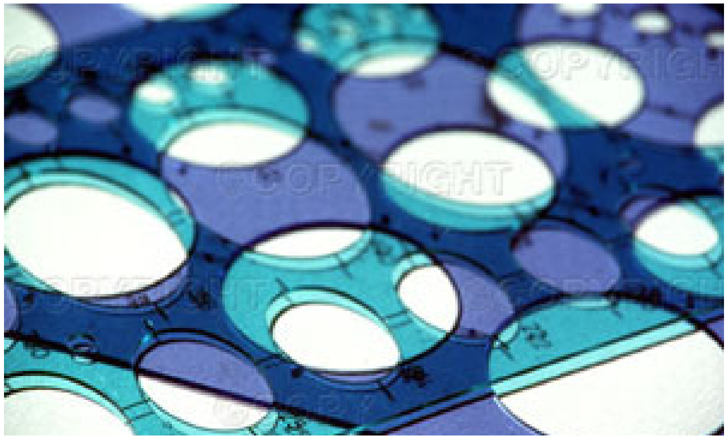
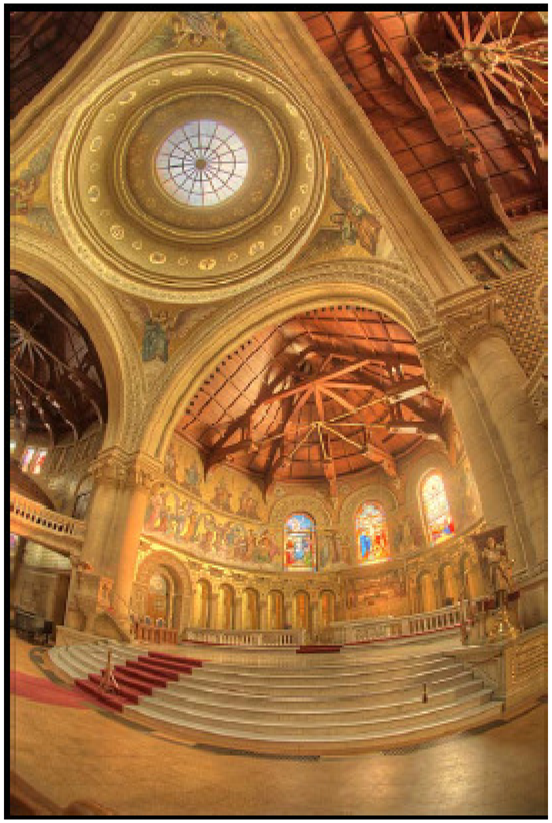
Ellipse Fitting

COMP 4900D

Winter 2006

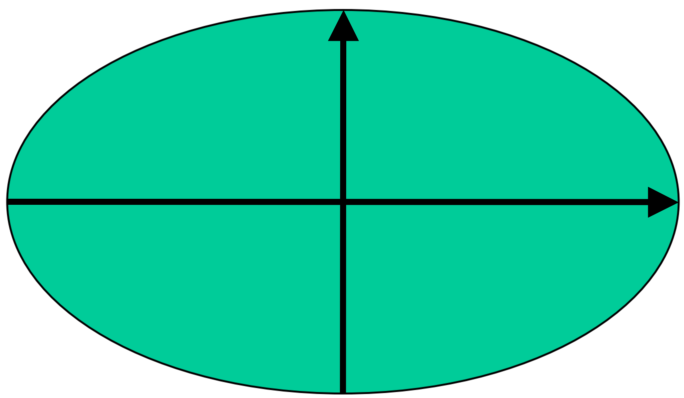
Ellipses in Images





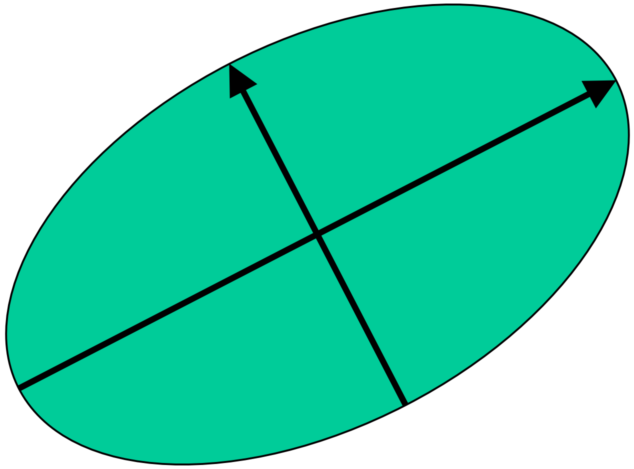
Perspective projection of circles form ellipses in the images.

Equations of Ellipse



$$\frac{x^2}{r_1^2} + \frac{y^2}{r_2^2} = 1$$

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$



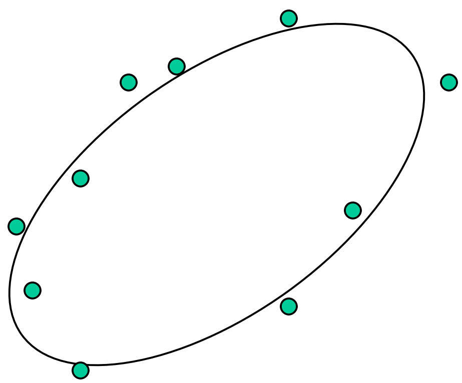
Let $\mathbf{x} = [x^2, xy, y^2, x, y, 1]^T$

$\mathbf{a} = [a, b, c, d, e, f]^T$

Then $\mathbf{x}^T \mathbf{a} = 0$

Ellipse Fitting: Problem Statement

Given a set of N image points $\mathbf{p}_i = [x_i, y_i]^T$
find the parameter vector $\mathbf{a_0}$ such that the ellipse



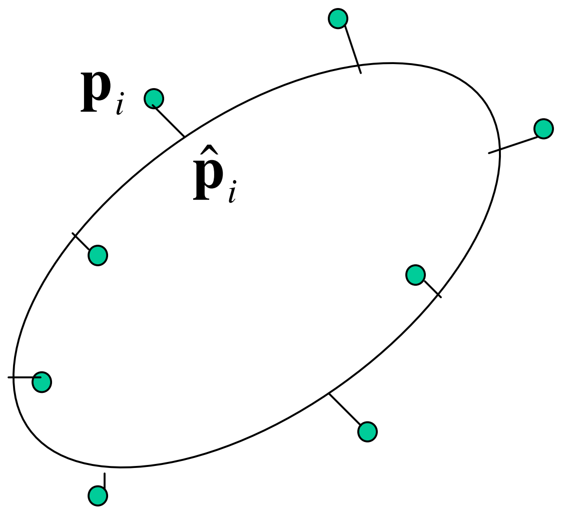
$$f(\mathbf{p}, \mathbf{a}) = \mathbf{x}^T \mathbf{a} = 0$$

fits $\mathbf{p}_i$ best in the least square sense:

$$\min_{\mathbf{a}} \sum_{i=1}^{N} \left[D(\mathbf{p}_i, \mathbf{a}) \right]^2$$

Where $D(\mathbf{p}_i, \mathbf{a})$ is the distance from $\mathbf{p}_i$ to the ellipse.

Euclidean Distance Fit



$$D(\mathbf{p}_i, \mathbf{a}) = \|\hat{\mathbf{p}}_i - \mathbf{p}_i\|$$

$\hat{\mathbf{p}}_i$ is the point on the ellipse that is nearest to $\mathbf{p}_i$

$$f(\hat{\mathbf{p}}_i, \mathbf{a}) = 0$$

$\hat{\mathbf{p}}_i - \mathbf{p}_i$ is normal to the ellipse at $\hat{\mathbf{p}}_i$

Compute Distance Function

Computing the distance function is a constrained optimization problem:

$$\min_{\hat{\mathbf{p}}_i} \left\| \hat{\mathbf{p}}_i - \mathbf{p}_i \right\|^2 \qquad \text{subject to} \quad f(\hat{\mathbf{p}}_i, \mathbf{a}) = 0$$

Using Lagrange multiplier, define:

$$L(x, y, \lambda) = \left\| \hat{\mathbf{p}}_i - \mathbf{p}_i \right\|^2 - 2\lambda f(\hat{\mathbf{p}}_i, \mathbf{a})$$

where $\hat{\mathbf{p}}_i = [x, y]^T$

Then the problem becomes: $\min_{\hat{\mathbf{p}}_i} L(x, y, \lambda)$

Set $\frac{\partial L}{\partial x} = \frac{\partial L}{\partial y} = 0$ we have $\hat{\mathbf{p}}_i - \mathbf{p}_i = \lambda \nabla f(\hat{\mathbf{p}}_i, \mathbf{a})$

Two Approximations

1. First-order approximation at $\mathbf{p}_i$

$$f(\hat{\mathbf{p}}_i, \mathbf{a}) \approx f(\mathbf{p}_i, \mathbf{a}) + (\hat{\mathbf{p}}_i - \mathbf{p}_i)^T \nabla f(\mathbf{p}_i, \mathbf{a}) = 0$$

2. Assume $\mathbf{p}_i$ is close to $\hat{\mathbf{p}}_i$, then

$$\nabla f(\hat{\mathbf{p}}_i, \mathbf{a}) \approx \nabla f(\mathbf{p}_i, \mathbf{a})$$

Approximate Distance Function

$$f(\hat{\mathbf{p}}_i, \mathbf{a}) \approx f(\mathbf{p}_i, \mathbf{a}) + (\hat{\mathbf{p}}_i - \mathbf{p}_i)^T \nabla f(\mathbf{p}_i, \mathbf{a}) = 0$$

$$\hat{\mathbf{p}}_i - \mathbf{p}_i = \lambda \nabla f(\hat{\mathbf{p}}_i, \mathbf{a}) \approx \lambda \nabla f(\mathbf{p}_i, \mathbf{a})$$

Solve for λ

$$\lambda = -\frac{f(\mathbf{p}_i, \mathbf{a})}{\left\| \nabla f(\mathbf{p}_i, \mathbf{a}) \right\|^2}$$

Substitute back

$$\hat{\mathbf{p}}_i - \mathbf{p}_i = -\frac{f(\mathbf{p}_i, \mathbf{a}) \nabla f(\mathbf{p}_i, \mathbf{a})}{\left\| \nabla f(\mathbf{p}_i, \mathbf{a}) \right\|^2}$$

$$D(\mathbf{p}_i, \mathbf{a}) = \left\| \hat{\mathbf{p}}_i - \mathbf{p}_i \right\| = \frac{\left| f(\mathbf{p}_i, \mathbf{a}) \right|}{\left\| \nabla f(\mathbf{p}_i, \mathbf{a}) \right\|}$$

Ellipse Fitting with Euclidean Distance

Given a set of N image points $\mathbf{p}_i = [x_i, y_i]^T$
find the parameter vector $\mathbf{a_0}$ such that

$$\min_{\mathbf{a}} \sum_{i=1}^{N} \frac{\left| f(\mathbf{p}_i, \mathbf{a}) \right|^2}{\left\| \nabla f(\mathbf{p}_i, \mathbf{a}) \right\|^2}$$

This problem can be solved by using a numerical nonlinear optimization system.