An $O(\sqrt{n})$ TIME ALGORITHM FOR THE ECDF SEARCHING PROBLEM FOR ARBITRARY DIMENSIONS ON A MESH-OF-PROCESSORS

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Dehne (1986) presented an optimal $O(\sqrt{n})$ time parallel algorithm for solving the ECDF searching problem for a set of $n$ points in two- and three-dimensional space on a mesh-of-processors of size $n$. However, it remained an open problem whether such an optimal solution exists for the $d$-dimensional ECDF searching problem for $d \geq 4$.

In this paper we solve this problem by presenting an optimal $O(\sqrt{n})$ time parallel solution to the $d$-dimensional ECDF searching problem for arbitrary dimension $d = O(1)$ on a mesh-of-processors of size $n$. The algorithm has several interesting implications. Among others, the following problems can now be solved on a mesh-of-processors in (asymptotically optimal) time $O(\sqrt{n})$ for arbitrary dimension $d = O(1)$: the $d$-dimensional maximal element determination problem, the $d$-dimensional hypercube containment counting problem, and the $d$-dimensional hypercube intersection counting problem. The latter two problems can be mapped to the $2d$-dimensional ECDF searching problem but require an efficient solution to this problem for at least $d \geq 4$.

Keywords: ECDF searching, mesh-of-processors, parallel computational geometry

1. Introduction

A set $S = \{ p_1, p_2, \ldots, p_n \}$ of $n$ points in $d$-dimensional space is given. A point $p_i$ dominates a point $p_j$ (denoted $p_i > p_j$) if and only if $p_i[k] > p_j[k]$ for all $k \in \{1, 2, \ldots, d\}$ where $p[k]$ denotes the $k$th coordinate of a point $p$. The $d$-dimensional ECDF searching problem consists of computing for each $p \in S$ the number $D(p, S)$ of points of $S$ dominated by $p$. (For more details about this problem, consult, e.g., [7,8].)

An efficient solution to the ECDF searching problem has several interesting applications [3,7,8].

One of these is the well-known transformation of the rectangle containment counting problem to the ECDF searching problem [3,8]. The rectangle containment counting problem consists of counting for each rectangle $R$ of a set of iso-oriented rectangles the number of rectangles which are contained in $R$. If we map each rectangle $R = [x_1, x_2] \times [y_1, y_2]$ into the four-dimensional point $R' = (-x_1, x_2 - y_1, y_2)$, then a rectangle $R_1$ contains a rectangle $R_2$ if and only if $R'_2 \leq R'_1$; hence, the problem is easily transformed into a four-dimensional ECDF searching problem.

In [2], an optimal $O(\sqrt{n})$ time parallel algorithm was introduced for solving the two- and three-dimensional ECDF searching problem on a mesh-of-processors of size $n$, i.e., a set of $n$ processing elements (PEs) arranged on a $\sqrt{n} \times \sqrt{n}$.

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grid where each PE is connected to its direct neighbors by bidirectional communication links. (For a more detailed description of the mesh-of-processors architecture and basic algorithm design techniques on these machines, consult, e.g., [5,10].)

However, the algorithm in [2] did not solve, in $O(\sqrt{n})$ time, the $d$-dimensional ECDF searching problem for $d \geq 4$, and the existence of an optimal solution for $d \geq 4$ remained an open problem [4].

In this paper we will solve this problem by introducing an optimal $O(\sqrt{n})$ time solution to the $d$-dimensional ECDF searching problem for arbitrary dimension $d = O(1)$. Miller and Stout [6] have also considered this problem and, independently, suggested the same solution based on Bentley's multidimensional divide-and-conquer technique [1].

2. Description and analysis of the proposed algorithm

In order to obtain a convenient description of the algorithm, we introduce the following definitions:

1. Let $p, q$ be two points in $d$-space and let $1 \leq k \leq d$; then $q <_k p$ if and only if $q[2] \prec p[2]$, $\ldots$, $q[k] \prec p[k]$.
2. Let $p$ be a point in $d$-space, $S_1$ be a subset of $S$, and $1 \leq k \leq d$; then $M^k(p, S_1)$ denotes the number of those $q \in S_1$ such that $q <_k p$.
3. Let $S_1, S_2$ be two subsets of $S$ and let $1 \leq k \leq d$; then $k$-dimensional dominance merge, denoted $\text{MERGE}^k(S_1, S_2)$, consists of computing the value $M^k(p, S_1)$ for all $p \in S_2$.

2.1. Global structure of the algorithm

Initially, each processing element of the mesh contains the $d$ coordinates of one point of $S$. Each PE is assumed to have a register $D$ which will contain the value $D(p, S)$, where $p$ is the point stored in the respective PE, after the algorithm has terminated.

The global structure of the proposed algorithm is a divide-and-conquer mechanism which solves the problem as follows:

(I) Divide

Partition $S$ into two subsets $S_1$ and $S_2$ by comparing the $d$th coordinate of the points with the median $d$th coordinate (points in $S_2$ have larger $d$th coordinate). $S_1$ and $S_2$ are stored in one half of the mesh-of-processors, each. (This step is easily obtained by sorting $S$ with respect to the $d$th coordinate (see, e.g., [9]). The mesh is split into two submeshes of equal size by either a vertical or a horizontal line to minimize the diameter of the submeshes.)

(II) Recur

Solve the $d$-dimensional ECDF searching problem for $S_1$ and $S_2$, respectively, on each half of the mesh-of-processors in parallel.

(III) Merge

(a) Solve the $(d-1)$-dimensional dominance merge problem $\text{MERGE}^{d-1}(S_2, S_1)$.

(b) Update

Each PE updates its register $D$ as follows:

$$
D(p, S) = \begin{cases} 
D(p, S_1) & \text{for } p \in S_1, \\
D(p, S_2) + M^{d-1}(p, S_1) & \text{for } p \in S_2.
\end{cases}
$$

The following subsection shows how to solve the $k$-dimensional dominance merge problem $\text{MERGE}^k(S_2, S_1)$, $1 \leq k \leq d$, as required for step III(a).

2.2. $k$-dimensional dominance merge

$\text{MERGE}^k(S_2, S_1)$

The structure of the $k$-dimensional dominance merge algorithm is again a divide-and-conquer mechanism. In each iteration, $k$ decreases by one, i.e., the merge step for $k$-dimensional dominance merge involves the solution of a $(k-1)$-
dimensional dominance merge problem. This process is iterated until \( k = 1 \).

Each PE is assumed to have a register \( M \) which will finally contain the value \( M^k(p, S_i) \) for \( p \in S_2 \) where \( p \) is the point stored in the respective PE.

**Case \( k \geq 2 \)**

(I) **Divide**

Partition \( S_1 \) into two subsets \( S_{11} \) and \( S_{12} \) and, simultaneously, \( S_2 \) into two subsets \( S_{21} \) and \( S_{22} \) by comparing the \( k \)th coordinate of the points with the median \( k \)th coordinate of \( S_1 \cup S_2 \) (points in \( S_{12} \) and \( S_{22} \) have larger \( k \)th coordinate). Store \( S_{21} \cup S_{11} \) and \( S_{22} \cup S_{12} \) in one half of the current submesh, each. (Again, split the current submesh into two submeshes of equal size using either a vertical or a horizontal split line to minimize the diameter of the submeshes.)

(II) **Recur**

Solve the \( k \)-dimensional dominance merge problems denoted \( \text{MERGE}^k(S_{21}, S_{11}) \) and \( \text{MERGE}^k(S_{22}, S_{12}) \), respectively, on each half of the mesh-of-processors in parallel.

(III) **Merge**

(a) Solve the \((k - 1)\)-dimensional dominance merge problem \( \text{MERGE}^{k-1}(S_{22}, S_{11}) \).

(b) **Update**

Each PE updates its register \( M \) as follows:

\[
M^k(p, S_i) = \begin{cases} 
M^k(p, S_{11}) & \text{for } p \in S_{21}, \\
M^k(p, S_{12}) + M^{k-1}(p, S_{11}) & \text{for } p \in S_{22}. 
\end{cases}
\]

**Case \( k = 1 \)**

Sort \( S_1 \cup S_2 \) with respect to the first coordinate in snake-like ordering [9]. For each \( p \in S_2 \), the value \( M^k(p, S_1) \) is equal to the number of \( q \in S_1 \) with lower rank.

2.3. **Time complexity of the proposed algorithm**

Let \( T_{\text{ECDF}}(n) \) and \( m_k(n) \) denote respectively the time complexity for solving the \( d \)-dimensional ECDF searching problem for a set of \( n \) points and the \( k \)-dimensional dominance merge problem \( \text{MERGE}^k(S_2, S_1) \) for \( |S_2 \cup S_1| = n \), as described above.

With these definitions, the following recurrence relations are easily observed:

1. \[
T_{\text{ECDF}}(n) = T_{\text{ECDF}}\left(\frac{n}{2}\right) + m_{d-1}(n) + O(\sqrt{n}),
\]
   \[
T_{\text{ECDF}}(n) = O(1).
\]

2. \[
m_k(n) = m_k\left(\frac{n}{2}\right) + m_{k-1}(n) + O(\sqrt{n}),
\]
   \[
m_1(n) = O(1).
\]

From (2) it follows that

\[
m_k(n) = O(\mu^k\sqrt{n}),
\]

\[
\mu = \frac{\sqrt{2}}{2 - 1} + \epsilon \approx 3.414213 \ldots + \epsilon \quad (\epsilon > 0).
\]

Hence, \( m_{d-1}(n) = O(\mu^{d-1}\sqrt{n}) \) and, therefore, it follows from (1) that

\[
T_{\text{ECDF}}(n) = O(\mu^{d-1}\sqrt{n}).
\]

For any \( d = O(1) \), i.e., any fixed dimension \( d \), this yields the following result.

**Theorem.** The \( d \)-dimensional ECDF searching problem, \( d = O(1) \), for a set of \( n \) points can be solved on a mesh-of-processors of size \( n \) in \( O(\sqrt{n}) \) time which is asymptotically optimal.

3. **Applications**

The above algorithm has several interesting applications. Among other, the following problems, for arbitrary dimension \( d = O(1) \), can now be solved on a mesh-of-processors of linear size in (asymptotically optimal) time \( O(\sqrt{n}) \):

- The \( d \)-dimensional maximal element determination problem: compute the set of points which are not dominated by any other point.
- The \( d \)-dimensional hypercube containment counting problem, i.e., the \( d \)-dimensional generalization of the rectangle containment counting problem described in Section 1 (mapping the
**References**


