Sensor Network Connectivity with Multiple Directional Antennae of a Given Angular Sum

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Abstract

We investigate the problem of converting sets of sensors into strongly connected networks of sensors using multiple directional antennae. Consider a set S of n points in the plane modeling sensors of an ad hoc network. Each sensor uses a fixed number, say $1 \leq k \leq 5$, of directional antennae modeled as a circular sector with a given spread (or angle) and range (or radius). We give algorithms for orienting the antennae at each sensor so that the resulting directed graph induced by the directed antennae on the nodes is strongly connected. We also study trade-offs between the total angle spread and range for maintaining connectivity.

1 Introduction

Sensors in use today can have a variety of antennae enabling them to vary their transmission range and orientation. A *directional* (or *beam*) antenna radiates greater power towards one or more directions thus allowing for increased performance on transmit and receive as well as reduced interference from unwanted sources. For example, ESPAR (Electronically Steerable Passive Array Radiator) antennae consisting of a *steerable* central source radiating within a region that can be approximated by a circular sector have been proposed by the engineering community. [13] and [16] study configuration and design of such multiple element antennae and present simulation results on their beam/null steering capability; they also found that 7-element ESPAR antennae can achieve continuous steering in almost all directions.

Directional antennae are known to enhance ad hoc network capacity and performance. The theoretical model presented in [7] shows that when n omnidirectional antennae are optimally placed and assigned optimally chosen traffic patterns the transport capacity is $\Theta(\sqrt{W/n})$, where each antenna can transmit W bits per second over the common channel(s). When both transmission and reception is directional [19] proves an $\sqrt{2\pi/\alpha\beta}$ capacity gain as well as corresponding throughput improvement factors, where α is the transmission angle and β is a parameter indicating that $\beta/2\pi$ is the average proportion of the number of receivers inside the transmission zone that will get interfered with. Additional experimental studies confirm the importance of using directional antennae in ad hoc networking. For example, [15] provides experimental evidence of throughput increase in ad hoc networks with beam forming antennas, [18] and [17] investigate bounds (depending on the specific antenna type and its parameters) on the amount of capacity gain that can be achieved when directional antennae perform independent communications in parallel, while [1] looks at transmission scheduling in ad hoc networks with directional antennae. See also [5] for MAC layer protocols that exploit the use of directional antennae.

1.1 Antennae spreads and strong connectivity

In this paper we are interested in studying connectivity properties of the underlying communication graph of a set of sensors using directional antennae. Unlike omnidirectional antennae, directional antennae give rise to a more complex directed graph whose connectivity varies depending on the direction of the beam. A directed edge (u, v) exists between sensors u and v if and only if v lies within the spread and range of u. It is important to maintain strong connectivity using the minimum possible antenna spread and range. Assuming each beam forming sensor has a given number of steerable directional antennae we are interested in providing an algorithm that minimizes the total antenna spread required so that by an appropriate rotation of the antennae the resulting network becomes strongly connected. Thus, in the problems studied in the sequel, we assume that the antenna spread is part of the input.

Consider a set S of n points in the plane modeling sensors of an ad hoc network. Each sensor uses a directional antenna modeled by a circular sector with a given spread (or angle) and reach (or radius). A bound is given on the the spread of the antennae. Given a bound on the reach, the connectivity problem is to decide whether or not it is possible to orient the antennae so that the resulting directed graph induced by the directed antennae on the nodes is strongly connected. A variant of this problem concerns sensors with multiple antennae each, i.e., each sensor has k antennae, for some given integer k. For a given bound on the sum of the angles of the antennae, it is of interest to know if there is a way to direct antennae with a given bound on their radius so that the resulting graph is strongly connected. Surprisingly, it will be shown that strong connectivity can still be maintained by appropriately rotating the antennae, even though the sum of angles of the directional antennae at each sensor is significantly less than the full angular spread 2π of an omnidirectional antenna. We are interested in showing trade-offs between the number k of antennae being used per sensor, the total spread of the antennae, and the maximum radius required per antennae, such that for suitable rotations of the antennae the network becomes strongly connected.

1.2 Notation

Let S be a set of n points in the plane. As usual, $d(x, y)$ is the Euclidean distance between two points x and y in the plane. Let φ_k be a given non-negative value in $[0, 2\pi)$ such that the sum of angles of k antennae at each sensor location is bounded by φ_k. Denote by r_{k,φ_k} the minimum radius (or range) of directional antennae for a given k and φ_k that achieves strong connectivity under some rotation of the antennae. T is a MST of S with max degree ≤ 5, $l_{\max}$ is the maximum length of edges in T. Note that $l_{\max}$ is a lower bound on r_{k,φ_k} for any k and φ_k. Without loss of generality, we assume that $l_{\max} = 1$. This will be useful in the simplifying notation throughout the paper without compromising generality. Let $\delta(v)$ be the degree of vertex $v \in S$ in T. A degree-one vertex is arbitrarily chosen to be the root vertex of T, denoted by R_T. That is, $\delta(R_T) = 1$. For each $v, v \in S$, let T_v denote the subtree rooted at vertex v and $v(1), v(2), \ldots, v(\delta(v)-1)$ denote its children sorted in counterclockwise order. For each $v, v \in S \setminus \{R_T\}$, let $p(v)$ be the parent of vertex v. Finally, $\widehat{uvw}$ denotes the counterclockwise angle between rays $\vec{vu}$ and $\vec{vw}$.

1.3 Related work

The problem of converting (connected) networks of omnidirectional sensors to strongly connected networks of sensors with directional antennae was first addressed in [4] where it was assumed that sensors had only one antenna each. They present a polynomial time algorithm achieving the optimal radius for the case when the sector angle of the antennae is at least $8\pi/5$. For smaller angles ϕ with $\pi \leq \phi < 8\pi/5$ they give a polynomial time algorithm that computes an orientation of sectors of angle ϕ and radius $2\sin(\pi - \phi/2)$ so that the transmission graph is strongly connected. When the sector angle is smaller than $2\pi/3$, they show that the problem of determining the minimum radius in order to achieve connectivity is NP-hard. When antennae have spread 0 this problem turns out to be the bottleneck traveling salesman problem studied in [14]. It is shown in [14] that there is an algorithm which for any set of points in the plane computes antennae orientations such that the sensors have range ≤ 2. Since $l_{\max} = 1$ is a lower bound on the radius, the algorithms with radius greater than 1 above may thought of as approximation algorithms for the minimum radius required to achieve connectivity.

Other related papers include [9] on minimum range for energy consumption, [2] on energy-efficient broadcasting in wireless networks, [3] on energy efficient wireless network design and [6] on the complexity of computing minimum energy consumption broadcast subgraphs. Topology control for ad hoc networks with directional antennae is studied in [8], while directional versus omnidirectional antennae for energy consumption and k-connectivity of networks of sensors is considered in [11]. Coverage and connectivity in networks with directional sensors is investigated in [10]. Power consumption and throughput in mobile ad hoc networks using directional antennas is investigated in [12] while the performance of ad hoc networks with beamforming antennas is studied in [15]. However, to the best of our knowledge the problem of using directional antennae to achieve connectivity has not appeared in the literature before for the case of multiple antennae per sensor.

1.4 Outline and results of the paper

In Section 2 we provide necessary and sufficient conditions on antennae spreads so as to form a strongly connected graph with range equal to $l_{\max} = 1$. These results provide optimal algorithms for large spread sum. In Section 3 we provide algorithms for two antennae of a given sum and prove the main result for two antennae per sensor. In Section 4 we discuss the case of more than two antennae. Table 1 summarizes the results obtained and

#	Antennae Spreads	Antennae Range	Reference
1	$\varphi_1 \geq 0$	2	[14]
	$\pi \leq \varphi_1 < 8\pi/5$	$2\sin\left(\pi - \frac{\varphi_1}{2}\right)$	[4]
	$\varphi_1 \geq 8\pi/5$	1	[4]
2	$\varphi_2 \geq 0$	2	[14]
	$\frac{2\pi}{3} \leq \varphi_2 < \pi$	$2\sin(\frac{\pi}{2} - \frac{\varphi_2}{4})$	Theorem 3
	$\varphi_2 \geq \pi$	$2\sin(2\pi/9)$	Theorem 3
	$\varphi_2 \geq 6\pi/5$	1	Theorem 2
3	$\varphi_3 \geq 0$	$\sqrt{3}$	Theorem 5
	$\varphi_3 \geq 4\pi/5$	1	Theorem 2
4	$\varphi_4 \geq 0$	$\sqrt{2}$	Theorem 6
	$\varphi_4 \geq 2\pi/5$	1	Theorem 2
5	$\varphi_5 \geq 0$	1	Folklore

Table 1. Upper bounds on antenna range for various specified sums of antennae.

also mentions previous results from the literature.

2 Optimal Antennae Spreads and Connectivity

For the given set of points, we will be referring to an arbitrary minimum weight spanning tree (MST) induced when edges between any two points are weighted by their corresponding Euclidean distance. Well-known geometric considerations imply that an MST of maximum degree 5 can be shown to exist. The first useful result we prove relates the node degree with the sum of the spreads of all the antennae.

Lemma 1 (Node degree and sum of antennae spreads) *Assume that a node u has degree d and the sensor at u is equipped with k antennae, where $1 \leq k \leq d$, of range at least the maximum edge length of an edge from u to its neighbors. Then $2(d-k)\pi/d$ is always sufficient and sometimes necessary bound on the sum of the angles of the antennae at u so that there is an edge from u to all its neighbors in an MST.*

Proof. The result is trivially true for $k = d$ since we can satisfy the claim by directing a separate antenna to each node adjacent to u. So we can assume that $k \leq d-1$. To prove the necessity of the claim take a point at the center of a circle and with d adjacent neighbors forming a regular d-gon on the perimeter of the circle of radius equal to the maximum edge length of the given spanning tree on S. Thus each angle formed between two consecutive neighbors on the circle is exactly $2\pi/d$. It is easy to see that for this configuration a sum of $2(d-k)\pi/d$ is always necessary.

To prove that sum $2(d-k)\pi/d$ is always sufficient we argue as follows. Consider the point u which has d neighbors and consider the sum σ of the largest k angles formed by $k+1$ consecutive points of the regular polygon on the perimeter of the circle. We claim that $\sigma \geq 2k\pi/d$. Indeed, let the d consecutive angles be $\alpha_0, \alpha_1, \ldots, \alpha_{d-1}$. (see Figure 1). Consider the d sums $\alpha_i + \alpha_{i+1} + \cdots + \alpha_{i+k-1}$, for $i = 0, \ldots, d-1$, where addition on the indices is modulo d. Observe, that

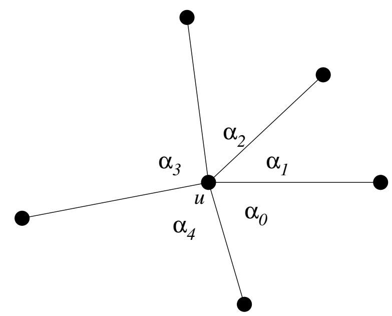

Figure 1. Example vertex with $d = 5$.

$$2k\pi = \sum_{i=0}^{d-1}(\alpha_i + \alpha_{i+1} + \cdots + \alpha_{i+k-1}) \leq d\sigma.$$

It follows that the remaining angles sum to at most $2\pi - \sigma \leq 2\pi - 2k\pi/d = 2\pi(d-k)/d$. Now consider the $k+1$ consecutive points, say $p_1, p_2, \ldots, p_{k+1}$, such that the sum σ of the k consecutive angles formed is at least $2k\pi/d$. Use $k-1$ antennae each of size 0 radians to cover each of the points $p_2, \ldots, p_k$, respectively, and an angle of size $2\pi(d-k)/d$ to cover the remaining $n-k+1$ points. This proves the lemma. ■

The next result we prove shows how antennae spreads affect the range in order to accomplish strong connectivity.

Theorem 2 *For any $1 \leq k \leq 5$, if $\varphi_k \geq \frac{2(5-k)\pi}{5}$ then $r_{k,\varphi_k} = 1$.*

Proof. We prove the theorem by that if $\phi_k \geq 2(5-k)\pi/5$ then the antennae can be oriented in a way such that for every vertex u there is a directed edge from u to all its neighbors. Let d be the degree of a vertex u in an MST with max degree 5. If $d \leq k$ then for sure we have enough antennae.

Otherwise, $k < d \leq 5$. Then, $\phi_k \geq 2(5-k)\pi/5 \geq 2(d-k)\pi/d$. We know from Lemma 1 that for k antennae $2(d-k)\pi/d$ is always sufficient and sometimes necessary on the sum of the angles of the antennae at u so that there is a directional antenna from u pointing to all its neighbors. Therefore, if $\phi_k \geq 2(5-k)\pi/5$ then we can orient the antennae such that for every vertex u there is a directed edge from u to all its neighbors. ■

3 Two Antennae

In this section we prove the main theorem which provides a more detailed tradeoff between sum of antennae spreads and antennae range for the case of two antennae.

Theorem 3 *Consider a set of n sensors in the plane with two antennae each. There is an algorithm for directing the antennae so that the resulting graph is strongly connected such that the sum of angles and range of the antennae are as follows:*

1. *if $\varphi_2 = \pi$ then $r_{2,\pi} \leq 2\sin(\frac{2\pi}{9})$, and*
2. *if $\frac{2\pi}{3} \leq \varphi_2 < \pi$ then $r_{2,\varphi_2} \leq 2\sin(\frac{\pi}{2} - \frac{\varphi_2}{4})$.*

Proof. (Theorem 3) We begin the proof by stating two simple facts.

Fact 1 *(Refer to Figure 2(a).) For any vertices $u, v, w \in S$ such that u and w are adjacent neighbors of v in T,*

1. *the angle $\widehat{uvw}$ is at least $\pi/3$,*
2. *$d(u,w) \leq 2\sin(\frac{\widehat{uvw}}{2})$, and*
3. *the triangle $\triangle uvw$ is empty.*

From Fact 1, we can easily derive the following fact.

Fact 2 *(Refer to Figure 2(b).) Let v_1, v_2, v_3, v_4, v_5 be neighbors of a vertex $v \in S$ of degree 5 in T, sorted in counterclockwise order. Then,*

1. *the angles $\widehat{v_1vv_2}, \widehat{v_2vv_3}, \ldots, \widehat{v_5vv_1}$ are in $[\frac{\pi}{3}, \frac{2\pi}{3}]$.*
2. *the angles $\widehat{v_1vv_3}, \widehat{v_2vv_4}, \ldots, \widehat{v_5vv_2}$ are in $[\frac{2\pi}{3}, \pi]$.*

Next we prove the results stated in the theorem by induction. In all cases (i.e., combination of k and φ_k) we say that a rooted subtree T_v ($v \in S$) satisfies **Property 1** if for any imaginary point p with $d(v,p) \leq r_{k,\varphi_k}$ (we use an imaginary point to simulate the parent or a sibling vertex of v), there exists a way to direct antennae located at vertices in T_v such that the resulting graph is strongly connected and p is covered by an antenna located at v. We have the following lemma.

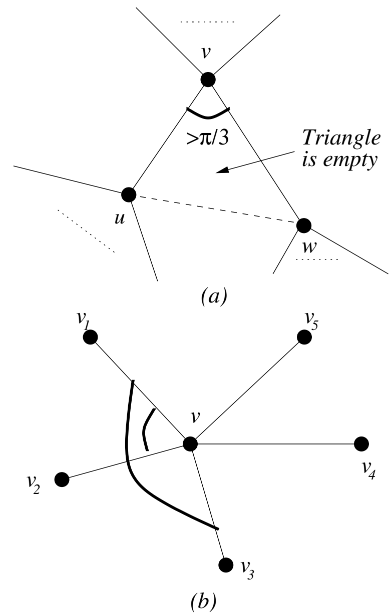

Figure 2. Examples for Facts 1 and 2.

Lemma 4 *For each case in Theorem 3, if the rooted tree T (rooted at R_T) satisfies* Property 1 *then there exists a way to direct antennae, where each antenna has radius r_{k,φ_k} and each sensor has k antennae whose angle sum is bounded by φ_k, such that the resulting graph is strongly connected.*

Proof of Part 1: if $\varphi_2 = \pi$ then $r_{2,\varphi_2} \leq 2\sin(\frac{2\pi}{9})$

In this part of the proof, let us define $R_{2,\varphi_2} := 2\sin(\frac{2\pi}{9})$. Trivially, for any leaf $v \in S$, T_v has Property 1 since we can use one antenna (located at v) of angle 0 and radius R_{2,φ_2} to cover any imaginary point p with $d(v,p) \leq R_{2,\varphi_2}$. For an internal vertex u, we assume that all rooted subtrees $T_{u(1)}, \ldots, T_{u(\delta(u)-1)}$ have Property 1. Next, we prove that T_u has Property 1. Based on the degree of u in T, we consider the following cases.

Case $\delta(u) = 1, 2$**:** In this case, u has one child $u(1)$. In fact, if $\delta(u) = 1$ then $u = R_T$. According to assumption, $T_{u(1)}$ has Property 1. Therefore, there is a way to direct antennae in $T_{u(1)}$ such that the resulting graph of vertices in $T_{u(1)}$ is strongly connected and u is covered by an antenna located at $u(1)$. Refer to Figure 3(a). We then use two antennae (located at u) of angle 0 and radius R_{2,φ_2}. One antenna covers a given imaginary point p with $d(u, p) \leq R_{2,\varphi_2}$ and the other one covers $u(1)$. In this way, the resulting graph of vertices in T_u is strongly connected and p is covered by an antenna located at u.

In the following, without loss of any generality, we assume that $u(1)$ is the first neighbor of u when rotating the ray $\vec{up}$.

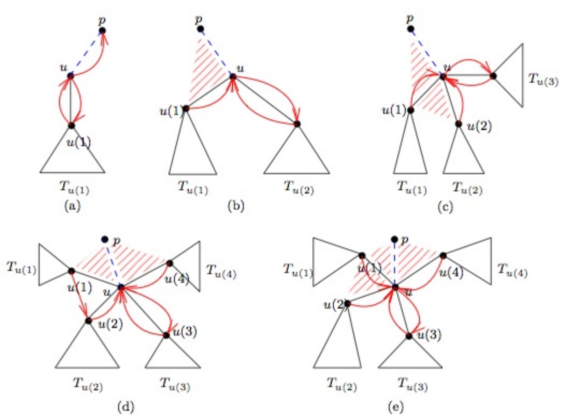

Figure 3. Case $\varphi_2 = \pi$.

Case $\delta(u) = 3$**:** According to assumption, there is a way to direct antennae in $T_{u(1)}$ (resp. $T_{u(2)}$) such that the resulting graph of vertices in $T_{u(1)}$ (resp. $T_{u(2)}$) is strongly connected and u is covered by an antenna located at $u(1)$ (resp. $u(2)$). We can see that $\min\{\widehat{puu(1)}, \widehat{u(1)uu(2)}, \widehat{u(2)up}\} \leq \frac{2\pi}{3}(< \varphi_2)$. Suppose that $\widehat{puu(1)} \leq \frac{2\pi}{3}$. Refer to Figure 3(b). We then use two antennae (located at u) of radius R_{2,φ_2}. One antenna is of angle $\frac{2\pi}{3}$ that covers the sector between rays $\vec{up}$ and $\overrightarrow{uu(1)}$. The other one (of angle 0) covers $u(2)$. In this way, the resulting graph of vertices in T is strongly connected and p is covered by an antenna located at u.

Case $\delta(u) = 4$**:** It is easy to see that either $\widehat{puu(2)}$ or $\widehat{u(2)up}$ is no more than $\pi(= \varphi_2)$. Suppose $\widehat{puu(2)} \leq \pi$. Refer to Figure 3(c). We then use two antennae (located at u) of radius R_{2,φ_2}. One antenna is of angle π that covers the sector between rays $\vec{up}$ and $\overrightarrow{uu(2)}$. The other one (of angle 0) covers $u(3)$.

Case $\delta(u) = 5$**:** Recall that p lies in the sector between rays $\overrightarrow{uu(4)}$ and $\overrightarrow{uu(1)}$ (counterclockwise). We consider two cases depending on the location of u's predecessor.

If $p(u)$ lies in the sector between rays $\overrightarrow{uu(4)}$ and $\overrightarrow{uu(1)}$, then the angle $\widehat{u(4)uu(1)}$ is in $[\frac{2\pi}{3}, \pi]$ (by Fact 2). Accordingly,

$$\min\{\widehat{u(1)uu(2)}, \widehat{u(2)uu(3)}, \widehat{u(3)u(4)}\} \leq \frac{4\pi}{9}.$$

Suppose that $\widehat{u(1)uu(2)} \leq \frac{4\pi}{9}$. Refer to Figure 3(d). We then use two antennae (located at u) of radius R_{2,φ_2}. One antenna is of angle π that covers the sector between rays $\overrightarrow{uu(4)}$ and $\overrightarrow{uu(1)}$. The other one (of angle 0) covers $u(3)$. According to assumption and $d(u(1), u(2)) \leq 2\sin(\frac{2\pi}{9})(= R_{2,\varphi_2})$, there is a way to direct antennae in $T_{u(1)}$ such that the resulting graph of vertices in $T_{u(1)}$ is strongly connected and $u(2)$ is covered by an antenna located at $u(1)$.

Otherwise, $p(u)$ does not lie in the sector between rays $\overrightarrow{uu(4)}$ and $\overrightarrow{uu(1)}$. Clearly, either the sector between rays $\overrightarrow{uu(1)}$ and $\overrightarrow{uu(2)}$ or the sector between rays $\overrightarrow{uu(3)}$ and $\overrightarrow{uu(4)}$ does not contain $p(u)$. Without loss of any generality, assume that the sector between rays $\overrightarrow{uu(1)}$ and $\overrightarrow{uu(2)}$ does not contain $p(u)$ (see Figure 3(e)). By Fact 2, the angle $\widehat{u(4)uu(2)}$ is in $[\frac{2\pi}{3}, \pi]$. We then use two antennae (located at u) of radius R_{2,φ_2}. One antenna is of angle π that covers the sectors between rays $\vec{uu(4)}$ and $\vec{uu(2)}$. The other one (of angle 0) covers $u(3)$.

From the above discussions, we can see that T_u has Property 1. This completes the proof of the result for the case where $k = 2$ and $\varphi_2 = \pi$, that is, we are able to direct antennae in T such that the resulting graph is strongly connected where radii of antennae is no more than $R_{2,\varphi_2} = 2\sin(\frac{2\pi}{9})$. This completes the proof of Part 1 of Theorem 3 in all cases.

Proof of Part 2: if $\frac{2\pi}{3} \le \varphi_2 < \pi$ then $r_{2,\varphi_2} \le 2\sin(\frac{\pi}{2} - \frac{\varphi_2}{4})$

In this part of the proof, let us define $R_{2,\varphi_2} := 2\sin(\frac{\pi}{2} - \frac{\varphi_2}{4})$. The proof of the result for this case is similar to the one for the case $\varphi_2 = \pi$. Here we only present the details for an internal vertex u with $\delta(u) \ge 4$. Still, we assume that all rooted subtrees $T_{u(1)}, \dots, T_{u(\delta(u)-1)}$ have Property 1 and that $u(1)$ is the first neighbor of u when rotating the ray $\vec{up}$.

Case $\delta(u) = 4$: There are three cases to consider depending on the size of the angle $\widehat{u(3)uu(1)}$.

If $\widehat{u(3)uu(1)} \le \varphi_2$, then we use two antennae (located at u) of radius R_{2,φ_2}. Refer to Figure 4(a). One antenna is of angle φ_2 that covers the sector between rays $\vec{uu(3)}$ and $\vec{uu(1)}$. The other one (of angle 0) covers $u(2)$.

Else if $\widehat{u(1)uu(3)} \le \varphi_2$, similar to the above. One antenna is of angle φ_2 that covers the sector between rays $\vec{uu(1)}$ and $\vec{uu(3)}$. The other one (of angle 0) covers p.

Otherwise, $\widehat{u(3)uu(1)} > \varphi_2$ and $\widehat{u(1)uu(3)} > \varphi_2$. In this case, $\min\{\widehat{u(3)up}, \widehat{puu(1)}\}$ is at most $\frac{2\pi}{3}$ which in turn is $\le \varphi_2$. Suppose that $\widehat{u(3)up} \le \varphi_2$. We then use two antennae (located at u) of radius R_{2,φ_2}. Refer to Figure 4(b). One antenna is of angle φ_2 that covers the sector between rays $\vec{uu(3)}$ and $\vec{up}$. The other one (of angle 0) covers $u(1)$. We can see that $\min\{\widehat{u(1)uu(2)}, \widehat{u(2)uu(3)}\} \le \pi - \frac{\varphi_2}{2}$ since $\widehat{u(3)uu(1)} > \varphi_2$. Therefore, either $u(1)$ or $u(3)$ is able to cover $u(2)$ with radius $\le 2\sin(\frac{\pi}{2} - \frac{\varphi_2}{4})$.

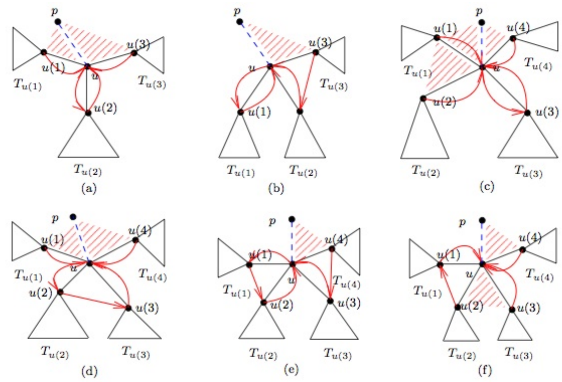

Figure 4. Case $\frac{2\pi}{3} \le \varphi_2 < \pi$.

Case $\delta(u) = 5$: Recall that p lies in the sector between rays $\vec{uu(4)}$ and $\vec{uu(1)}$ (counterclockwise). There are two cases to consider depending on the location of u's predecessor.

In the first case, if $p(u)$ does not lie in the sector between rays $\vec{uu(4)}$ and $\vec{uu(1)}$, then $\widehat{u(4)uu(1)} \le \frac{2\pi}{3} \le \varphi_2$ (Fact 2).

- If the angle $\widehat{u(4)uu(2)} \le \varphi_2$ then one antenna is of angle φ_2 that covers the sector between rays $\vec{uu(4)}$ and $\vec{uu(2)}$. The other one (of angle 0) covers $u(3)$. (See Figure 4(c).)
- Otherwise, $\widehat{u(4)uu(2)} > \varphi_2$. It implies that $\min\{\widehat{u(1)uu(2)}, \widehat{u(2)uu(3)}\} \le \pi - \frac{\varphi_2}{2}$. Therefore, we can use one antenna (of angle φ_2) to cover the sector between rays $\vec{uu(4)}$ and $\vec{uu(1)}$ and the other one (of angle 0) to cover $u(2)$. Either $u(4)$ or $u(2)$ has one antenna to cover $u(3)$. (See Figure 4(d).)

In the second case, $p(u)$ also lies in the sector between rays $\vec{uu(4)}$ and $\vec{uu(1)}$ and there are two cases to consider.

1. If $\min\{\widehat{u(3)up}, \widehat{u(4)uu(1)}, \widehat{puu(2)}\} \le \varphi_2$, *w.l.a.g.*, we assume that $\widehat{u(3)up} \le \varphi_2$. Then we can use one antenna located at u (of angle φ_2) to cover the sector between rays $\vec{uu(3)}$ and $\vec{up}$ and the other one (of angle 0) to cover $u(1)$. Either $u(3)$ or $u(1)$

has one antenna to cover $u(2)$ (note that $\min\{d(u(1),u(2)), d(u(3),u(2))\} \leq \sqrt{2}$ since $\widehat{u(1)uu(3)} \leq \pi$). $u(2)$ (resp. $u(4)$) has one antenna to cover u.

2. Otherwise, $\widehat{u(3)up}, \widehat{u(4)uu(1)}, \widehat{puu(2)}$ are larger than φ_2. We have $\widehat{u(3)up} \leq \pi$ and $\widehat{puu(2)} \leq \pi$ since $\varphi_2 \geq \frac{2\pi}{3}$ and $\widehat{u(2)uu(3)} \geq \frac{\pi}{3}$. Without loss of generality we may assume that $\widehat{u(4)up} \leq \widehat{puu(1)}$. Then $\widehat{u(4)up} \leq \varphi_2$ since $\varphi_2 \geq \frac{2\pi}{3}$.

   (a) If $\widehat{u(4)up} \geq \frac{\varphi_2}{2}$, then $\widehat{puu(1)} \geq \frac{\varphi_2}{2}$. We can see that

   $$\begin{aligned} & \widehat{u(3)up} \leq \pi, \widehat{puu(2)} \leq \pi \\ \Rightarrow \quad & \widehat{u(1)uu(2)} \leq \pi - \frac{\varphi_2}{2}, \\ & \widehat{u(3)uu(4)} \leq \pi - \frac{\varphi_2}{2} \\ \Rightarrow \quad & d(u(1),u(2)) \leq 2\sin\left(\frac{\pi}{2} - \frac{\varphi_2}{4}\right), \\ & d(u(4),u(3)) \leq 2\sin\left(\frac{\pi}{2} - \frac{\varphi_2}{4}\right). \end{aligned}$$

   Therefore, we use one antenna (of angle φ_2) to cover the sector between rays $\vec{uu(4)}$ and $\vec{up}$ and the other one (of angle 0) to cover $u(1)$. Also, $u(1)$ (resp. $u(4)$) has one antenna to cover $u(2)$ (resp. $u(3)$). See Figure 4(e).

   (b) Otherwise, $\widehat{u(4)up} < \frac{\varphi_2}{2}$. We can see that $\widehat{u(4)uu(1)} \geq \varphi_2 \Rightarrow \widehat{puu(1)} > \frac{\varphi_2}{2} \Rightarrow \widehat{u(1)uu(2)} < \pi - \frac{\varphi_2}{2} \Rightarrow d(u(1),u(2)) < 2\sin\left(\frac{\pi}{2} - \frac{\varphi_2}{4}\right)$.

      i. If $\widehat{u(2)uu(3)} \leq \frac{\varphi_2}{2}$ then we use one antenna (of angle $\frac{\varphi_2}{2}$) to cover the sector between rays $\vec{uu(4)}$ and $\vec{up}$ and the other one (of angle $\frac{\varphi_2}{2}$) to cover the sector between rays $\vec{uu(2)}$ and $\vec{uu(3)}$. Also, $u(2)$ has one antenna to cover $u(1)$. See Figure 4(f).

      ii. Otherwise, $\widehat{u(2)uu(3)} > \frac{\varphi_2}{2}$. Then, $\widehat{u(3)uu(4)} < \pi - \frac{\varphi_2}{2} \Rightarrow d(u(3),u(4)) < 2\sin\left(\frac{\pi}{2} - \frac{\varphi_2}{4}\right)$. Therefore, we use one antenna (of angle φ_2) to cover the sector between rays $\vec{uu(4)}$ and $\vec{up}$ and the other one (of angle 0) to cover $u(1)$. Also, $u(1)$ (resp. $u(4)$) has one antenna to cover $u(2)$ (resp. $u(3)$). See Figure 4(d).

This completes the proof of Part 2 of Theorem 3 in all cases and therefore the proof of the main theorem is now complete. ∎

4 More than two Antennae

In this section we consider the case of more than two antennae. To begin with, observe that since every set of points in the plane has a Euclidean spanning tree of degree at most 5, it is easy to see that for any set S of points in the plane, with five antennae per sensor, we can strongly connect the sensors using a range of at most 1. Therefore it remains to consider the cases of three and four antennae.

Three antennae per sensor

Theorem 5 *For any set S of points in the plane, with three antennae per sensor, we can strongly connect the sensors using a range of at most $\sqrt{3}$.*

Proof. Assume we have three antennae with angles 0 at each sensor. The proof is by induction on the height of the tree. The inductive hypothesis that must be maintained throughout is the following: "Given a rooted directional tree we can assign antennae so that the resulting graph is strongly connected while the out degree of the root never exceeds 2." Assuming the inductive hypothesis is true we argue as follows. Take as root any vertex, say u, of an MST. Vertex u has at most $d \leq 5$ subtrees, with corresponding roots $u_1, u_2, \ldots, u_d$, respectively. By the induction hypothesis we can assign antennae in the subtrees so that the resulting graphs are strongly connected and the vertices u_i use at most two antennae each. It follows that we can direct the remaining antenna at u_i towards the root u.

It remains to show how to direct two antennae from u to its d children so that the range being used does not exceed $\sqrt{3}$. By the previous construction, if it is satisfied for all the subtrees of u it must also be satisfied for the tree rooted at u. Therefore it remains to examine the initial cases of the induction hypothesis. To this end consider a tree consisting of the root u and the d vertices $u_1, \ldots, u_d$ adjacent to u. The result is trivial if $d \leq 2$. If $d = 3$ then as depicted in Figure 5(a) we find an angle between two children of u forming an angle $\leq 2\pi/3$ and connect them with a directed edge. If $d = 4$ then as depicted in Figure 5(b) we find two adjacent angles among three children of u each forming an angle $\leq 2\pi/3$ and connect them with directed edges. Finally, if $d = 5$ then as depicted in Figure 5(c) we find three angles among the children of u each forming an angle $\leq 2\pi/3$ and connect them with directed edges. This completes the proof of Theorem 5. ■

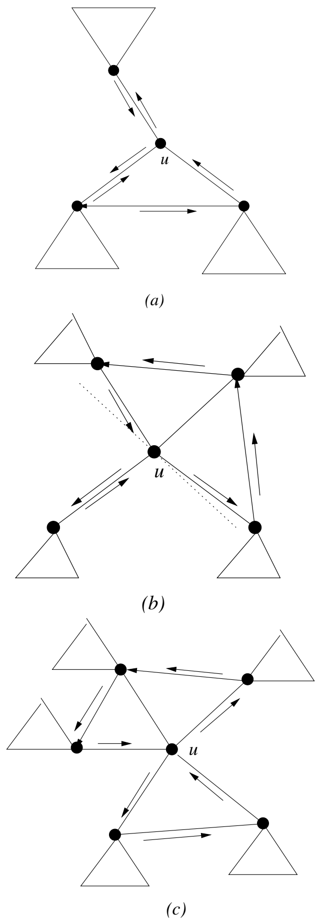

Figure 5. Directing antennae among u and its children so as to guarantee that the maximum range does not exceed $\sqrt{3}$. Notice that in all three cases the out degree of u is 2.

Four antennae per sensor

Theorem 6 *For any set S of points in the plane, with four antennae per sensor, we can strongly connect the sensors using a range of at most $\sqrt{2}$.*

Proof. This is similar to the proof of Theorem 5. Assume we have four antennae with angles 0 at each sensor. The proof is by induction on the height of the tree. The inductive hypothesis that must be maintained throughout is the following: "Given a rooted directional tree we can assign antennae so that the resulting graph is strongly connected while the out degree of the root never exceeds 3." Assuming the inductive hypothesis is true we argue as follows. Take as root any vertex, say u, of an MST. Vertex u has at most $d \leq 5$ subtrees, with corresponding roots $u_1, u_2, \ldots, u_d$, respectively. By the induction hypothesis we can assign antennae in the subtrees so that the resulting graphs are strongly connected and the vertices u_i use at most three antennae each. It follows that we can direct the remaining antenna at u_i towards the root u.

It remains to show how to direct two antennae from u to its d children so that the range being used does not exceed $\sqrt{2}$. By the previous construction, if it is satisfied for all the subtrees of u it must also be satisfied for the tree rooted at u. Therefore it remains to examine the initial cases of the induction hypothesis. To this end consider a tree consisting of the root u and the d vertices $u_1, \ldots, u_d$ adjacent to u. The result is trivial if $d \leq 3$. If $d = 4$ then as depicted in Figure 6(a) we find an angle between two children of u forming an angle $\leq \pi/2$ and connect them with a directed edge. Finally, if $d = 5$ then as depicted in Figure 6(b) we find two angles among

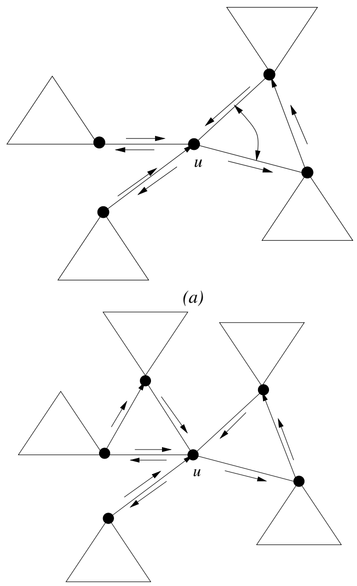

Figure 6. Directing antennae among u and its children so as to guarantee that the maximum range does not exceed $\sqrt{2}$. Notice that in both cases the out degree of u is 3.

the children of u each forming an angle $\leq \pi/2$ and connect them with directed edges. This completes the proof of Theorem 6. ■

5 Conclusion

We gave several tradeoffs between the total antenna spread and range when each sensor has k antennae, $k = 2, 3, 4$, so as to guarantee the resulting network is strongly connected. In addition to studying tradeoffs arising from results obtained in this paper several problems remain open. Lower bounds are lacking from our study and it remains open to prove NP completeness results for the case of multiple antennae per sensor. Another interesting question concerns ensuring that for a given integer c the resulting network is strongly c-connected, i.e., it remains strongly connected after the deletion of any $c - 1$ nodes.

In a real network, one has to consider interference from nearby links to be able to judge the connectivity of the network. In this study the system model assumes that there is no interference. It is a challenge work to obtain similar results that can be applied to the real world.

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