Communication in Wireless Networks with Directional Antennae

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Abstract

We study the problem of maintaining connectivity in a wireless network where the network nodes are equipped with directional antennae. Nodes correspond to points on the plane and each uses a directional antennae modeled by a sector with a given angle and radius. The connectivity problem is decide whether or not it is possible to orient the antennae so that the directed graph induced by the node transmissions is strongly connected. We present algorithms for simple polynomial-time solvable cases of the problem, show that the problem is NP-complete in the 2-dimensional case when the sector angle is small, and present algorithms that approximate the minimum radius to achieve connectivity for sectors with a given angle. We also discuss several extensions to related problems. To the best of our knowledge, the problem has not been studied before in the literature.

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1 Introduction

Various types of antennas are used in wireless networking. For our purposes we distinguish between directional and omnidirectional. The former emit greater power in one direction thus allowing for increased transmission range and performance at the receiver's end as well as reduced interference from unwanted sources. They are different from the latter which radiate power uniformly in all directions in the plane. The set of neighbors of an omnidirectional antenna with transmission radius r consists of all points within a disk of radius r centered at the source. In addition to the range r, directional antennas have a transmission angle ϕ (measured in radians) that specifies how wide from the source is the spread of the antenna. The neighborhood of a node is determined by which nodes appear in the resulting sector of the disk (which in turn depends upon its orientation on the plane).

An important issue in wireless networking is attaining connectivity with minimum communication cost. Regardless of the type of antenna being used the communication cost is determined by the transmission cost of each antenna, which in turn is proportional to the coverage area of the antenna. For omnidirectional antennas the coverage area is πr^2, while for directional antennas it is proportional to the square of the range r multiplied by ϕ. Therefore for the same coverage area a directional antenna can reach further than an omnidirectional antenna in the direction of a target node. Moreover, the smaller the transmission spread ϕ the lower the communication cost. This is a great advantage of directional antennas over omnidirectional ones in terms of single edge cost. However, maintaining network connectivity in this scenario becomes more complex. For omnidirectional antennae of the same range, the resulting network has bi-directional links, while in the case of directional antennae this may not be the case and the resulting network has to be modeled by a directed graph. To ensure connectivity the antenna orientations of the individual nodes must be chosen so that the resulting directed graph is strongly connected.

Directional antennas in use today have a variety of capabilities that enable them to vary their transmission range and orientation. In this paper we model our antennae after those such as the ESPAR (Electronically Steerable Passive Array Radiator) antenna [13] which consists of a steerable central source that can radiate in a region reasonably approximated by a sector of a circle. (The number of sector sizes is fixed by the choice of the number of radiator elements typically set at six, i.e., in this case only multiples of $\pi/3$ are possible. In the problems we consider the sector size is part of the input.) We consider the problem of maintaining connectivity using the minimum possible range for a given angular spread. Specifically, for a set of sensors located in the plane at established positions and with a given angular spread we are interested in providing an algorithm that minimizes the range required so that by an appropriate rotation of each of the antennae the resulting network becomes strongly connected.

Preliminaries and notation. In the sequel we introduce the formal model and basic definitions that will be used throughout the paper. Given a set X of points on the plane and $r > 0$, consider the *proximity* graph $G_r(X)$ containing a node for each point of X and an edge for each pair of nodes if the distance of the corresponding points is at most r. Lets call the *transmission* graph the directed graph defined by containment of sectors of radius r and angle ϕ at the nodes with a given orientation. In particular, the proximity graph is an undirected graph representing the communication network underlying a set of sensors with identical transmission range r, while the transmission graph is obtained from the proximity graph by taking into account the orientation of the antennas at the nodes and as such it is a directed graph. Given a set of points X on the plane, denote by $r^*(X, \phi)$ the smallest radius such that there exists an orientation of sectors of angle ϕ and radius $r^*(X, \phi)$ so that the resulting transmission graph is strongly connected.

Given a set of points X on the plane consider the minimum spanning tree $MST(X)$ connecting the points. Denote by $r(MST(X))$ the edge of $MST(X)$ of maximum distance. Clearly, $MST(X)$ is also spanning tree on the proximity graph $G_r(X)$. Also, for any $r' < r$, the proximity graph $G_{r'}(X)$ is disconnected, otherwise $MST(X)$ would not be a minimum spanning tree for X. It follows that given any angle ϕ, $r^*(X, \phi)$ must be bounded from below by $r(MST(X))$.

Related work There has been a fair deal of research on employing directional antennas for routing and topology control in wireless ad hoc networks [5, 12, 14, 15, 17]. Since nodes in such networks are energy constrained, energy efficiency is an important parameter along which these algorithms are compared. The connectivity problem for omni-directional antennae has been studied extensively by a number of authors including [1, 2, 7, 11]. For the case of directional antennae, [9] makes a direct comparison of the energy consumption of directional and omnidirectional antennas for achieving k-connectivity for randomly distributed nodes in the unit square and [8] looks at coverage and connectivity problems in the same scenario. Directional antennae seem to have wide applicability not only for improving energy consumption but also in security of adhoc networks. They have been used for preventing wormhole attacks [3] and for mitigating the broadcast storm problem [4].

The problem studied in this paper concerns how to maintain connectivity in a network with directional antennae with a given angular spread while achieving the minimum possible transmission range. To the best of our knowledge the problem explored here has not been proposed in the literature before.

Outline and results of the paper. In this paper, we provide a first set of results for the connectivity problem in wireless networks using directional antennae. We present simple polynomial time algorithms for the linear case and the 2-dimensional case when the sector angle of the antennae is large (i.e., at least $8\pi/5$). For smaller sector angles, we present algorithms that approximate the minimum radius. When the sector angle is smaller than $2\pi/3$, we show that the problem of determining the minimum radius in order to achieve connectivity is NP-hard.

The rest of the paper is structured as follows. The two polynomial-time solvable instances are presented in Section 2. Our NP-completeness result is proved in Section 3. We present the approximation algorithm in Section 4. We conclude with interesting extensions and open problems in Section 5.

2 Polynomial-time solvable cases

First we consider the simpler linear case whereby all the sensors are located on a straight line. Observe that when X consists of points on a line and $\phi \geq \pi$, there exists an orientation of sectors of angle ϕ and radius $r(MST(X))$ at each point p in such a way that the sector of each point covers both the left and the right closest point (if any) to p. The next theorem gives a better bound for $0 \leq \phi \leq \pi$ and its proof can be found in the Appendix.

Theorem 1 *Consider a set of $n > 2$ points $i = 1, 2, \ldots, n$ sorted according to their location on the line. For any $\phi \geq 0$ and $r > 0$, there exists an orientation of sectors of angle ϕ and radius r at the points so that the transmission graph is strongly connected if and only if the distance between points i and $i + 2$ is at most r, for any $i = 1, 2, \ldots, n - 2$.*

An important ingredient of the construction is related to the minimum spanning tree $MST(X)$ of the set of points X. The following theorem turns out to be easy to prove and its proof can be found in the Appendix.

Theorem 2 *Given $\phi \geq 8\pi/5$, $r > 0$ and a set of points on the plane, an orientation of sectors of angle ϕ and radius r so that the transmission graph is strongly connected can be computed (if it exists) in polynomial time.*

3 The complexity of the 2-dimensional case

In this section we prove that the problem on the 2-dimensional case is NP-complete for sector angles smaller than $2\pi/3$. A weaker statement for sector angles smaller than $\pi/2$ follows by the same reduction used by Itai et al. [6] in order to prove that the hamilton circuit problem in grid graphs is NP-complete. Consider an instance of the problem consisting of points with integer coordinates on the euclidean plane (these can be thought of as the nodes of the grid proximity graph between them). Then, if there exists an orientation of sector angles of radius 1 and angle $\phi < \pi/2$ at the nodes so that the corresponding transmission graph is strongly connected, then this must also be a hamilton circuit of the proximity graph. The construction of [6] can be thought of as reducing the hamilton circuit problem on bipartite planar graphs of maximum degree 3 (which is proved in [6] to be NP-complete) to an instance of the problem with a grid graph as a proximity graph such that there exists a hamilton circuit in the grid graph if and only if the original graph has a hamilton circuit. Below we use a slightly more involved reduction with different gadgets in order to show that the problem is NP-complete for sector angles smaller than $2\pi/3$. In particular, we prove the following statement.

Theorem 3 *For any constant $\epsilon > 0$, given ϕ such that $0 \leq \phi < 2\pi/3 - \epsilon$, $r > 0$, and a set of points on the plane, determining whether there exists an orientation of sectors of angle ϕ and radius r so that the transmission graph is strongly connected is NP-complete.*

In order to prove Theorem 3 we will show that the hamilton circuit problem in a special class of (bipartite) planar graphs of degree 3 having a particular embedding on the euclidean plane; these graphs are called ϵ-hexagon graphs.

Definition 4 *Let $\epsilon > 0$. An ϵ-hexagon graph $G = (V, E)$ is a bipartite planar graph of maximum degree 3 which has an embedding on the plane with the following properties:*

- *Each node of the graph corresponds to a point in the plane.*
- *The euclidean distance between the points corresponding to two nodes v_1, v_2 of G is in $[1 - \epsilon, 1]$ if $(v_1, v_2) \in E$ and larger than $\sqrt{3} - 3\epsilon$ otherwise.*
- *The angle between any two line segments corresponding to edges adjacent to the same node of G is at least $2\pi/3 - \epsilon/2$.*

Clearly, an ϵ-hexagon graph is the proximity graph for an instance of the problem and any orientation of sector of radius 1 and angle $\phi = 2\pi/3 - \epsilon$ that induces a strongly connected transmission graph actually corresponds to a hamiltonian circuit of the proximity graph. The opposite also holds, i.e., given a hamilton circuit in the proximity graph, there exist (two) orientations of sectors of radius 1 and angle $2\pi/3 - \epsilon$ that induce the hamilton circuit as a transmission graph (with the two possible opposite directions). Hence, in order to prove Theorem 3, it suffices to prove that the hamilton circuit problem in ϵ-hexagon graphs is NP-complete.

Theorem 5 *For any constant $\epsilon > 0$, the hamilton circuit problem in ϵ-hexagon graphs is NP-complete.*

Proof. We will use a reduction from the hamilton circuit problem on bipartite planar graphs of maximum degree 3 which is known to be NP-complete [6]. Given such a graph $G = (V_0, V_1, E)$ with n nodes, we will construct an ϵ-hexagon graph H (together with its embedding) which has a hamilton circuit if and only if G has a hamilton circuit.

Edges of G are simulated by *necklaces* in H. The main building block of a necklace is a *cell*, i.e., a 6-node cycle with nodes labeled 0, 1, 2, 3, 4, and 5. A necklace consists of consecutive cells so that nodes 4 and 5 of the i-th cell coincides with nodes 0 and 1 of the $(i+1)$-th cell, respectively. We note that the node labels in consecutive cells have opposite (clockwise or counter-clockwise) orders. The endpoints of a necklace with k cells are nodes 0 and 1 of the first cell and node 5 of the last cell (see Figure 1). Observe that the only hamilton path of a necklace ending at node 5 of its last cell originates from node 0 of its first cell. We call this a *cross path*. Also, the only hamilton path of a necklace ending at node 1 of its first cell originates from node 0 of the first cell. Such a path is called a *return path*.

The representation of a necklace with regular hexagons as cells (with nodes corresponding to the corners of the hexagons) has a particular orientation (see Figure 1). We use regular hexagons of side 1 to represent the first and the last cell of a necklace and irregular hexagons with sides of length in $[1-\epsilon, 1]$ and angle between adjacent sides in $[2\pi/3 - \epsilon/2, 2\pi/3 + \epsilon/4]$, to represent the intermediate cells of a necklace. In this way, we can implement turns of necklaces and achieve different orientations provided that there is enough space for them (see Figure 1). We also note that the distance between points corresponding to non-adjacent nodes is more than $\sqrt{3} - 3\epsilon$.

Each node of G is simulated by a *diamond*, i.e., by the 13-node construction of Figure 2 consisting of three mutually adjacent regular hexagons of side 1. The three nodes p_1, p_2, and p_3 as well as their incident edges e_1, e_2, and e_3 are used to connect the diamond to the necklaces corresponding to edges of G incident to the node of G corresponding to the diamond. Observe that any hamilton path between node p_i to p_j (with $1 \leq i \neq j \leq 3$) crosses all the three edges e_1, e_2, and e_3.

The connection of edges to nodes of G are simulated by connecting necklaces to diamonds. The special edges of diamonds corresponding to nodes of V_0 are used to connect to necklaces (the different necklaces that are connected to the same diamond use different special edges). The edge between nodes 0 and 1 of a necklace coincides with edge e_i so that node p_i coincides in with 0 (see Figure 2). Similarly, the special nodes of diamonds corresponding to nodes of V_1 are used to connect to necklaces (again, the necklaces that are connected to the same diamond use different special nodes). Node 5 of the last cell of a necklace is located at distance 1 from node p_i of the diamond so that the angle between the edge between p_i and node 5 of

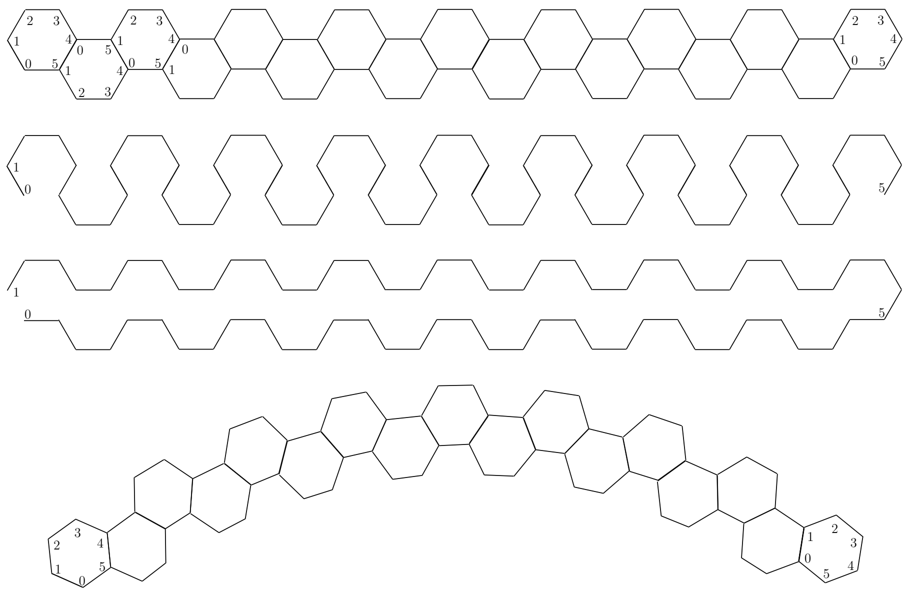

Figure 1: The first figure shows the fixed orientation of a necklace represented with regular hexagons of side 1. The second and third figures show the cross and return path, respectively. The fourth figure shows the representation of the necklace using irregular hexagons of sides between 0.95 and 1 and with angles between sides from 115^o to 125^o.

the last cell of the necklace and any of its four adjacent edges is exactly $2\pi/3$ (see Figure 2). We also note that since each necklace has an odd number of cells, the order of the node labels in the first and the last cell of each necklace is the same. Hence, the points of the cell of the necklace attached to the last one (and any other point of the necklace) will be at distance more than $\sqrt{3} - 3\epsilon$ from any point of a diamond (see the dotted lines in Figure 2).

Now, in order to embed the whole graph on the euclidean plane, we first use an embedding of G on the rectangular grid $n \times n$. Such constructions are well-known in the literature. The nodes of G are embedded on points of the grid and edges correspond to vertex-disjoint paths on the grid. Each grid point can correspond to a node of V_0, a node of V_1, a edge corner, an horizontal segment of an edge, or a vertical segment of an edge, or not used at all. Now, we use a square of the Euclidean plane with side $\kappa_\epsilon n$ where κ_ϵ is a constant depending on ϵ. This square is divided into n^2 squares of side κ_ϵ, one square for each grid point. Squares corresponding to grid points serving as edge corners, horizontal or vertical segments of edges of G host segments of necklaces crossing the square. Squares corresponding to grid points corresponding to nodes of V_0 host a diamond and the beginning of necklaces connected to it while squares corresponding to grid points corresponding to nodes of V_0 host a diamond and the end of necklaces connected to it. Squares corresponding to unused grid points are empty. See Figure 3 for an example.

We have completed the description of the construction. In order to complete the proof, first observe that given a hamilton circuit in G, we can construct a hamilton circuit in H as follows. Edges of G which are included in the hamilton circuit of G are covered by a cross path in the corresponding necklace in H. Edges of G which are not included in the hamilton circuit of G are covered by a return path in the corresponding necklace in H. In this way, we have

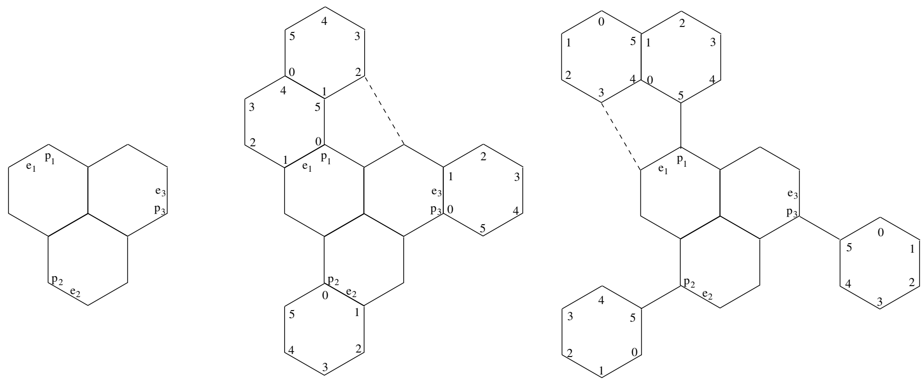

Figure 2: A diamond (left) and its connection to necklaces when it corresponds to a node of V_0 (middle) or V_1 (right).

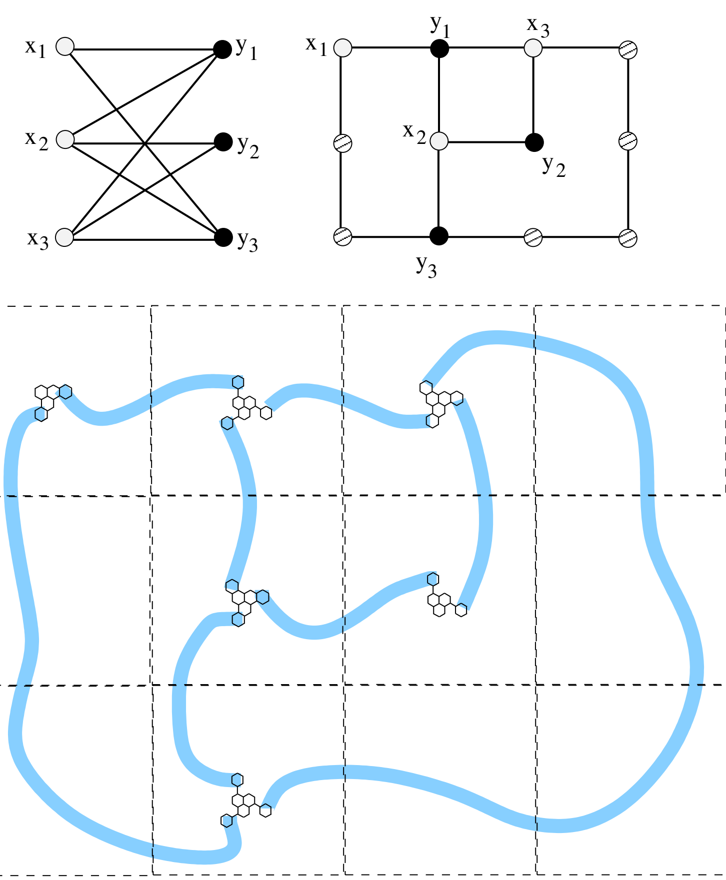

Figure 3: A bipartite planar graph of maximum degree 3, its embedding on the rectangular grid, and the corresponding ϵ-hexagon graph. Each thick line represents a necklace.

constructed the part of a hamilton circuit which enters each diamond of H through two cross paths of necklaces connected to it. It remains to complete the hamilton circuit by covering all the nodes of the diamond. The reader may see Figure 2 again in order to be convinced by the following argument. If a diamond corresponds to a node of V_0, assume without loss of generality that the two necklaces whose nodes are covered by cross paths are connected to the special edges e_1 and e_2 and there is possibly a return path on the third necklace starting at node 0 and ending at node 1 of its first cell (which are the endpoints of e_2). In order to complete the hamilton circuit and cover all nodes of the diamond, we use the hamilton path from node p_1 to node p_3 which starts with the special edge e_1 (and, in this way, is connected to the hamilton path that covers the nodes of the necklace connected to the special edge e_1) and ends with the special edge e_3 (and, in this way, is connected to the hamilton path that covers the nodes of the necklace connected to the special edge e_3). We replace edge e_2 with the return path in the necklace (if any) connected to edge e_2 of the diamond. If a diamond corresponds to a node in V_1, assume assume without loss of generality that nodes 5 of the two necklaces connected to nodes p_1 and p_2 are the endpoints of the corresponding cross paths. Then, in order to complete the hamilton circuit and cover the nodes of the diamond we use the hamilton path covering the nodes of the diamond that starts with p_1 and ends with p_2 and the edges connecting p_1 and p_2 to nodes 5 of the last cells of the two necklaces adjacent to them.

By our construction, in any hamilton circuit in H, for each diamond, the nodes of exactly two necklaces connected to it are covered by cross paths and the nodes of the third necklace (if any) are covered by a return path. Then, the tour obtained by the edges of G corresponding to necklaces of H covered by cross paths in the hamilton circuit of H is a hamilton circuit in G. ■

Recall that Definition 4 states that the distance between any two points corresponding to non-adjacent nodes of an ϵ-hexagon graph is larger than $\sqrt{3} - 3\epsilon$. Hence, Theorem 5 also implies the following statement.

Corollary 6 *For any constant $\epsilon > 0$, given ϕ such that $0 \leq \phi < 2\pi/3 - \epsilon$, and a set of points X on the plane, determining whether there exists an orientation of sectors of angle ϕ and radius $(\sqrt{3} - 3\epsilon)r^*(X, \phi)$ so that the transmission graph is strongly connected is NP-complete.*

4 Approximating the minimum radius

In this section we present algorithms that uses sectors of slightly larger radius than the optimal one. In particular, we prove the following theorem.

Theorem 7 *Given an angle ϕ with $\pi \leq \phi < 8\pi/5$ and a set of points in the plane, there exists a polynomial time algorithm that computes an orientation of sectors of angle ϕ and radius $2\sin\left(\pi - \frac{\phi}{2}\right) \cdot r^*(X, \phi)$ so that the transmission graph is strongly connected.*

Proof. Consider a set X of nodes on the Euclidean plane and let T be a minimum spanning tree of X. Let $r = r(MST(X))$ be the longest edge of T. We will use sectors of angle ϕ and radius $d(\phi) = 2r\sin\left(\pi - \frac{\phi}{2}\right)$ and we will show how to orient them so that the transmission graph induced is a strongly connected subgraph over X. The theorem will then follow since r is a lower bound on $r^*(X, \phi)$.

We first construct a matching M consisting of (mutually non-adjacent) edges of T with the following additional property: any non-leaf node of T is adjacent to an edge of M. This can be done as follows. Initially, M is empty. We root T at an arbitrary node s. We pick an edge between s and one of its children and insert it in M. Then, we visit the remaining nodes of T in a BFS manner. When visiting a node u, if u is either a leaf-node or a non-leaf node such that the edge between it and its parent is in M, we do nothing. Otherwise, we pick an edge between u and one of its children and insert it to M.

We denote by Λ the leaves of T which are not adjacent to edges of M. We also say that the endpoints of an edge in M form a *couple*. We use sectors of angle ϕ and radius $d(\phi)$ at each point and orient them as follows. At each node $u \in \Lambda$, the sector is oriented so that it induces the directed edge from u to its parent in T in the corresponding transmission graph G. For each two points u and v forming a couple, we orient the sector at u so that it contains all points p at distance $d(\phi)$ from u for which the counter-clockwise angle $v\hat{u}p$ is in $[0, \phi]$. See Figure 4.

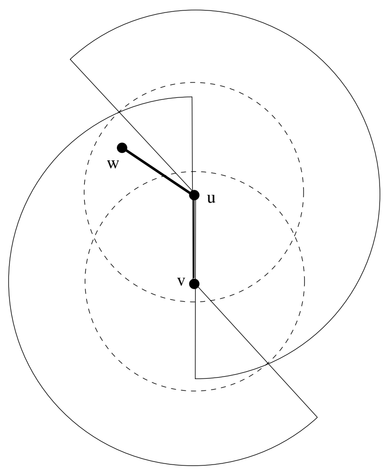

Figure 4: The orientation of sectors at two nodes u, v forming a couple, and a neighbor w of u that is not contained in the sector of u. The dashed circles have radius r and denote the range in which the neighbors of u and v lie.

We first show that the transmission graph G defined in this way has the following property (P): for each two points u and v forming a couple, G contains the two opposite directed edges between u and v, and, for each neighbor w of either u or v in T, it contains a directed edge from either u or v to w. Consider a point w corresponding to a neighbor of u in T (the argument for the case where w is a neighbor of v is symmetric). Clearly, w is at distance $|uw| \leq r$ from u. Also, note that since $\phi < 8\pi/5$, we have that the radius of the sectors is $d(\phi) = 2r \sin\left(\pi - \frac{\phi}{2}\right) \geq 2r \sin\frac{\pi}{5} > 2r \sin\frac{\pi}{6} = r$. Hence, w is contained in the sector of u if the counter-clockwise angle $v\hat{u}w$ is at most ϕ; in this case, the graph G contains a directed edge from u to w. Now, assume that the angle $v\hat{u}w$ is $x > \phi$ (see Figure 4). By the law of cosines in the triangle defined by points u, v, and w, we have that

$$\begin{aligned} |vw| &= \sqrt{|uw|^2 + |uv|^2 - 2|uw||uv|\cos x} \\ &\leq r\sqrt{2 - 2\cos x} \\ &= 2r \sin \frac{x}{2} \\ &\leq 2r \sin\left(\pi - \frac{\phi}{2}\right) \\ &= d(\phi). \end{aligned}$$

Since the counter-clockwise angle $v\hat{u}w$ is at least π, the counter-clockwise angle $u\hat{v}w$ is at most $\pi \leq \phi$ and, hence, w is contained in the sector of v; in this case, the graph G contains a directed edge from v to w. In order to complete the proof of property (P), observe that since $|uv| \leq r \leq d(\phi)$ the point v is contained in the sector of u (and vice-versa).

Now, in order to complete the proof of the theorem, we will show that for any two neighbors u and v in T, there exist a directed path from u to v and a directed path from v to u in G. Without loss of generality, assume that u is closer to the root s of T than v. If the edge between u and v belongs in M (i.e., u and v form a couple), property (P) guarantees that there exist two opposite directed edges between u and v in the transmission graph G. Otherwise, let w_1 be the node with which u forms a couple. Since v is a neighbor of u in T, there is either a directed edge from u to v in G or a directed edge from w_1 to v in G. Then, there is also a directed edge from u to w_1 in G which means that there exists a directed path from u to v. If v is a leaf (i.e., it belongs to Λ), then its sector is oriented so that it induces a directed edge to its parent u. Otherwise, let w_2 be the node with which v forms a couple. Since u is a neighbor of v in T, there is either a directed edge from v to u in G or a directed edge from w_2 to u in G. Then, there is also a directed edge from v to w_2 in G which means that there exists a directed path from v to u. ■

As corollaries we obtain approximation ratios 2, $\sqrt{3}$, $\sqrt{2}$, and approaching $2\sin\frac{\pi}{5} \approx 1.1756$ from below for angles π, $4\pi/3$, $3\pi/2$, and slightly smaller than $8\pi/5$, respectively.

A 3-approximation algorithm of the minimum radius for any angle $\phi \geq 0$ follows by a folklore result that any n-node ring can be embedded on a n-node connected tree with dilation 3. This result is attributed to Sekanina [16] in [10] (Theorem 3.15, page 470). Hence, given the minimum spanning tree among a set of points X on the plane, we can compute in polynomial time a hamilton tour over the points in X such that the distance between any two consecutive points in the tour is at most $3r(MST(X))$. We obtain the following corollary.

Corollary 8 *Given an angle $\phi \geq 0$ and a set of points in the plane, there exists a polynomial time algorithm that computes an orientation of sectors of angle ϕ and radius $3 \cdot r^*(X, \phi)$ so that the transmission graph is strongly connected.*

We note that the results of Theorem 7 cannot be significantly improved by using only $r(MST(X))$ as a lower bound for $r^*(X, \phi)$. Figures 5a, 5b, 5c, and 5d show examples where $r^*(X, \phi)$ is at least 2, $\sqrt{3}$, $\sqrt{2}$, and $2\sin\frac{\pi}{5} \approx 1.1756$ times $r(MST(X))$ for sector angles ϕ smaller than π, $4\pi/3$, $3\pi/2$, and $8\pi/5$, respectively. For angles smaller than $\pi/3$, Figure 5e shows an example where $r^*(X, \phi) = \sqrt{7} \cdot r(MST(X))$.

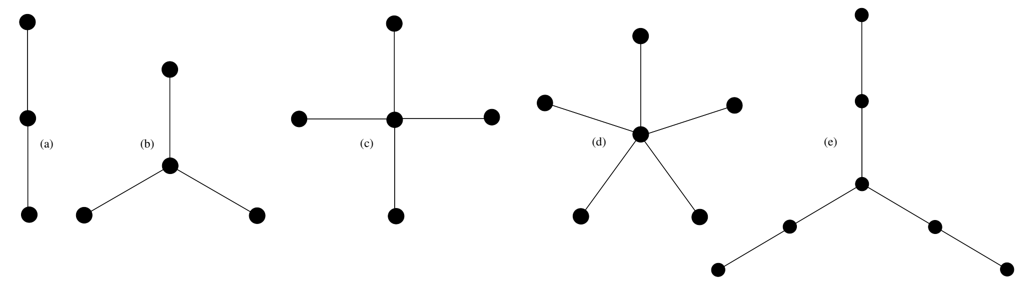

Figure 5: Examples indicating that our results cannot be significantly improved by using $r(MST(X))$ as lower bound for $r^*(X, \phi)$. The figures show the minimum spanning tree among sets of points and all edges are equal to 1. The angles between edges incident to the central point are π, $2\pi/3$, $\pi/2$, and $2\pi/5$ in (a), (b), (c), and (d), respectively. Hence, using sectors of angle ϕ smaller than π, $4\pi/3$, $3\pi/2$, and $8\pi/5$, respectively, at the central node at least one of the leaves must be reached directly by another leave. Such a transmission requires a sector of radius at least $2\sin\left(\pi - \frac{\phi}{2}\right)$. In (e), the angle between two edges incident to the central node is $2\pi/3$ and any other angle is π. It is not difficult to see that if sectors have angles smaller than $\pi/3$ (i.e., if no node adjacent to the central one can transmit to the other two nodes adjacent to the central one), then transmission from some leaf that reaches a point which is 3 hops away (i.e., at distance $\sqrt{7}$) is necessary.

5 Extensions and open problems

Our work has revealed several interesting open questions: The most challenging one is to determine the threshold on the sector angle above which the connectivity problem can be solved in polynomial time and below which the problem is NP-complete. Our results indicate that this threshold is between $2\pi/3$ and $8\pi/5$.

Designing more efficient algorithms for approximating (or even achieving) the minimum radius is an interesting problem as well. The approximation algorithms presented in Section 4 use $r(MST(X))$ as lower bound on the minimum radius. Although there is a small gap between 3 and $\sqrt{7}$ for the case of small sector angle (see Corollary 8 and the example in Figure 5e), improving the approximation ratios further will require new techniques since there are instances where our results are tight (see Figure 5).

We have considered the case where there is a single antenna per node. We may define interesting combinatorial problems by having more antennae per node. The antennae may have the same sector angles or the sum of the sector angles of the antennae of each node is fixed. It is interesting to investigate the complexity of these variants of the problem.

The related problem of broadcasting from a single node to any other node of the network also deserves investigation. The problem can be shown to be NP-complete for sector angle smaller than $2\pi/3$ using a proof similar to that of Theorem 5 (and using a reduction from the hamilton path problem instead of hamilton circuit) while the problem can be proved to be solvable in polynomial time for sector angles at least $4\pi/3$ (using a slightly more involved argument than that used in the proof of Theorem 2). Again, algorithms that approximate the minimum radius sufficient to perform broadcasting can be designed using minimum spanning trees like those presented in Theorem 7 for the connectivity problem.

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Appendix

Proof. (of Theorem 1) Assume $d(x_i, x_{i+2}) > r$, for some $i \leq n-2$. Consider the antenna at x_{i+1}. There are two cases to consider. First, if the antenna at x_{i+1} is directed to the left then the portion of the graph to its left cannot be connected to the portion of the graph to the right; second, if the antenna at x_{i+1} is directed to the right then the portion of the graph to its right cannot be connected to the portion of the graph to the left. In either case the graph becomes disconnected. Conversely, assume $d(x_i, x_{i+2}) \leq r$, for all $i \leq n-2$. Consider the following antenna orientation:

1. antennas $x_1, x_3, x_5, \ldots$ labeled with odd integers are oriented right, and
2. anntennas $x_2, x_4, x_6, \ldots$ labeled with even integers are oriented left.

It is easy to see that the resulting orientation leads to a strongly connected graph. This completes the proof of Theorem 1. ■

Proof. (of Theorem 2) If the proximity graph is not connected, then clearly no orientation of the sectors that defines a strongly connected transmission graph can be found. If the proximity graph is connected, consider a minimum spanning tree on it. Since the edge costs are Euclidean, each node on this spanning tree has degree at most 5 (it may also have 6 but in this case we can transform it to a spanning tree of the same cost and degree 5). Hence, for each node u, there are two consecutive neighbors v, w in the spanning tree so that the angle $\angle(vuw)$ is at least $2\pi/5$. Hence, by using sectors of angle $8\pi/5$ at each node and by orienting it so that it covers all neighbors in the spanning tree, we obtain a transmission graph that contains two opposite directed edges per undirected) edge of the spanning tree and thus, it is strongly connected. This completes the proof of Theorem 2. ■