

Algorithmic and mathematical methods for cryptology in France

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Outline

- 1 Mathematical methods and algorithms in cryptology
- 2 French research on arithmetic/mathematical methods for cryptology
- 3 Zoom on two groups
- 4 Two recent results from the Nancy group

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Warning :

- Primitives = *building blocks* ;
- MUST be combined with symmetric techniques to prevent access to underlying structure.

Number-based primitives

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- Evaluating security \Leftrightarrow try to invert those mappings ;

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Security

- Extract e -th roots or factor N (harder) in the first case ; best algorithm is NFS, complexity

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- Hard problems based on lattices (SVP, CVP) and on polynomial systems are used to build primitives.

A map of existing teams



Topics addressed

-  : teams partially supported by INRIA.
- Factoring, primality proving : Nancy , Palaiseau  ;
 - Discrete logs over finite fields : Rennes, Versailles ;
 - Algebraic cryptanalysis : Paris , Versailles ;
 - Algebraic curves for cryptology : Marseille, Nancy , Palaiseau , Rennes ;
 - Computer arithmetic, hardware for cryptology : Lyon , Montpellier, Nancy .
 - Lattices : Caen, Lyon , Paris .
 - Coding theory : Rocquencourt .

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 - Algorithms for fast and efficient arithmetic.

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- Strong involvement in (efficient) software development.

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 - Applications of pairings to e-cash.

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- Can be used as a cryptographic group over a finite field of size $\approx 2^{96}$ (“cheap” field arithmetic).

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Goal : compute $n \cdot P$; basic operations = addition, doubling.

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Practice : some implementations faster than elliptic curve for same security level (but pre-“Edwards coordinates”)

Forging RSA signatures with affine padding

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- Some RSA-based signature schemes are of the form

$$S(m) = (h(m) + c)^d \bmod N = (h(m) + c)^{1/e} \bmod N,$$

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- For small padding, previous results show that forgery is possible in polynomial time ;

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Extra results : computing e -th roots faster than factoring given access to different type of oracles (and thus “general forgery”).