Connectivity Trade-offs in 3D Wireless Sensor Networks using Directional Antennae

Evangelos Kranakis
School of Computer Science Carleton University
Ottawa, Canada
Email: kranakis@scs.carleton.ca

Danny Krizanc
Department of Mathematics Wesleyan University
Middletown, USA
Email: dkrizanc@wesleyan.edu

Ashish Modi
Department of Electronics and Electrical Communication Engineering
IIT Kharagpur, India
Email: ashishmodi.kgp@gmail.com

Oscar Morales-Ponce
School of Computer Science Carleton University
Ottawa, Canada
Email: oscarmponce@gmail.com

***Abstract*—We consider a 3D antenna orientation problem for maintaining connectivity of a wireless network in 3D space using only directional antennae. Sensors are located at points in 3D space and are equipped with directional antennae. The *strong connectivity antenna orientation problem* is concerned with deciding whether or not for given solid angle Ω and range r it is possible to orient the antennae so as to ensure that the sensor network resulting from the induced transmissions is strongly connected. In this paper we 1) present an algorithm ensuring optimal antenna range for the case when $\Omega \geq \frac{18\pi}{5}$, 2) show that determining whether or not there exists a strong orientation of directional sensors of solid angle $\Omega < \pi - \varepsilon$ having optimal range is NP-complete, for any $\varepsilon > 0$, and 3) provide an algorithm for approximating the antennae range so as to ensure strong connectivity of the resulting graph, provided the solid angle of the antennae is $2\pi \leq \Omega < \frac{18\pi}{5}$. In addition, we study the effect of replacing omnidirectional antennae with directional antennae on the hop stretch factor of the resulting network of directional antennae and present some simulation results on the variation of hop stretch factor with different network sizes and solid angles of directional antennae. This is the first paper concerning the strong connectivity antenna orientation problem in 3D space.**

***Keywords*-Algorithm, Directional Antenna, Kissing number, MST, Orientation, Sensor Network, Tammes Problem.**

I. Introduction

Directional antennae are being used in wireless networks not only for reducing energy consumption and interference, but also for improving routing efficiency and security. Sensors rely on the use of antennae to configure and operate an ad hoc network. Numerous types of antennae are in practical use in various settings today. In our subsequent analysis it will suffice to compare two types of antennae. *Omnidirectional* antennae which transmit the signal in all directions in the three-dimensional space and *directional* antennae which can transmit the signal towards a specific direction. Omnidirectional antennae usually incur more interference than directional antennae thus hampering nodes from receiving data from other transmitters and causing overall performance degradation of the sensor network. Sensor networks using directional antennae not only can have extended life-time since the consumption of energy in each antenna is proportional to the volume covered by the transmitting antennae, but also using a small antenna spread prevents unwanted nodes from listening to the communication and therefore, improving the overall security of the network. Hence, it is desirable to reduce not only the range, but also the angle of an antenna.

In this paper we consider a 3D antenna orientation problem for maintaining connectivity of a wireless network in 3D space using only directional antennae. Sensors are located at points in 3D space and are equipped with directional antennae. The *strong connectivity antenna orientation problem* is concerned with deciding whether or not for a given solid angle Ω and range r it is possible to orient the antennae so as to ensure that the sensor network resulting from the induced transmissions is strongly connected. We model *three-dimensional directional* antennae using a spherical circular sector of *solid angle* or *spherical beam width* Ω. The term solid angle and spherical beam width are used interchangeably. We propose algorithms for orienting the sensors in *three dimensional* space so as to maintain connectivity. In particular, we determine an upper bound on the increase in range required for maintaining connectivity in this case and show that there is a tradeoff between the increase in range required and solid angle of the *directional* antennae used.

A. Related Work

There is extensive theoretical literature documenting the performance improvements on a wireless network when using directional antennae. Motivated from the studies in [8] concerning the capacity and throughput of wireless networks the papers [15] and [20] investigate performance improvements when using directional antennae. There has also been some recent research concerning the advantages of using directional antennae. For example, in [14] they study the energy consumption of networks of omnidirectional antennae and compare it to the consumption of networks of directional antennae in *two dimensional* networks. They have modelled a *two dimensional* directional antennae using a circular sector of angle α and shown that in this case the range of antennae increases by a factor of $\sqrt{2\pi/\alpha}$. Related studies can also be found in [2], [3]. It is also worth noting that directional antennae can improve security because the narrower beam width of the directional antennae guarantees less exposure of the signal to adversaries during transmission [10].

Closely related to our work are topology control issues in wireless sensor networks, e.g., [9], [11] and [13], and is also directly related to the problem of understanding the trade-offs between antenna range and beam-width for attaining network connectivity. In particular, the problem of replacing omni-directional antennae with directional antennae while maintaining connectivity has been considered in the two dimensional space. In [4], the authors have modelled the directional antennae in two dimensional space as a circular sector of angle ϕ. The authors have proposed a polynomial time approximation algorithm for orienting antennae in sensor networks when the sector angle $\phi \geq \pi$ while maintaining connectivity. They have also determined that the optimal range of an omni-directional antennae network is sufficient for maintaining connectivity in the directional antennae network as well, provided that the sector angle satisfies $\phi \geq \frac{8\pi}{5}$. Further, they have proved that the problem of orienting directional antennae in two dimensional space while maintaining connectivity is NP-complete for sector angles $\phi < 2\pi/3 - \varepsilon$, for any $\varepsilon > 0$. Similarly, in [7] the authors give an analysis of antennae range for attaining strong connectivity when each sensor is equipped with k antennae, for a given value of $1 \leq k \leq 5$.

Observe that in a real setting, the two dimensional model used in [4] may result in a network that is not strongly connected due to distinct altitudes. To overcome this deficiency, we propose a 3 dimensional model to address the orientation problem having one antenna at each sensor. Thus, this work indeed generalizes the results given in [4].

B. Preliminaries and Notation

In this section, we will introduce the model for the directional antennae and some definitions which will be used throughout the paper. For any two points u, v let us represent the distance between the points u and v as $d(u,v)$. Given a set of sensors S equipped with omnidirectional antennae, we define the *Unit Ball Graph* (UBG) G as the graph whose vertex set is the set S of sensors in the three dimensional space and the edge set is defined as follows: if $d(u,v)$ is less than or equal to *unity*, then the straight line segment between u and v forms an edge in the unit ball graph. Here, without loss of generality, we assume that the range of an *omnidirectional* antennae is *normalized* to unity.

We model a *three dimensional* directional antenna as a spherical sector of solid angle Ω (see Definition 1) and depicted in Figure 1. A *directional* antennae is characterized by its *solid angle* and *range*.

Definition 1: The solid angle of a solid spherical sector is defined as the ratio of the area of the spherical surface and the square of the radius of the sphere of which it forms part. It is usually represented by Ω.

Figure 1: 3D directional antenna of solid angle $\Omega = 2\pi(1-\cos\theta)$.

Definition 2: The apex angle of a spherical sector with solid angle Ω is defined as the maximum planar angle between any two generatrices of the spherical sector. It is usually represented by 2θ.

Using the well-known relation of Archimedes, we note that the apex angle 2θ and the solid angle Ω are related by the following identity.

$$\Omega = 2\pi(1-\cos\theta) \tag{1}$$

Definition 3: Consider a set S of sensors located at points in 3D space. The optimal range of the set of sensors S having unit ball graph G is defined as the maximum length of an edge of the minimum spanning tree T of G and is denoted by $r_{MST}(S)$.

An interesting question relating to our analysis was proposed by the botanist Tammes in [18] and concerns "what is the length of the largest largest diameter of n equal circles that can be placed on the surface of the unit sphere without overlap" (see Figure 2). This is made precise in the following definition.

Definition 4: The Tammes' radius of the unit sphere for n circles is defined as the maximum radius of n equal non-overlapping circles on the surface of the sphere and is denoted by R_n.

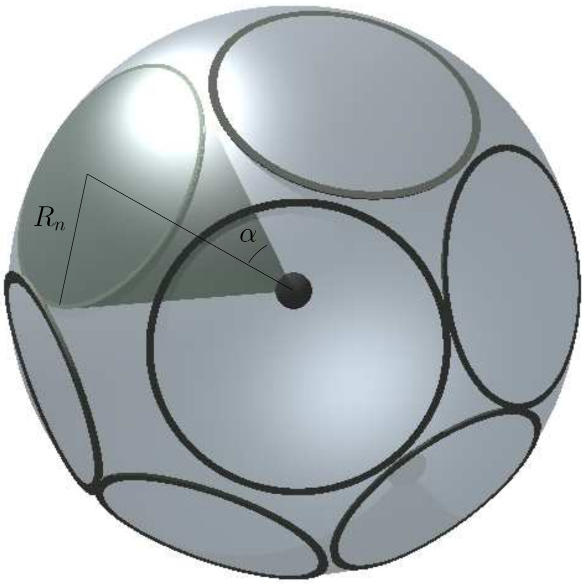

Figure 2: Tammes' problem.

The problem of finding the Tammes' radius for different values of n has been studied thoroughly in the literature. In [19], Tarnai et al. have found the values of the Tammes' radius for $n \leq 12$ and $n = 24$ in terms of the angle α it subtends at the center as shown in Figure 2.

Recall that in a graph G, a minimum spanning tree MST or minimum weight spanning tree is a spanning tree whose weight is less than or equal to the weight of every other spanning tree. When the weights are the Euclidean distances then it is called the Euclidean MST. In general the maximum degree of the Euclidean MST on G is bounded by the *Kissing number* [16], [17], [6] which is defined as the maximum number of disjoint unit spheres that can be simultaneously tanget to a given unit sphere. Hence, the maximum degree of a Euclidean MST in 3D is bounded by 12.

C. Outline and results of the Paper

In this paper, we provide a set of results for the problem of maintaining connectivity in wireless networks in 3D using directional antennae. In summary, we

1) present an algorithm ensuring optimal antenna range for the case when $\Omega \geq \frac{18\pi}{5}$,
2) show that determining whether or not there exists a strong orientation of directional sensors of solid angle $\Omega < \pi - \varepsilon$ having optimal range is NP-complete, for any $\varepsilon > 0$, and
3) provide an algorithm for approximating the antennae range so as to ensure strong connectivity of the resulting graph, provided the beam width of the antennae is $2\pi \leq \Omega < \frac{18\pi}{5}$.

An outline of the paper is as follows. In Section II, we give a lower bound on the solid angle of the directional antennae for the case where the optimal range for networks of omnidirectional antennae is sufficient for maintaining connectivity. In this section, we also present a simple algorithm for orienting the antennae. In Section IV, we determine an upper bound on the increase in range required for maintaining connectivity when the solid angle is less than that found in Section II. In this section, we also present a simple polynomial time algorithm for orienting the antennae in such a way that the resulting graph remains connected. In Section III, we use the results of [7] and prove that the problem of maintaining connectivity in the transmission graph when solid angle is less than π is NP-Hard. In Section V, we study the effect of replacing *omnidirectional* antennae with *directional* antennae on the *stretch factor* of the resulting network of directional antennae and present some simulation results on the variation of hop stretch factor with different network sizes and solid angles of directional antennae. We conclude with a discussion of possible extensions and interesting open problems in Section VI.

II. Lower Bound on Solid Angle and Optimal Range

In this section, we will derive a lower bound on the solid angle of the antennae for which optimal range (as defined in Definition 3) is sufficient for maintaining connectivity.

Theorem 5: Given a set of points S in the three dimensional space and a spherical angle $\Omega \geq \frac{18\pi}{5}$, there exists a polynomial time algorithm that computes a strong orientation of three dimensional antennae of spherical sector with solid angle Ω and having optimal range.

Proof: Let T be a minimum spanning tree on S and $r_{MST}(S)$ be the longest edge of minimum spanning tree T. For each point p of S we will show how to orient the antenna at p. Consider the sphere B_p centered at p of minimum radius r_p that covers all the neighbors of p in T. Observe that $r_p \leq r_{MST}(S)$. For each neighbor u of p in T, let u' be the intersection point u' of B_p with the ray emanating from p toward u. Let $N_{B_r}(p)$ be the set of points projected on the surface of B_p. Since p has maximum degree 12 in T, $|N_{B_r}(p)| \leq 12$. Let DT_p be the Delaunay Triangulation of $N_{B_r}(p)$ on the surface of B_p. Consider the largest triangle t_p of DT_p (In case of a tie, break it arbitrarily.) Orient the antenna at p with range r_p in such a way that t_p is not covered. This can be done by orienting the antennae toward the opposite direction to the center of t_p .

To prove the lower bound on the solid angle at each point p, observe that every edge of DT_p has length at least twice the *Tammes' radius* R_{12} which corresponds to the kissing number in 3D [17]. Therefore, every circumcircle is at least of radius $a = 2R_{12}/\sqrt{3}$ (see Figure 3) and the planar angle α at the center of the sphere B_p is at least $\arcsin(a)$. From

[5][Problem D7, pages 114-116], it is known that

$$R_{12} = \sin\left(\frac{63^\circ 26'}{2}\right).$$

Therefore, using Equation 1 we can calculate the solid angle of the antennae as follows:

$$\begin{aligned}\Omega &= 4\pi - 2\pi(1-\cos(\alpha)) \\ &= 2\pi(1+\cos(\alpha)) \\ &= 2\pi\left(1+\cos\left(\arcsin\left(\frac{2R_{12}}{\sqrt{3}}\right)\right)\right) \\ &< \frac{18\pi}{5},\end{aligned}$$

where the last inequality is obtained after numerical calculation. It is easy to see that the resulting transmission graph is strongly connected since T is connected and all the edges of T are covered by exactly two antennae at opposite endpoints. This completes the proof of the theorem.

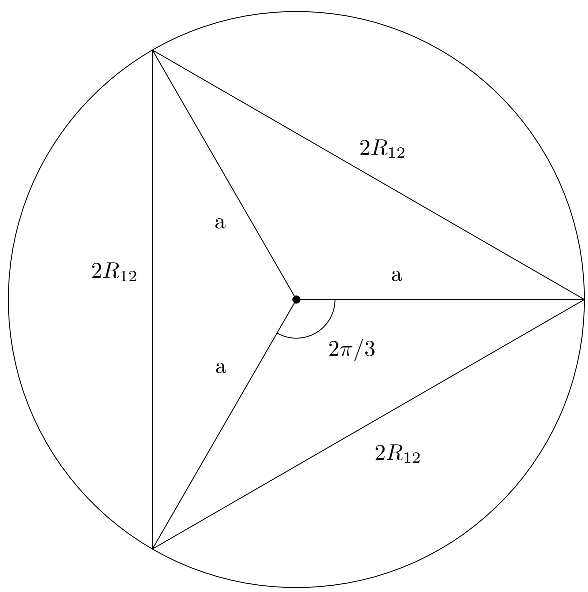

Figure 3: A circumcircle and the Tammes radius R_{12}.

■

Observe that Theorem 5 relies on the construction of the MST in 3D which takes $O((n\log(n))^{4/3})$ expected time [1]. Further every other step can be done in constant time. Therefore, if we do not insist on guaranteeing optimal range the algorithm can be implement in distributed manner to run in constant time by considering the k-Local MST of a connected UBG of S at distance 1 and doing only local computation.

III. The Complexity of the 3D Case

In this section, we will prove that when the solid angle of the directional antennae is less than $\pi - \varepsilon$, the problem of finding out an orientation of the sensors such that the transmission graph is connected is NP-Complete, for any $\varepsilon > 0$.

Theorem 6: Given a set of points S in 3D and $\Omega < \pi - \varepsilon$ for any $\varepsilon > 0$, determining whether there exists a strong orientation of directional sensors of solid angle Ω having optimal range is NP-Complete.

Proof: It is easy to see that the problem is in the class NP. We prove the NP-Hardness by using a result known for the 2D case. In [4], they have proved that in case of directional antennae in 2D, modelled as a circular sector, when the sector angle is less than $\pi - \varepsilon$, the problem of maintaining connectivity is NP-Complete, for any $\varepsilon > 0$.

Consider a set S of points in the plane. We will prove that the 2D problem is equivalent to the 3D problem. From Definition 2 and Archimedes' relation, any plane that cuts the coverage area of any 3D directional antennae through the apex with angle Ω has plane angle satisfying $\cos(\theta) \leq 1 - \frac{\Omega}{2\pi}$. Therefore $\theta \leq 2\pi/3$ if and only if $\Omega \leq \pi$. Assume a strong orientation of the planar directional antenna of S with angle at most $2\pi/3$. Clearly, we can orient the 3D directional antennae with angle π in such a way that it covers the planar angle. Similarly, If there exists an algorithm that creates a strong orientation of the 3D directional antenna of S with angle π then we can strongly orient the 2D directional antenna of S with angle $2\pi/3$. This completes the proof of the theorem.

■

IV. Approximation Algorithm and Upper Bound

In this section, first we will propose a linear algorithm for orienting the directional antennae when the solid angle is within a specified range. Following this we will prove that the transmission graph generated is strongly connected.

Theorem 7: Given a set S of n points in 3D and a solid angle Ω such that $2\pi \leq \Omega < \frac{18\pi}{5}$, it is possible to orient the antennae at each sensor with solid angle Ω and range $r(\Omega)$ in $O((n\log(n))^{4/3})$ time so that the transmission graph is connected, where

$$r(\Omega) = \frac{\sqrt{\Omega(4\pi-\Omega)}}{\pi} \cdot r_{MST}(S) \tag{2}$$

Proof: Let T be a minimum spanning tree on S and $r_{MST}(S)$ be the longest edge of minimum spanning tree T. We will use directional antennae with solid angle Ω and range,

$$r(\Omega) = r_{MST} \cdot \frac{\sqrt{\Omega(4\pi-\Omega)}}{\pi}$$

for constructing the transmission graph.

Consider a matching M of T with the following property: every internal node of T is incident to an edge in M. The matching M can be constructed as follows. Initially, M is empty. We root T at an arbitrary non-leaf node s. We pick an edge between s and one of its children and insert it to M. Then, we visit the remaining nodes of T in a BFS (Breadth

First Search) manner. When visiting a node u, if u is either a leaf-node or a non-leaf node such that the edge between it and its parent is in M, we do nothing. Otherwise, we pick an edge between u and one of its children and insert it to M.

We say that the endpoints of an edge in M form a couple. We use sectors of solid angle Ω and radius $r(\Omega)$ at each point and orient them as follows: At each node $u \in S$ not incident to an edge of M, the sector is oriented so that it induces the directed edge from u to its parent in T in the corresponding transmission graph. For each pair of points u and v forming a couple, we orient the sector at u so that it contains all points p at distance $r(\Omega)$ from u for which the counter-clockwise angle $\angle vup$ is in $[0, 2\theta]$ where 2θ is the apex angle of the directional antenna. See Figure 4.

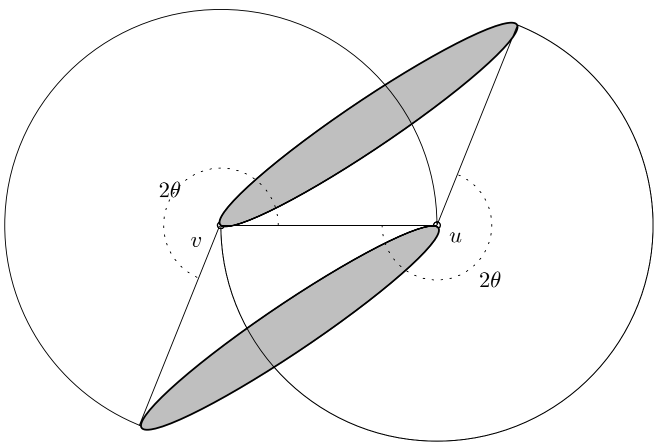

Figure 4: Orientation of antennae at u and v such that (u,v) is in the matching M.

Now, we will prove that the resulting transmission graph is connected. To prove this, we first prove that the transmission graph has the following property (P):

> for each pair of points u and v forming a couple, the transmission graph G contains two opposite directed edges between u and v and for each neighbor w of either u or v in T, it contains a directed edge from either u or v to w.

Consider a point w corresponding to a neighbor of u in T (the argument for the case where w is a neighbor of v is symmetric). Let θ be the apex angle of Ω. Clearly, w is at distance $d(u,w) \leq r_{MST}$ from u. Since $\Omega \leq \frac{18\pi}{5}$, using Equation 2, we have the antenna range, $r(\Omega) \geq 6/5 \cdot r_{MST}$. Hence, w is contained in the spherical sector of u if the counter-clockwise planar angle $\angle vuw$ is at most θ. Now, assume that the angle $\angle vuw > \theta$. By the law of cosines in the triangle defined by points u, v and w, we have that $d(v,w)$ is equal to

$$\sqrt{d(v,w)^2 + d(u,v)^2 - 2d(u,v)d(u,w)\cos(\angle vuw)}$$

It follows that

$$\begin{aligned} d(v,w) &\leq r_{MST} \cdot \sqrt{2 - 2\cos(2\theta)} \\ \text{or } d(v,w) &\leq r_{MST} \cdot |2\sin(\theta)| \\ \text{or } d(v,w) &\leq r_{MST} \cdot \left|2\sqrt{1 - \cos^2(\theta)}\right| \\ \text{or } d(v,w) &\leq r_{MST} \cdot \left|2\sqrt{1 - \left(1 - \frac{\Omega}{2\pi}\right)^2}\right| \\ \text{or } d(v,w) &\leq r_{MST} \cdot \frac{\sqrt{\Omega(4\pi - \Omega)}}{\pi}, \end{aligned}$$

since the apex angle satisfies $2\theta > \pi$, for all $\Omega \geq 2\pi$. Hence, in the above argument, either $\angle vuw$ or $\angle uvw$ is definitely less than π which means the transmission graph contains an edge from either u or v to w. This completes the proof of property P.

To proof that the resulting transmission graph is strongly connected we will show that for any two neighbors u and v in T, there exists a directed path from u to v and a directed path from v to u in unit ball graph G. Without loss of generality, assume that u is closer to the root s of T than v. If the edge between u and v belongs to M, i.e. u and v form a couple), property (P) guarantees that there exist two opposite directed edges between u and v in the transmission graph G. Otherwise, let w_1 be the node with which u forms a couple. Since v is a neighbor of u in T, there is either a directed edge from u to v in G or a directed edge from w_1 to v in G. Then, there is also a directed edge from u to w_1 in G which means that there exists a directed path from u to v. If v is a leaf, then its sector is oriented so that it induces a directed edge to its parent u. Otherwise, let w_2 be the node with which v forms a couple. Since u is a neighbor of v in T, there is either a directed edge from v to u in G or a directed edge from w_2 to u in G. Then, there is also a directed edge from v to w_2 in G which means that there exists a directed path from v to u.

Regarding the complexity, the construction of the minimum spanning tree can be done in $O((n\log(n))^{4/3})$ time (see [1]). It is not difficult to see that all the rest of the steps can be implemented in linear time. This completes the proof of theorem 7. ■

V. Simulation Results

In this section, we study the impact of replacing 3D omnidirectional antennae with 3D directional antennae. In our experiments we randomly generate sets of n points in the three dimensional space. For each instance S we construct the directional spanner, say G, with antenna beam-width Ω according to Theorem 7 as well as the unit ball graph UBG($r_{MST}(S)$). We then compare the average minimum path length of G with that of UBG($r_{MST}(S)$).

In the first simulation we fix Ω to $13\pi/5$ and vary the number of vertices from 30 to 100 in increments of 10. We ran 30 times each value of n and plot the ratio with the box-diagram as depicted in Figure 5. Observe that in all the cases the third quantile of the ratio is less than one and just a few values are greater than one. Thus, in general we can conclude that with $\Omega = 13\pi/5$ the directed network behaves better than the UBG($r_{MST}(S)$). A possible explanation is because the radius of the 3D directional antennae is greater than $r_{MST}(S)$ and UBG($r_{MST}(S)$) has optimal range.

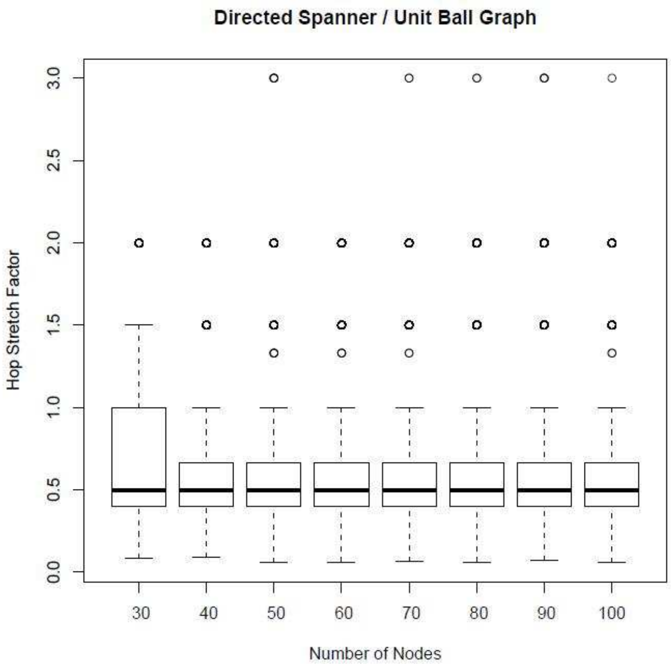

Figure 5: Direced Spaner/Unit Ball Graph varying n.

In the second simulation we fix n to 70 and consider distinct values of the solid angle Ω in the range $(2\pi, \frac{18\pi}{5})$. We ran 30 times each value of n and plot the ratio with the box-diagram as depicted in Figure 6. Similarly, in all the cases the third quantile of the ratio is less than one. However, it seems that when Ω grows, the ratio also increases. Thus, we can conclude that when Ω is small, the directed network behaves better than the UBG($r_{MST}(S)$). A possible reason for the phenomenon is that when Ω increases, $r(\Omega)$ converges to the optimal value.

VI. Extensions and Open Problems

In this paper we have considered connectivity trade-offs in 3D wireless sensor networks using directional antennae. In addition to improving the results presented in previous sections, several interesting questions remain open. For once, very little is known when the spherical antenna beam Ω is in the range $\pi \leq \Omega < 2\pi$. Another question, related to the Tammes' radius and Kissing number, is concerned with angle/range trade-offs when each sensor is equipped with a given number k of antennae, where $1 \leq k \leq 12$. This last problem is also related to the work (in the plane) in [12], [7] on strong connectivity with multiple directional antennae per sensor.

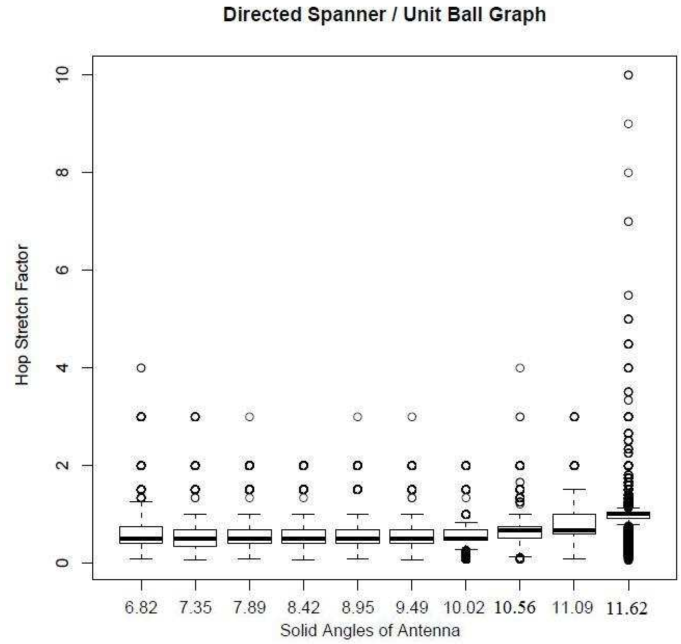

Figure 6: Direced Spaner/Unit Ball Graph varying Ω.

Acknowledgments

Research of Evangelos Kranakis was supported in part by NSERC and MITACS grants, of Ashish Modi by the MITACS Globalink program, and Oscar Morales by CONACyT graduate scholarship.

References

[1] P.K. Agarwal, H. Edelsbrunner, O. Schwarzkopf, and E. Welzl. Euclidean minimum spanning trees and bichromatic closest pairs. *Discrete and Computational Geometry*, 6(1):407–422, 1991.

[2] I. Caragiannis, C. Kaklamanis, and P. Kanellopoulos. New Results for Energy-Efficient Broadcasting in Wireless Networks. *Proc. of the 13th Annual International Symposium on Algorithms and Computation (ISAAC02)*, pages 332–343, 2002.

[3] I. Caragiannis, C. Kaklamanis, and P. Kanellopoulos. Energy-Efficient Wireless Network Design. *Theory of Computing Systems*, 39(5):593–617, 2006.

[4] I. Caragiannis, C. Kaklamanis, E. Kranakis, D. Krizanc, and A. Wiese. Communication in wireless networks with directional antennas. In *SPAA '08: Proceedings of the twentieth annual symposium on Parallelism in algorithms and architectures*, pages 344–351, New York, NY, USA, 2008. ACM.

[5] H.T. Croft, K.J. Falconer, and R.K. Guy. *Unsolved problems in geometry*. Springer, 1991.

[6] Eppstein D. Spanning trees and spanners. In Jörg-Rudiger Sack and Jorge Urrutia, editors, *Handbook of Computational Geometry*, chapter 9, pages 425–461. Elsevier, 2000.

[7] S. Dobrev, E. Kranakis, D. Krizanc, O. Morales, J. Opatrny, and L. Stacho. Strong connectivity in sensor networks with given number of directional antennae of bounded angle. In *Proceedings of the 4th Annual International Conference on Combinatorial Optimization and Applications (COCOA 10). Big Island, Hawaii, Dec 18-20, 2010. COCOA 2010, Part II, LNCS 6509*, pages 72–86. Springer, LNCS, 2010.

[8] P. Gupta and P. R. Kumar. The capacity of wireless networks. *Information Theory, IEEE Transactions on*, 46(2):388–404, 2000.

[9] Zhuochuan H., Chien chung S., Chavalit S., and Chaiporn J. Topology control for ad hoc networks with directional antennas. In *Proc. IEEE Int. Conference on Computer Communications and Networks*, pages 16–21, 2002.

[10] L. Hu and D. Evans. Using directional antennas to prevent wormhole attacks. In *Network and Distributed System Security Symposium (NDSS)*, pages 131–141. Internet Society, 2004.

[11] Z. Huang, C.C. Shen, C. Srisathapornphat, and C. Jaikaeo. Topology control for ad hoc networks with directional antennas. *Computer Communications and Networks, 2002. Proceedings. Eleventh International Conference on*, pages 16–21, 2002.

[12] E. Kranakis, D. Krizanc, and O. Morales. Maintaining connectivity in sensor networks using directional antennae. In S. Nikoletseas and J. Rolim, editors, *Theoretical aspects of Distributed Computing in Sensor Networks*. Springer, 2010.

[13] E. Kranakis, D. Krizanc, and J. Urrutia. Coverage and Connectivity in Networks with Directional Sensors. *proceedings Euro-Par Conference, Pisa, Italy, August*, pages 917–924, 2004.

[14] E. Kranakis, D. Krizanc, and E. Williams. Directional versus omnidirectional antennas for energy consumption and k-connectivity of networks of sensors. *In proceedings of OPODIS 2004, LNCS*, 3544:357–368, 2005.

[15] C. Peraki and S.D. Servetto. On the maximum stable throughput problem in random networks with directional antennas. In *Proceedings of the 4th ACM international symposium on Mobile ad hoc networking & computing*, pages 76–87. ACM, 2003.

[16] F. Pfender and G.M. Ziegler. Kissing numbers, sphere packings, and some unexpected proofs. *Notices-American Mathematical Society*, 51:873–883, 2004.

[17] G. Robins and J.S. Salowe. Low-degree minimum spanning trees. *Discrete and Computational Geometry*, 14(1):151–165, 1995.

[18] P.M.L. Tammes. On the origin of number and arrangement of the places of exit on the surface of pollen grains. *Recueil Traveaux Botaniques Néerlandais*, 27:1–84, 1930.

[19] T. Tarnai and Z. Gáspár. Covering a sphere by equal circles, and the rigidity of its graph. In *Mathematical Proceedings of the Cambridge Philosophical Society, Volume 110, Number 01*, pages 71–89. Cambridge University Press, 1991.

[20] S. Yi, Y. Pei, and S. Kalyanaraman. On the capacity improvement of ad hoc wireless networks using directional antennas. *Proceedings of the 4th ACM international symposium on Mobile ad hoc networking & computing*, pages 108–116, 2003.