + b2\,\underline{\mathbf{j}} + b3\,\underline{\mathbf{k}})$

- $\underline{\mathbf{a}} \times \underline{\mathbf{b}} =$

  $(a2b3 - a3b2)\underline{\mathbf{i}} + (a3b1 - a1b3)\underline{\mathbf{j}} + (a1b2 - a2b1)\underline{\mathbf{k}}$
- Cross product can also be written as multiplication by a matrix
- $\underline{\mathbf{a}} \times \underline{\mathbf{b}} = S\ \underline{\mathbf{b}}$

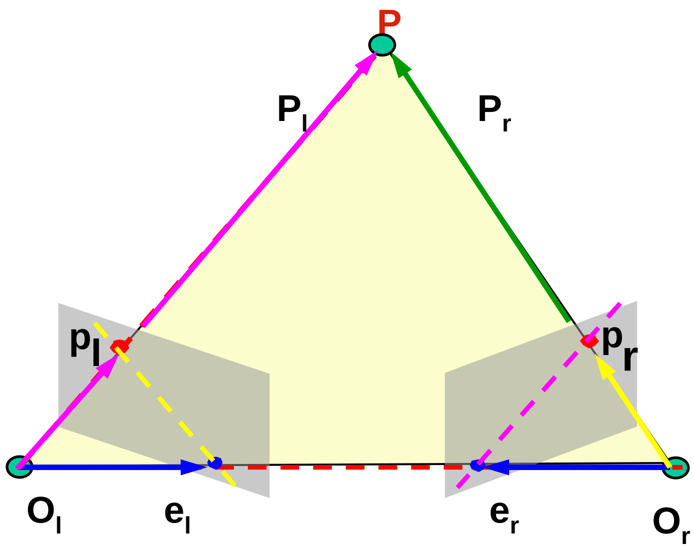
Cross product as matrix multiplication

- Define matrix S as

$$\begin{bmatrix} 0 & -a3 & a2 \\ a3 & 0 & -a1 \\ -a2 & a1 & 0 \end{bmatrix}$$

- **a** x **b** = S **b**
  - Try the program cross1.ch on the course web site

Essential Matrix



$$T \times P_l = SP_l$$

$$S = \begin{bmatrix} 0 & -T_z & T_y \\ T_z & 0 & -T_x \\ -T_y & T_x & 0 \end{bmatrix}$$

Coordinate Transformation: $P_r = R(P_l - T)$

T, P_l, P_l–T are coplanar $(P_l - T)^T T \times P_l = 0$

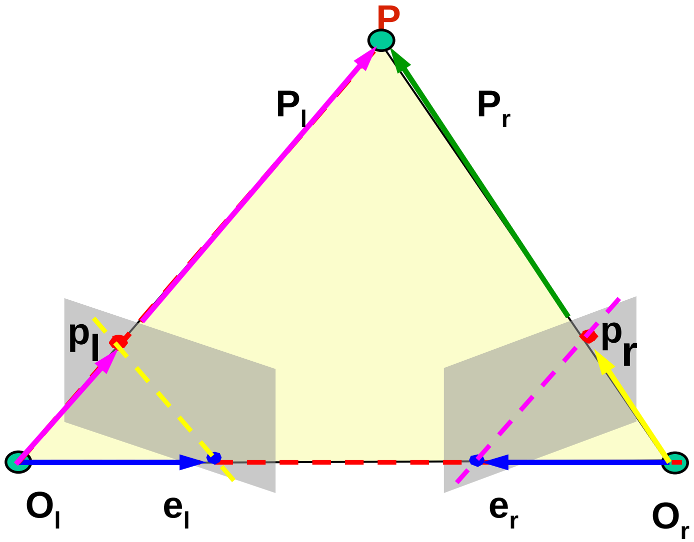
Resolves to $(R^T P_r)^T T \times P_l = 0$

$$(R^T P_r)^T SP_l = 0$$

$$P_r^T RSP_l = 0$$

Essential Matrix $E = RS$ $P_r^T EP_l = 0$

Essential Matrix



$$P_r^T E P_l = 0 \quad \Rightarrow \quad p_r^T E p_l = 0$$

$$\mathbf{P_r}^{\mathbf{T}} \mathbf{E} \mathbf{P_l} = 0$$

$$\mathbf{p}_l = \frac{f_l}{Z_l} \mathbf{P_l} \qquad \mathbf{p}_r = \frac{f_r}{Z_r} \mathbf{P}_r$$

$$\mathbf{p_r}^{\mathbf{T}} \mathbf{E} \mathbf{p_l} = 0$$

Essential Matrix

Essential Matrix E = RS

$$\mathbf{p_r}^{\mathbf{T}}\mathbf{E}\mathbf{p_l} = 0$$

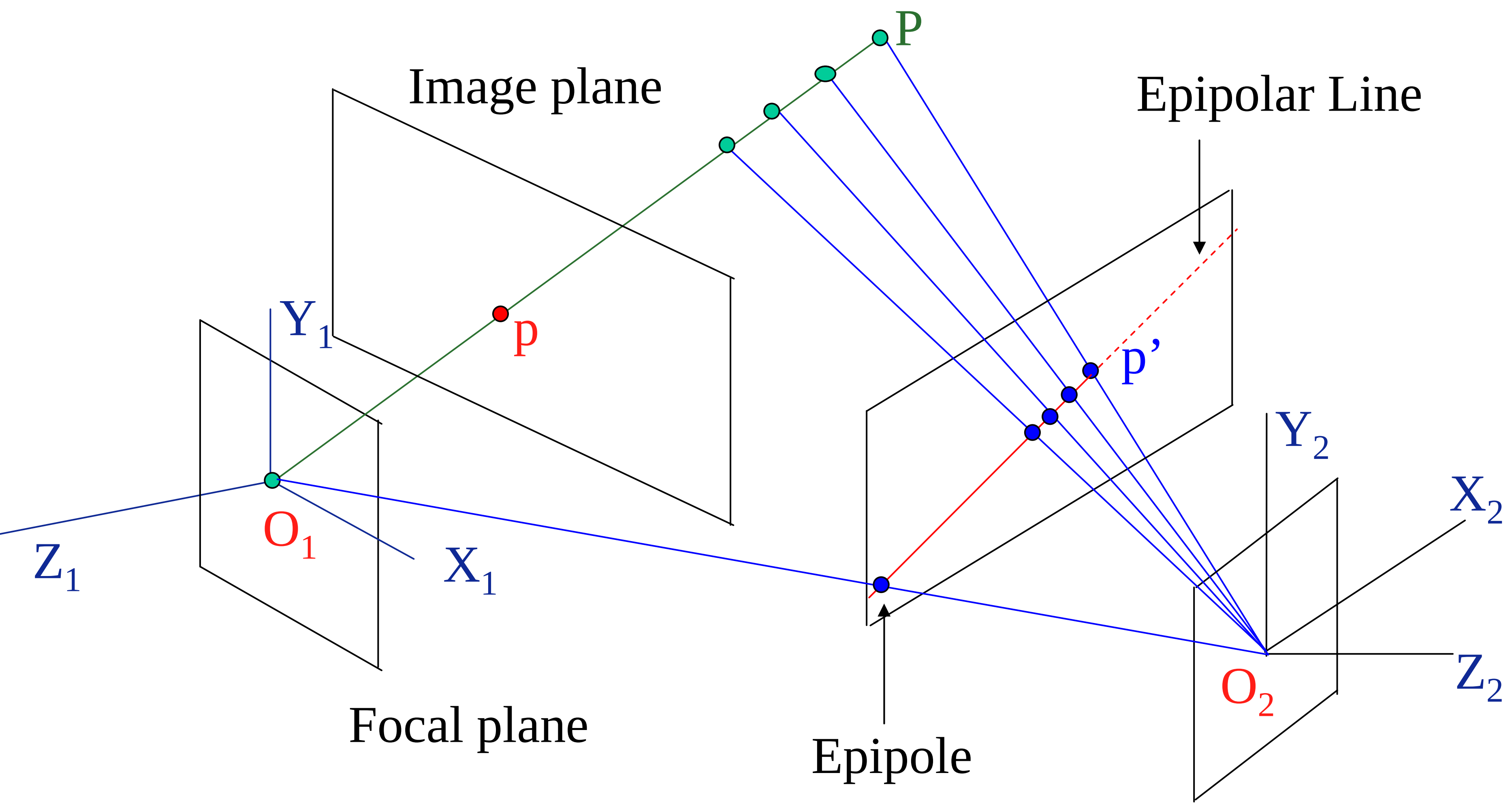
- A natural link between the stereo point pair and the extrinsic parameters of the stereo system
  - Can compute E from R, and T (S) but this is not always possible
  - Often E is computed from a set of correspondences
  - One correspondence -> a linear equation of 9 entries
  - Given 8 pairs of (pl, pr) -> E (will describe this process later)

- Mapping between points and epipolar lines we are looking for
  - Given p_l, E -> p_r on the projective line in the right plane
  - Equation represents the epipolar line of either pr (or pl) in the right (or left) image

Note:

- pl, pr are in the camera coordinate system, not pixel coordinates that we can measure

What does Essential Matrix Mean?



Projective Geometry

- Projective Plane - P^2
  - Set of equivalence classes of triplets of real numbers
  - $p = [x,y,z]^T$ and $p' = [x',y',z']^T$ are equivalent if and only if there is a real number k such that $[x,y,z]^T = k\,[x',y',z']^T$
  - Each projective point p in P^2 corresponds to a line through the origin in P^3
  - So points in P^2, the projective plane, and lines in P^3, ordinary space, are in a one to one correspondence
  - A line in the projective plane is called a projective line represented by $u = [ux, uy, uz]^T$
  - Set of points p that satisfy the relation $u^T \bullet p = 0$
  - A projective line u can be represented by a 3d plane through the origin, that is the line defined by the equation $u^T \bullet p = 0$
  - p is either a point lying on the line u, or a line going through the point u

Projective Line

- If we have a point in one image then this means the 3D point P is on the line from the origin through that point in the image plane
- So in the other image the corresponding point must be on the epipolar line
- What is the meaning of Ep_l ?
  - the line in the right plane that goes through p_r and the epipole e_r
- Therefore the essential matrix is a mapping between points and epipolar lines
- Ep_l defines the equation of the epipolar line in the right image plane through point p_r in the right image
- $E^T p_r$ defines the equation of the epipolar line in the left image through point p_l in the left image

Fundamental Matrix

Same as Essential Matrix but points are in pixel coordinates and not camera coordinates

$$p_r^T E p_l = 0$$

$$\Downarrow$$

$$\bar{p}_r^T F \bar{p}_l = 0$$

Pixel coordinates

$$F = M_r^{-T} E M_l^{-1}$$

Intrinsic parameters

Fundamental Matrix

Mapping between points and epipolar lines in the pixel coordinate systems

- With no prior knowledge of the stereo system parameters

From Camera to Pixels: Matrices of intrinsic parameters

$$\mathbf{M}_{\text{int}} = \begin{bmatrix} -f_x & 0 & o_x \\ 0 & -f_y & o_y \\ 0 & 0 & 1 \end{bmatrix}$$

Rank ($\mathbf{M}_{int}$) =3

$$\mathbf{p_r}^{\mathbf{T}}\mathbf{E}\mathbf{p_l} = 0$$

$$\mathbf{p_l} = \mathbf{M_l^{-1}}\overline{\mathbf{p_l}} \qquad \mathbf{p}_r = \mathbf{M}_r^{-\mathbf{1}}\overline{\mathbf{p}}_r$$

$$\overline{\mathbf{p_r}}^{\mathbf{T}}\mathbf{F}\overline{\mathbf{p_l}} = 0$$

$$\mathbf{F} = \mathbf{M}_r^{-\mathbf{T}}\mathbf{E}\mathbf{M}_l^{-1}$$

Essential/Fundamental Matrix

- Essential and fundamental matrix differ
- Relate different quantities
  - Essential matrix is defined in terms of camera co-ordinates
  - Fundamental matrix defined in terms of pixel co-ordinates
- Need different things to calculate them
  - Essential matrix requires camera calibration and knowledge of correspondences
    - – known intrinsic parameters, unknown extrinsic parameters
  - Fundamental matrix does not require any camera calibration, just knowledge of correspondences
    - – Unknown intrinsic and unknown extrinsic
- Essential and fundamental matrix are related by the camera calibration parameters

Essential/Fundamental Matrix

- We can compute the fundamental matrix from the 2d pixel co-ordinates of correspondences between the left and right image
- If we have the fundamental matrix it is possible to compute the essential matrix if we know the camera calibration
- But we can still compute the epipolar lines using only the fundamental matrix
- We can use the fundamental matrix to limit correspondence to s 1D search for general stereo camera positions in the same way as is possible for simple stereo

Essential Matrix

Essential Matrix E = RS

- 3x3 matrix constructed from R and T (extrinsic only)
  - Rank (E) = 2, two equal nonzero singular values

$$\mathbf{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$$S = \begin{bmatrix} 0 & -T_z & T_y \\ T_z & 0 & -T_x \\ -T_y & T_x & 0 \end{bmatrix}$$

Rank (R) =3 **Rank (S) =2**

- E has five degrees of freedom (3 rotation, 2 translation)
- If we know R and T it is easy to compute E
  - use the camera calibration method of Ch. 6 for two cameras
- We can compute E from correspondences between the two stereo cameras then (first compute F then E with calibration)

Fundamental Matrix

Fundamental Matrix

$$\mathbf{F} = \mathbf{M}_r^{-\mathrm{T}}\mathbf{E}\mathbf{M}_l^{-1}$$

- Rank (F) = 2
- Encodes info on both intrinsic and extrinsic parameters

- Enables full reconstruction of the epipolar geometry
- In pixel coordinate systems without any knowledge of the intrinsic and extrinsic parameters
- Linear equation of the 9 entries of F but only 8 degrees of freedom because of homogeneous nature of equations

$$\overline{\mathbf{p}}_{\mathbf{r}}^{\mathrm{T}}\mathbf{F}\overline{\mathbf{p}}_{\mathbf{l}} = 0 \Rightarrow \begin{pmatrix} x_{im}^{(l)} & y_{im}^{(l)} & 1 \end{pmatrix} \begin{bmatrix} f11 & f12 & f13 \\ f21 & f22 & f23 \\ f31 & f32 & f33 \end{bmatrix} \begin{pmatrix} x_{im}^{(r)} \\ y_{im}^{(r)} \\ 1 \end{pmatrix} = 0$$

Computing F: The Eight-point Algorithm

Input: n point correspondences (n >= 8)

$$\overline{\mathbf{p}}_{\mathbf{r}}^{\mathbf{T}}\mathbf{F}\overline{\mathbf{p}}_{\mathbf{l}} = 0$$

- Construct homogeneous system Ax= 0 from
  - x = (f_{11},f_{12}, ,f_{13}, f_{21},f_{22},f_{23} f_{31},f_{32}, f_{33}) : entries in F
  - Each correspondence give one equation
  - A is a nx9 matrix
- Obtain estimate F^ by Eigenvector with smallest eigenvalue
  - x (up to a scale) is column of V corresponding to the least singular value
- Enforce singularity constraint: since Rank (F) = 2
  - Compute SVD of F^ $\hat{\mathbf{F}} = \mathbf{U}\mathbf{D}\mathbf{V}^T$
  - Set the smallest singular value to 0: D -> D'
  - Correct estimate of F : $\mathbf{F'} = \mathbf{U}\mathbf{D'}\mathbf{V}^T$

Output: the fundamental matrix, F'

can then compute E given intrinsic parameters

Estimating Fundamental Matrix

The 8-point algorithm

$$u^T F u' = 0$$

Each point correspondence can be expressed as a linear equation

$$\begin{bmatrix} u & v & 1 \end{bmatrix} \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{bmatrix} \begin{bmatrix} u' \\ v' \\ 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} uu' & uv' & u & u'v & vv' & v & u' & v' & 1 \end{bmatrix} \begin{bmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \\ F_{33} \end{bmatrix} = 0$$

Homogeneous System

- M linear equations of form A**x** = 0
- If we have a given solution x1, s.t. Ax1 = 0 then c * x1 is also a solution A(c* x1) = 0
- Need to add a constraint on **x,**
  - Basically make **x** a unit vector $x^T x = 1$
- Can prove that the solution is the eigenvector correponding to the single zero eigenvalue of that matrix $A^T A$
  - This can be computed using eigenvector routine
  - Then finding the zero eigenvalue
  - Returning the associated eigenvector
- This is how we compute first estimate of F which is called F^

Singular Value Decomposition

- Any m by n matrix A can be written as product of three matrices $A = UDV^T$
- The columns of the m by m matrix U are mutually orthogonal unit vectors, as are the columns of the n by n matrix V
- The m by n matrix D is diagonal, and the diagonal elements, σ_i are called the singular values
- It is the case that $\sigma_1 \geq \sigma_2 \geq \ldots \sigma_n \geq 0$
- The rank of a square matrix is the number of linearly independent rows or columns
- For a square matrix (m = n) then the number of non-zero singular values equals the rank of the matrix

Locating the Epipoles from F

$$\bar{\mathbf{p}}_\mathbf{r}^{\mathbf{T}}\mathbf{F}\bar{\mathbf{p}}_\mathbf{l} = 0$$

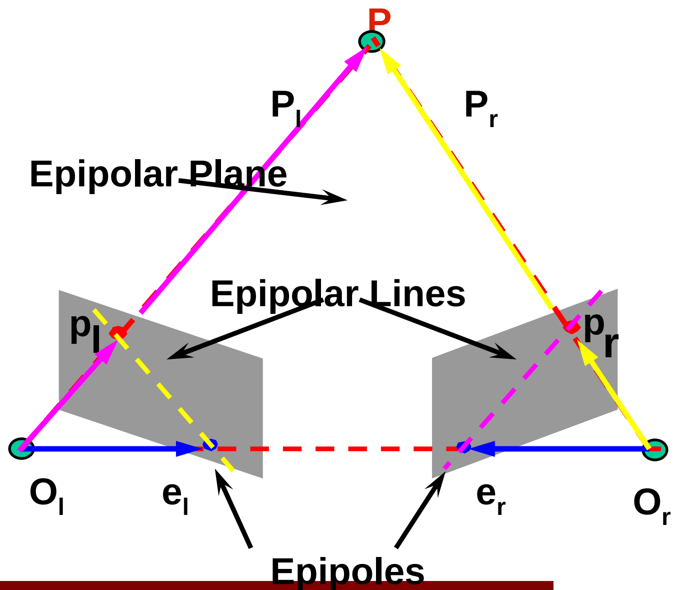
e_l lies on all the epipolar lines of the left image

$$\bar{\mathbf{p}}_\mathbf{r}^{\mathbf{T}}\mathbf{F}\bar{\mathbf{e}}_\mathbf{l} = 0$$

For every p_r

F is not identically zero

$$\mathbf{F}\bar{\mathbf{e}}_\mathbf{l} = 0$$



Input: Fundamental Matrix F

$$\mathbf{F} = \mathbf{UDV}^T$$

- Find the SVD of F
- The epipole e_l is the column of V corresponding to the null singular value (as shown above)
- The epipole e_r is the column of U corresponding to the null singular value

Output: Epipole e_l and e_r

Epipolar Geometry

- Basic constraint used to help correspondence
- Makes search for matching points into a 1D search along epipolar lines
- If you have intrinsic and extrinsic parameters
  - Then compute essential matrix and find epipolar lines
- If have intrinsic or extrinsic parameters but have at least 8 correct correspondences then
  - Compute fundamental matrix and find epipolar lines
- If have intrinsic but not extrinsic parameters and at least 8 correct correspondences then
  - Compute fundamental matrix, epipolar lines and use intrinsic parameters to compute E