Network Connectivity

Main Question

- Given a set of sensors with omnidirectional antennae forming a connected network:

  Question: How can omnidirectional antennae be replaced with directional antennae in such a way that the connectivity is maintained while the angle and range being used are the smallest possible?

Outline

- Motivation
- Orientation Problem
  - In 1D.
  - In 2D.
    * Complexity.
    * Optimal Range Orientation.
    * Approximate Range Orientation.
  - In 3D.
    * Complexity.
    * Optimal Range Orientation.
    * Approximate Range Orientation.
- Variations of the Antenna Orientation Problem.

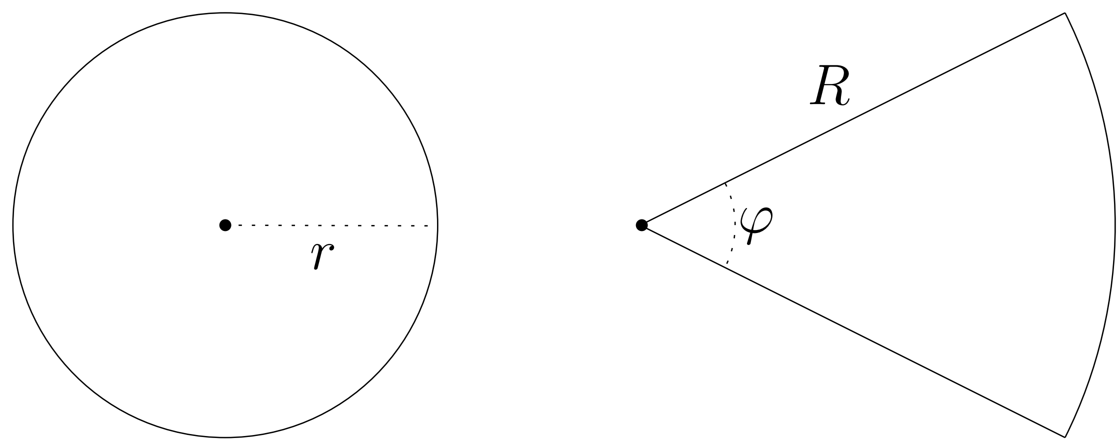
Motivation

Reasons for Replacing Antennae

- Energy Consumption
- Network Capacity

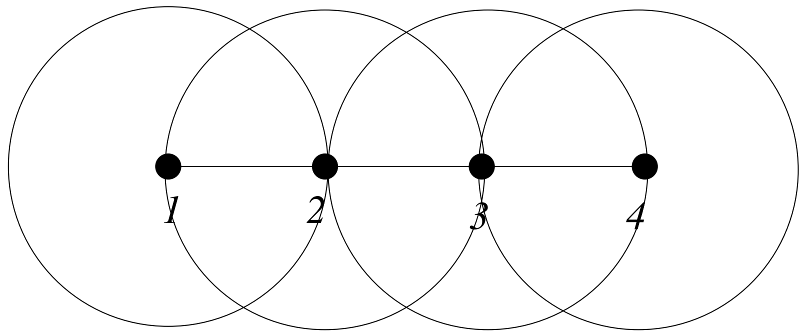
Energy

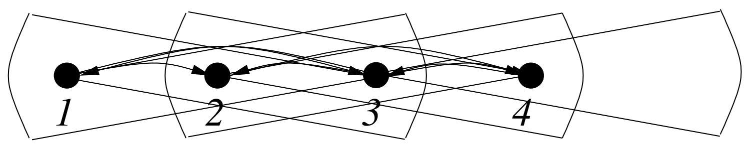
- The energy necessary to transmit a message is proportional to the coverage area.
  - An omnidirectional antenna with range r consumes energy proportional to πr^2.
  - A directional antenna with angle φ and range R consumes energy proportional to $\varphi R^2/2$.



Connectivity

- With the same amount of energy, a directional antenna with angle α can reach further.





Capacity of Wireless Networks

- Consider a set of sensors that transmit W bits per second with antennae having transmission beam of width α and a receiving beam width of angle β.

Sender	**Receiver**	
	Omnidirectional	Directional (β)
Omnidirectional	$\sqrt{\frac{1}{2\pi}}W\sqrt{n}$ [1]	-
Directional (α)	$\sqrt{\frac{1}{\alpha}}W\sqrt{n}$ [2]	$\sqrt{\frac{2\pi}{\alpha\beta}}W\sqrt{n}$ [2]

- **References:**

1. *Gupta and Kumar*. The capacity of wireless networks. 2000.
2. *Yi, Pei and Kalyanaraman*. On the capacity improvement of ad hoc wireless networks using directional antennas. 2003.

Capacity with Directional Antennae

- Consider a set of sensors that transmit W bits per second with antennae having transmission beam of width α and a receiving beam width of angle β.
- Assume that
  - sensors are placed in such a way that the interference is minimum, and
  - traffic patterns and transmission ranges are optimally chosen.
- Then the network capacity (amount of traffic that the network can handle) is at most $\sqrt{\frac{2\pi}{\alpha\beta}} W \sqrt{n}$ per second.

Enhancing Security with Directional Antennae

- The use of directional antennae enhances the network security since the radiation is more restricted.
  - Hu and Evans[a] designed several authentication protocols based on directional antennae.
  - Lu et al[b] employed the average probability of detection to estimate the overall security benefit level of directional transmission over the omnidirectional one.
  - Imai et al[c] examined the possibility of key agreement using variable directional antennae.

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[a] *Hu and Evans.* Using directional antennas to prevent wormhole attacks. 2004

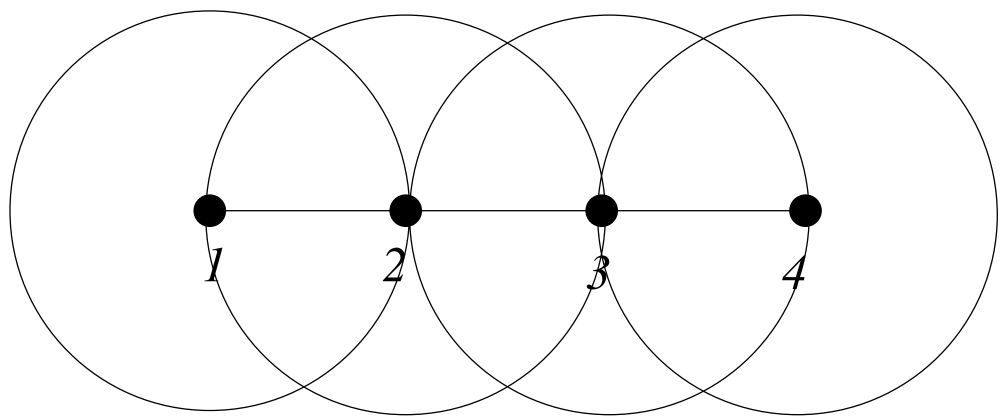
[b] *Lu, Wicker, Lio, and Towsley.* Security Estimation Model with Directional Antennas. 2008

[c] *Imai, Kobara, and Morozov.* On the possibility of key agreement using variable directional antenna. 2006

Antenna Orientation Problem in the Line

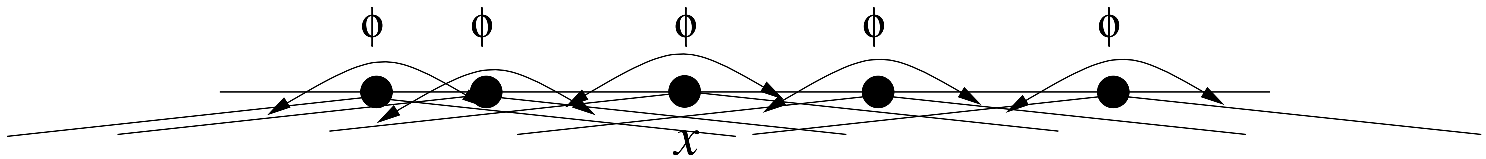
Antenna Orientation Problem in the Line

- Given a set of sensors in the line equipped with one directional antennae each of angle at most $\varphi \geq 0$.
- Compute the minimum range r required to form a strongly connected network by appropriately rotating the antennae.

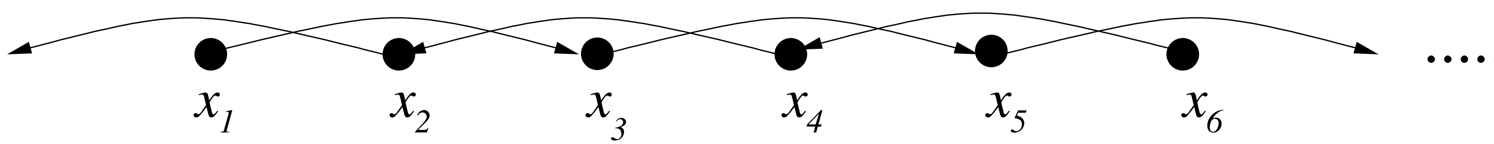


Antenna Orientation Problem in the Line

- Given $\varphi \geq \pi$. The orientation can be done trivially with the same range required when omnidirectional antennae are used.



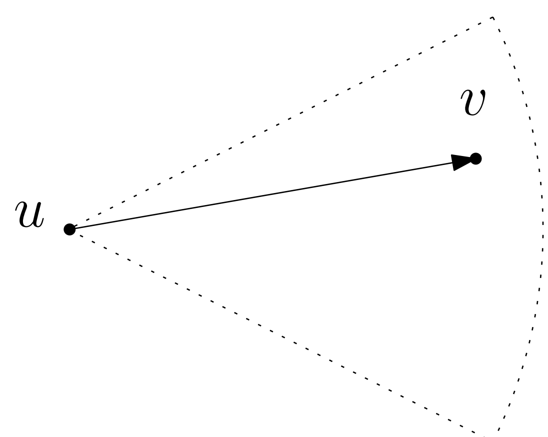
- Given $\varphi < \pi$. The strong orientation can be done with range bounded by two times the range required when omnidirectional antennae are used.



Antenna Orientation Problem in the Plane

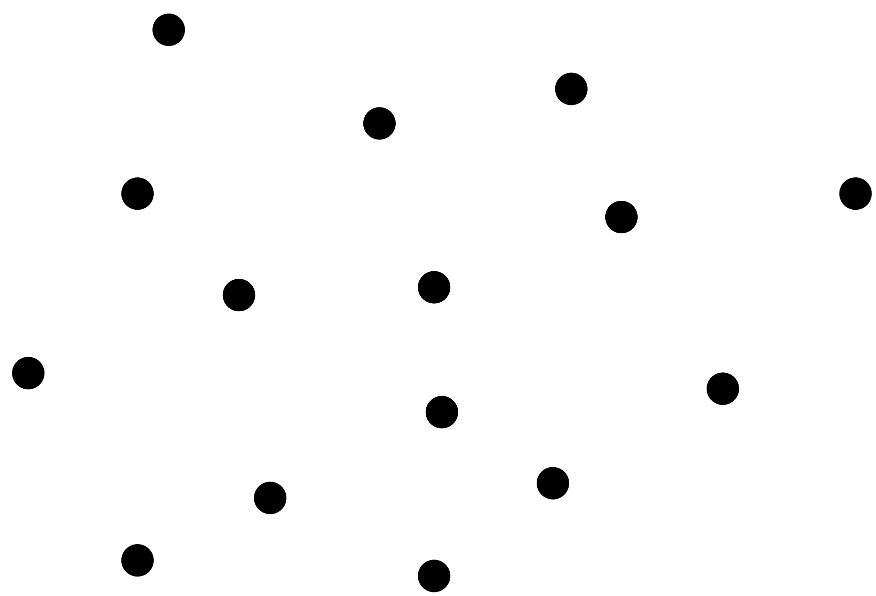
Antenna Orientation Problem

- Given a set of **identical** sensors in the plane equipped with one directional antenna each of angle at most φ.
- Compute the minimum range such that by appropriately rotating the antennae, a directed, strongly connected network on S is formed.

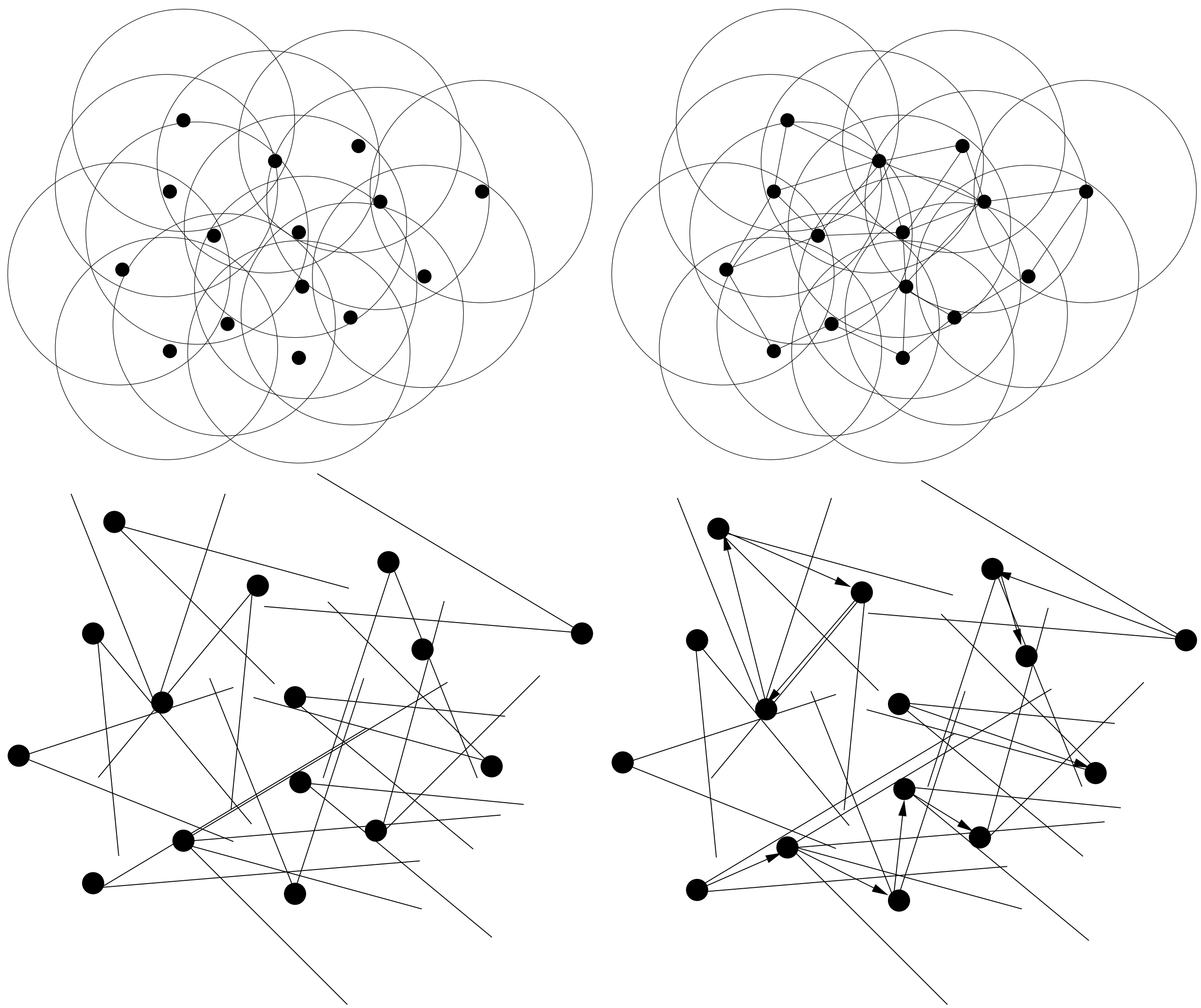


Example: Sensors in the Plane

Consider n sensors in the plane.



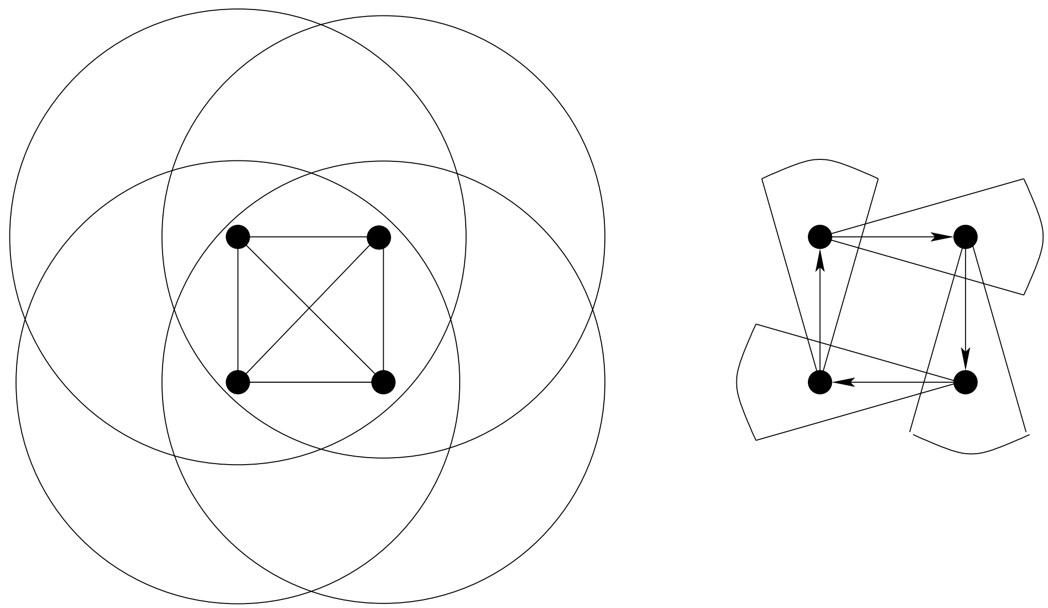
Example: Directional Antennae Affect Connectivity



Connectivity Issues

- When replacing omnidirectional with directional antennae the network topology changes!
- How do you maintain connectivity in a wireless network when the network nodes are equipped with directional antennae?
- Nodes correspond to points on the plane and each uses a directional antenna (modeled by a sector with a given angle and radius).
- The connectivity problem is to decide whether or not it is possible to orient the antennae so that the directed graph induced by the node transmissions is strongly connected.

Four sensors: Connectivity Example



Left: using omnidirectional antennae they form an underlying complete network on four nodes.

Right: using directional antennae they form an underlying cycle on four nodes.

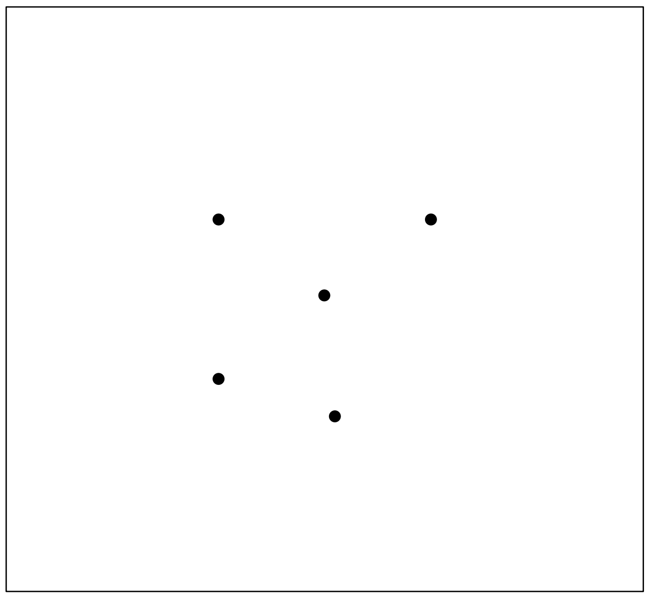
Connectivity Problem

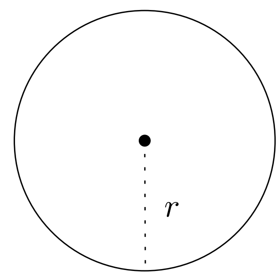
- We consider the problem of maintaining connectivity using the minimum possible range for a given angular spread.
- More specifically,

  For a set of sensors located in the plane at established positions and with a given angular spread we are interested in providing an algorithm that minimizes the range required so that by an appropriate rotation of each of the antennae the resulting network becomes strongly connected.

Antenna Orientation Problem: Distances

- Given n (identical) sensors in the plane with omnidirectional antennae, the optimal range can be computed in polynomial time.

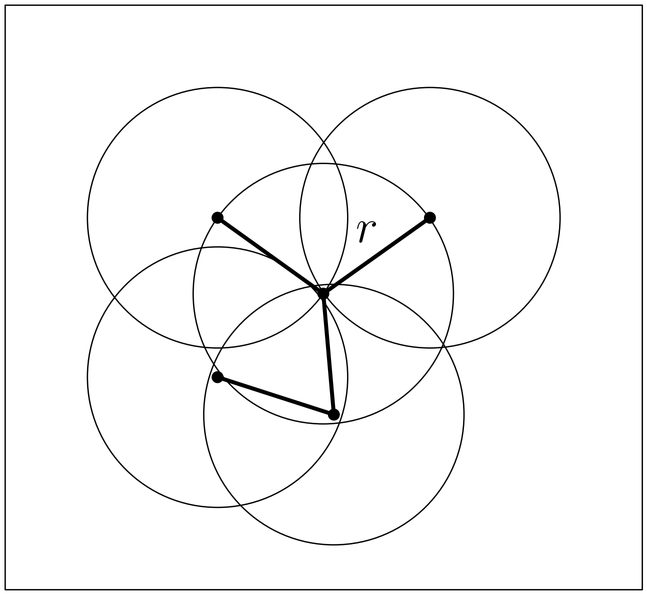


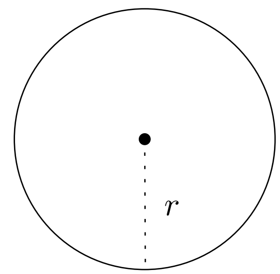


- Why?
- Try all possible (at most n^2) distances.

Antenna Orientation Problem: MST

- The sensors already form an omnidirectional network.

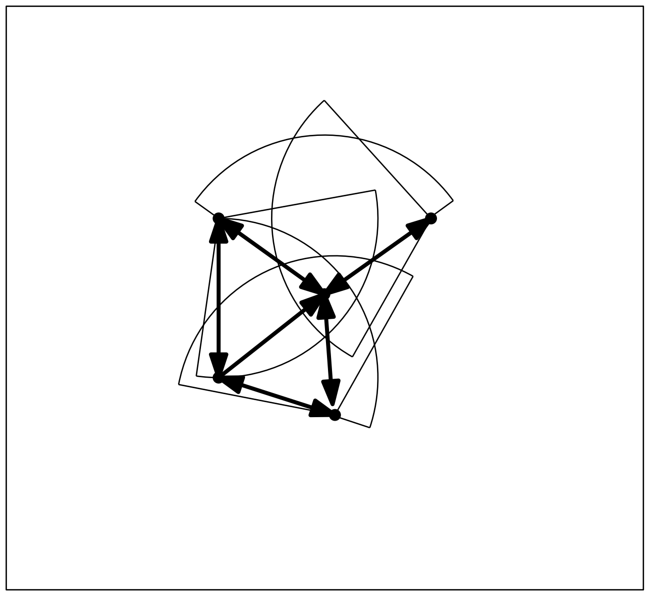


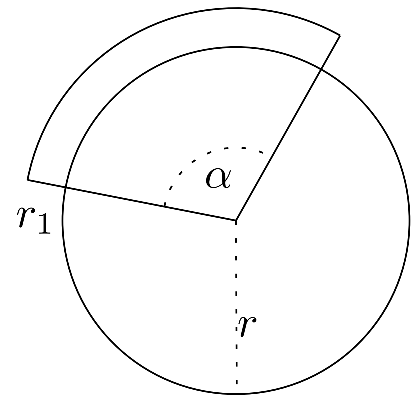


- Actually, the longest edge of the MST is the optimal range.
- Why?

Antenna Orientation Problem: Angle (1/2)

- Given a directional antenna with angle α.

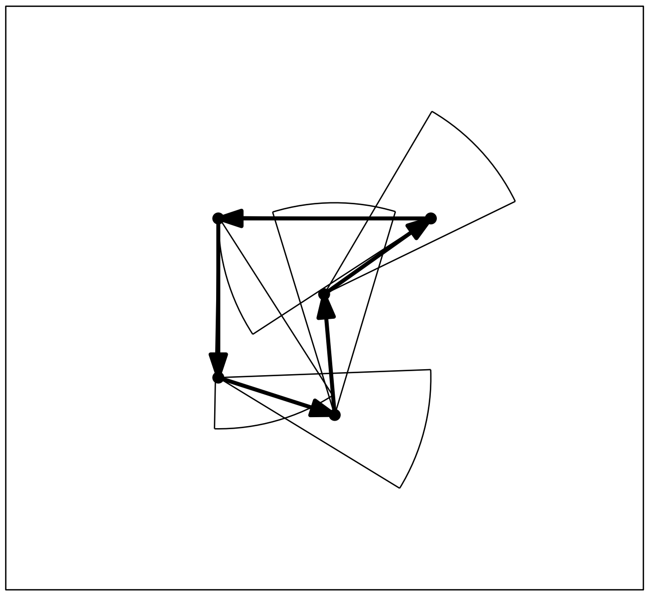


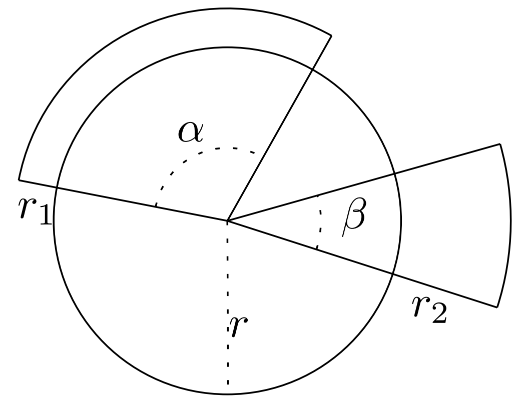


- What is the minimum radius r_1 to create a strongly connected network?

Antenna Orientation Problem: Angle (2/2)

- Given a directional antenna with angle β.



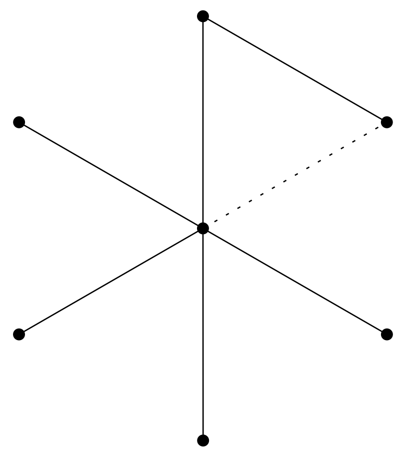


- What is the minimum radius r_2 to create a strongly connected network?

Upper Bound

Optimal Range Orientation (1/3)

- What is the minimum angle necessary to create a strongly connected network if the range of the directional antennae is the same as the omnidirectional antenna?
- Consider an MST T on the set of points.



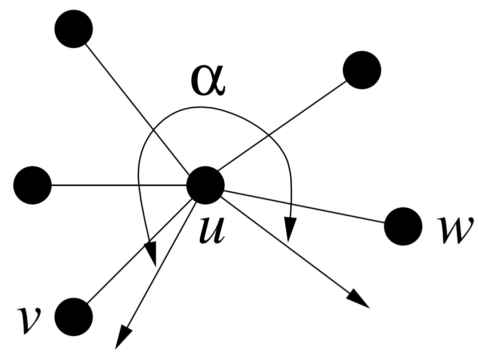
- If the maximum degree of T is 6, by a simple argument we can find an MST with the same weight and maximum degree 5.

Optimal Range Orientation (2/3)

- **If the proximity graph is not connected**, then clearly no orientation of the sectors that defines a strongly connected transmission graph can be found.
- **If the proximity graph is connected**, consider a MST.
- Since the edge costs are Euclidean, each node on this spanning tree has degree at most 5.

Optimal Range Orientation (3/3)

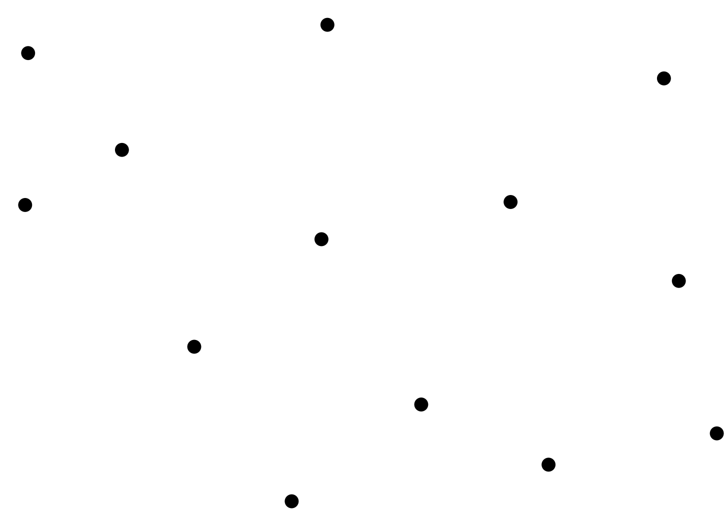
- For each node u, there are two consecutive neighbors v, w in the spanning tree so that the angle $\angle(vuw)$ is at least $2\pi/5$.



- **Theorem 2.** There exists an orientation of the directional antennae with optimal range when the angles of the antennae are at least $8\pi/5$.

Antenna Orientation With Approximation Range

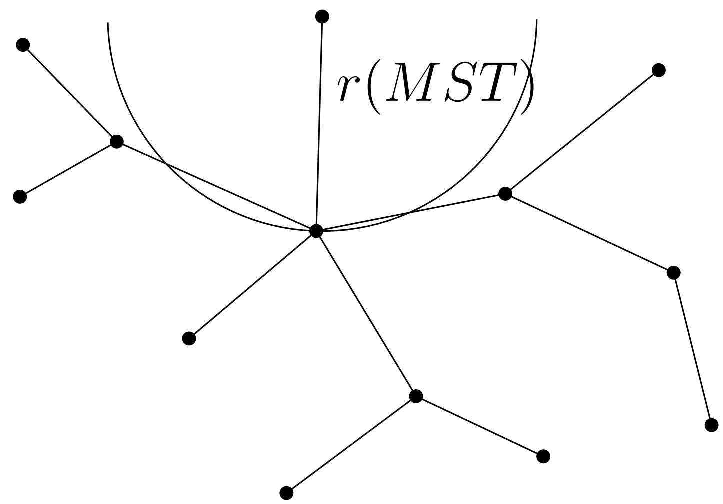
- **Theorem 3.** (Caragiannis et al[a].) There exists a polynomial time algorithm that given an angle φ with $\pi \leq \varphi < 8\pi/5$ and a set of points in the plane, computes a strong orientation with radius bounded by $2\sin(\varphi/2)$ times the optimal range.



[a] *Caragiannis, Kaklamanis,Kranakis, Krizanc and Wiese.* Communication in Wireless Networks with Directional Antennae. 2008

Proof (1/10)

- Consider a Minimum Spanning Tree on the Set of Points.

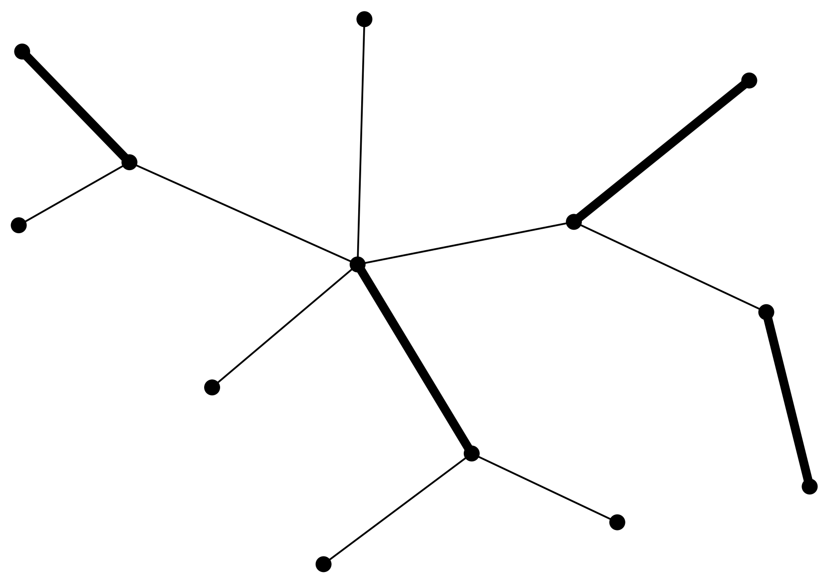


Proof (2/10)

- Let $r^*(\varphi)$ be the optimal range when the angle of the antennae is at most φ.
- Let $r(MST)$ be the longest edge of the MST on the set of points.
- Observe that for $\varphi \geq 0$, $r^*(\varphi) \geq r(MST)$.

Proof (3/10)

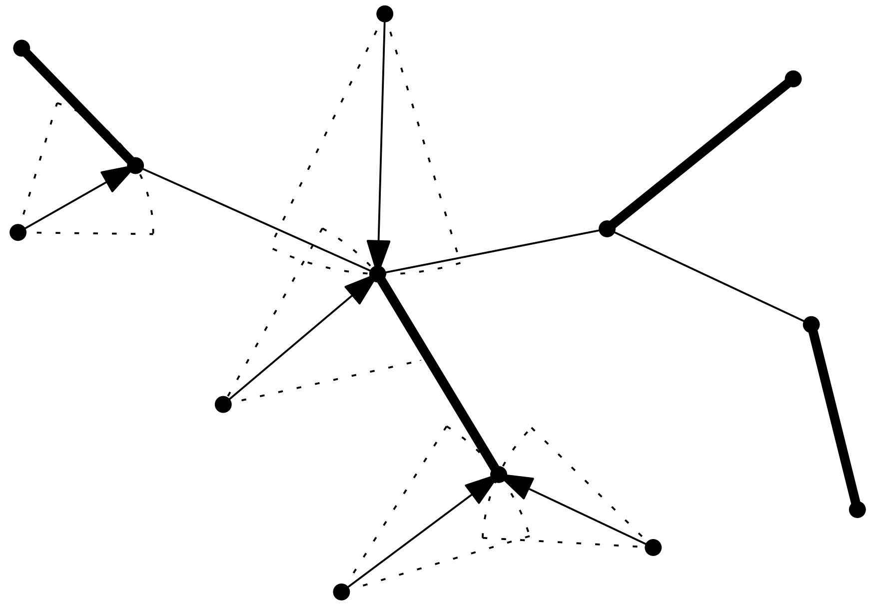
- Find a maximal matching such that each internal vertex is in the matching.



- This can be done by traversing T in BFS order.

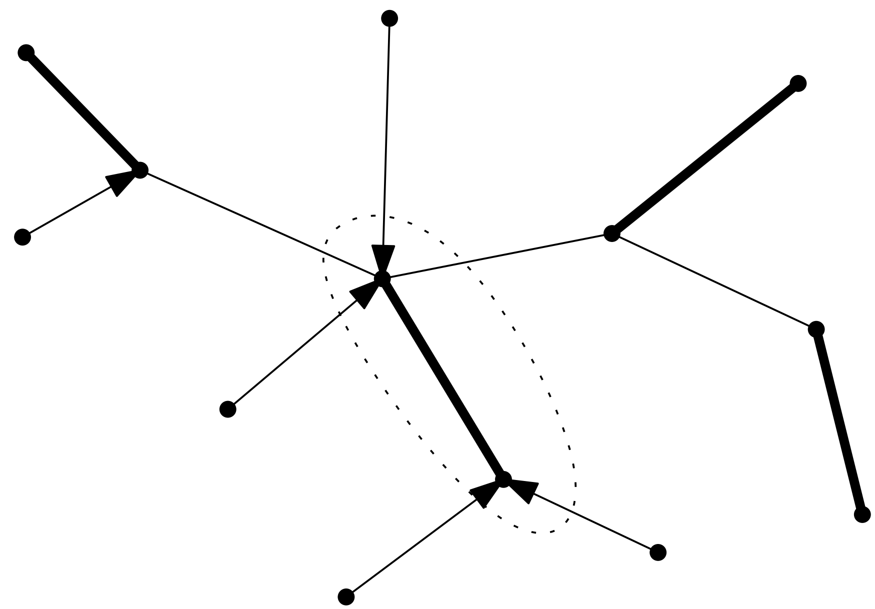
Proof (5/10)

- Orient unmatched leaves to their immediate neighbors.



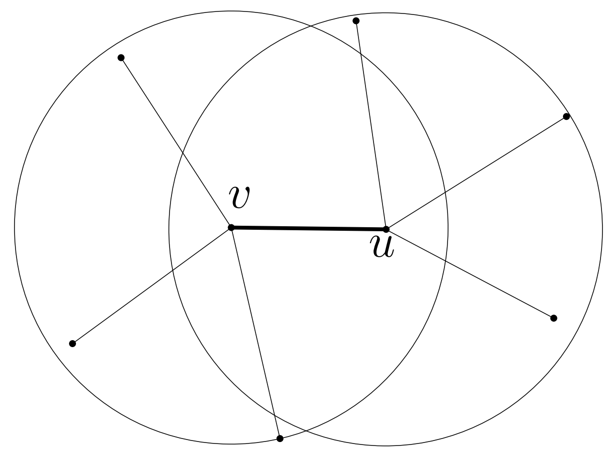
Proof (6/10)

- Consider a pair of matched vertices



Proof (7/10)

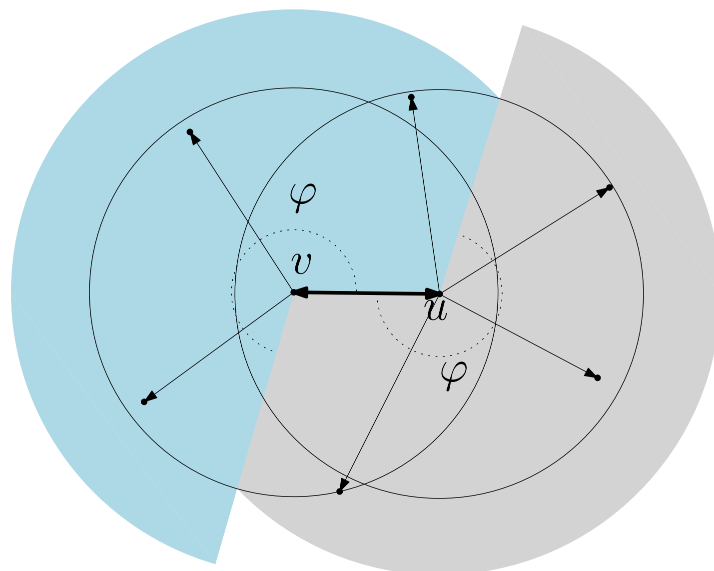
- Let $\{u, v\}$ be an edge in the matching.



- Consider the smallest disks of same radius centered at u and v that contain all the neighbors of u and v in the MST.

Proof (8/10)

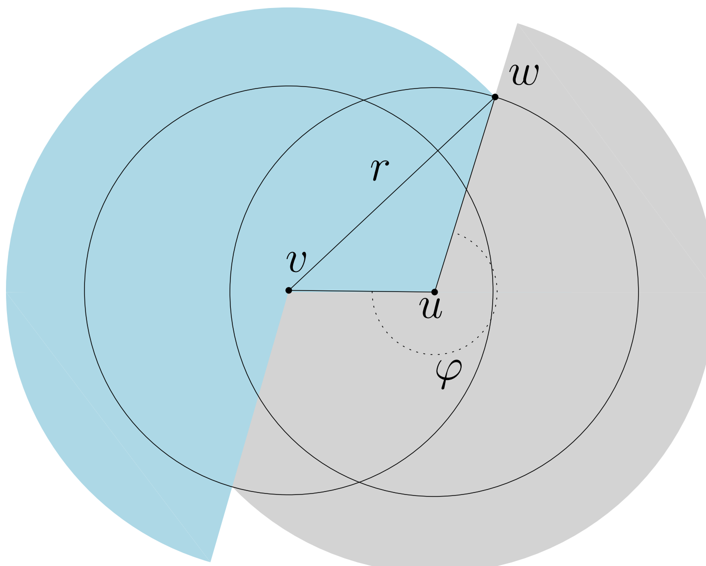
- Orient the directional antennae at u and v with angle φ in such a way that both disks are covered.



- What is the smallest radius necessary so that the union of the discs centered at u, v is covered "completely" by the directional antennae at u, v, respectively?

Proof (9/10)

- To calculate this smallest radius necessary to cover both disks, consider the triangle uvw.



- What is an upper bound on r?
- Observe that without loss of generality we can assume $|uv| = |uw| = 1$.

Proof (10/10)

- Recall the trigonometric identity

$$\sin(\alpha) = \sqrt{\frac{1-\cos(2\alpha)}{2}} \tag{1}$$

- From the law of cosines we can determine an upper bound on r.

$$\begin{aligned} r &\leq \sqrt{|uv|^2 + |uw|^2 - 2|uv||uw|\cos(2\pi - \varphi)} \\ &= \sqrt{2 - 2\cos(2\pi - \varphi)} \text{ (since } |uv| = |uw| = 1) \\ &= 2\sin(\tfrac{2\pi-\varphi}{2}) \text{ (by Equation (1))} \\ &= 2\sin(\pi - \varphi/2) \\ &= 2\sin(\varphi/2) \end{aligned}$$

Lower Bound

Related Work

- When the angle is small, the problem is equivalent to the bottleneck traveling salesman problem (**BTSP**) of finding the Hamiltonian cycle that minimizes the longest edge.
- A 2-approximation (on the antenna length) is given by Parker and Rardin[a].
- For which angles are the two problems equivalent?

[a] *Parker and Rardin.* Guaranteed performance heuristics for the bottleneck traveling salesman problem. 1984

Complexity

- **HCBPG**
  Hamiltonian Circuit Bipartite Planar Grid:
  - **Input:** Bipartite planar grid graph G of degree at most 3.
  - **Output:** Does G have a Hamiltonian circuit?
- HCBPG is NP-Complete[a].
- By reduction to the problem HCBPG, it can be proved that the problem is NP-Complete when the angle is less than $\pi/2$ and an approximation range less than $\sqrt{2}$ times the optimal range.
- We can prove something stronger.

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[a] *Itai, Papadimitriou, and Szwarcfiter.* Hamilton Paths in Grid Graphs. 1982

Computational Complexity

- **Theorem 1 (Caragiannis et al[a].)** Deciding whether there exists an orientation of one antenna at each sensor with angle less that $2\pi/3$ and optimal range is NP-Complete. The problem remains NP-complete even for approximation range less than $\sqrt{3}$ times the optimal range.
- By reduction to the problem of finding Hamiltonian circuit in bipartite planar graphs of maximum degree 3. [b]
- Given a bipartite planar graph $G = (V_0 \cup V_1, E)$ of degree ≤ 3 with n nodes, we construct an ϵ-hexagon graph H (together with its embedding) which has a hamilton circuit if and only if G has a hamilton circuit.

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[a] *Caragiannis, Kaklamanis,Kranakis, Krizanc and Wiese.* Communication in Wireless Networks with Directional Antennae. 2008

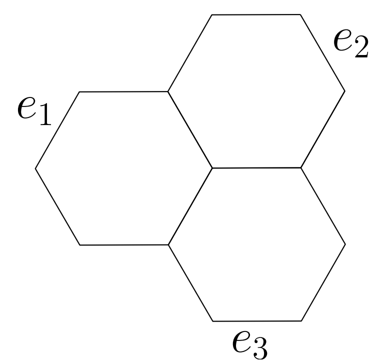
[b] *Itai, Papadimitriou, and Szwarcfiter.* Hamilton Paths in Grid Graphs. 1982

Main Idea: ϵ-Hexagon Graphs

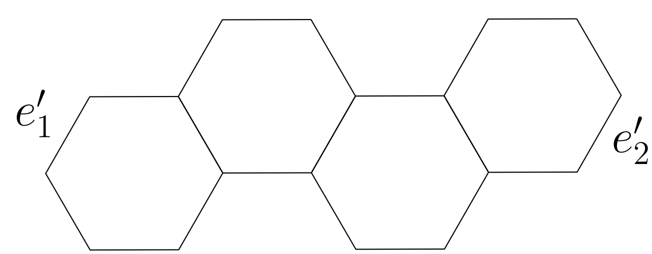
- Let $\epsilon > 0$. An ϵ-hexagon graph $G = (V, E)$ is a bipartite planar graph of maximum degree 3 which has an embedding on the plane with the following properties:
  1. Each node of the graph corresponds to a point in the plane.
  2. The euclidean distance between the points corresponding to two nodes v_1, v_2 of G is in $[1 - \epsilon, 1]$ if $(v_1, v_2) \in E$ and larger than $\sqrt{3} - 3\epsilon$ otherwise.
  3. The angle between any two line segments corresponding to edges adjacent to the same node of G is at least $2\pi/3 - \epsilon/2$.
- An ϵ-hexagon graph is the proximity graph for an instance of the problem and any orientation of sector of radius 1 and angle $\phi = 2\pi/3 - \epsilon$ that induces a strongly connected transmission graph actually corresponds to a hamiltonian circuit of the proximity graph, and vice versa.

Meta Vertices/Edges

- (**Meta vertex:**) Replace every vertex by a diamond (three hexagons)

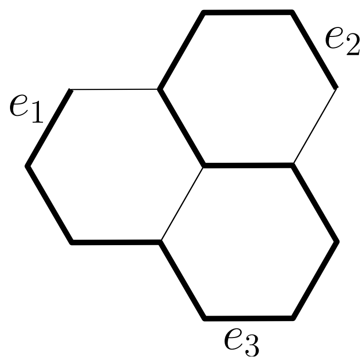


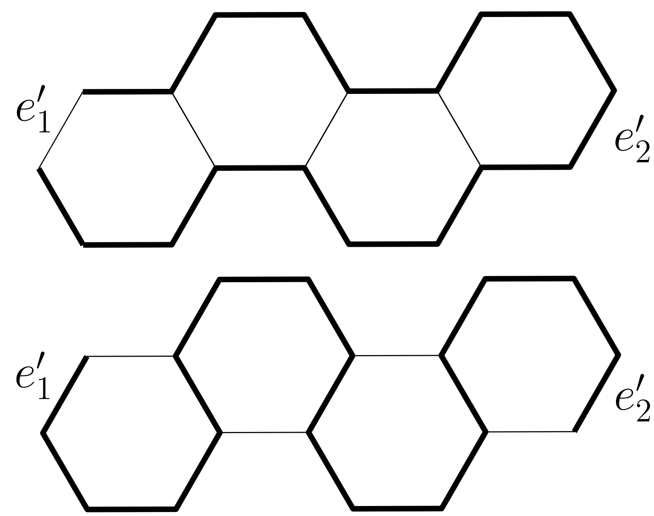
- (**Meta edge:**) Replace every edge by a necklace (path of hexagons)



Hamiltonian Paths

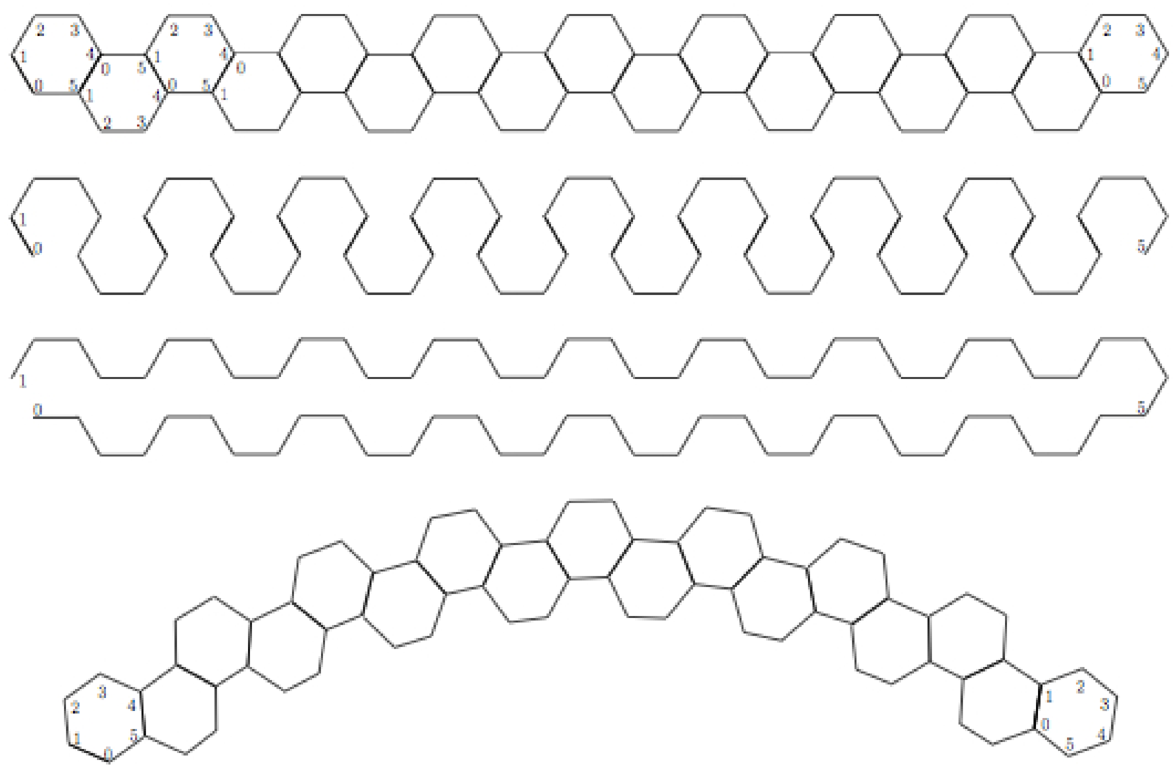
- The meta vertices and necklaces have the following Hamiltonian paths.





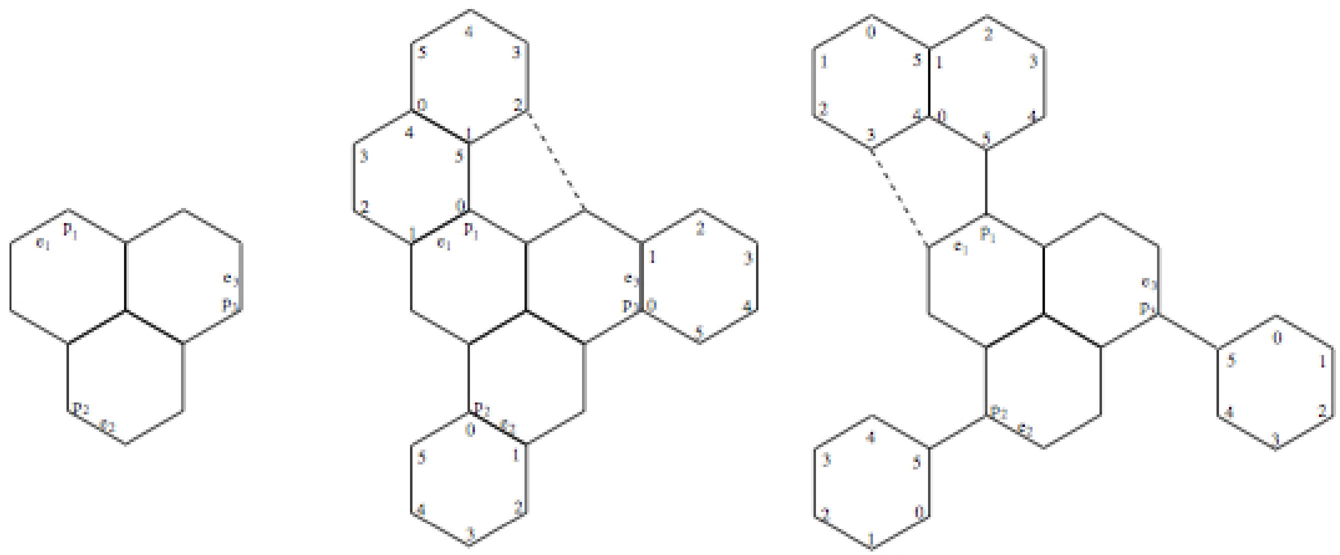
Necklaces, Cross and Return Paths (Examples)

- Top to bottom: 1) Orientation of a necklace, 2) cross path, 3) return path, and 4) representation of the necklace using irregular hexagons of sides between 0.95 and 1 and with angles between sides from 115^o to 125^o.



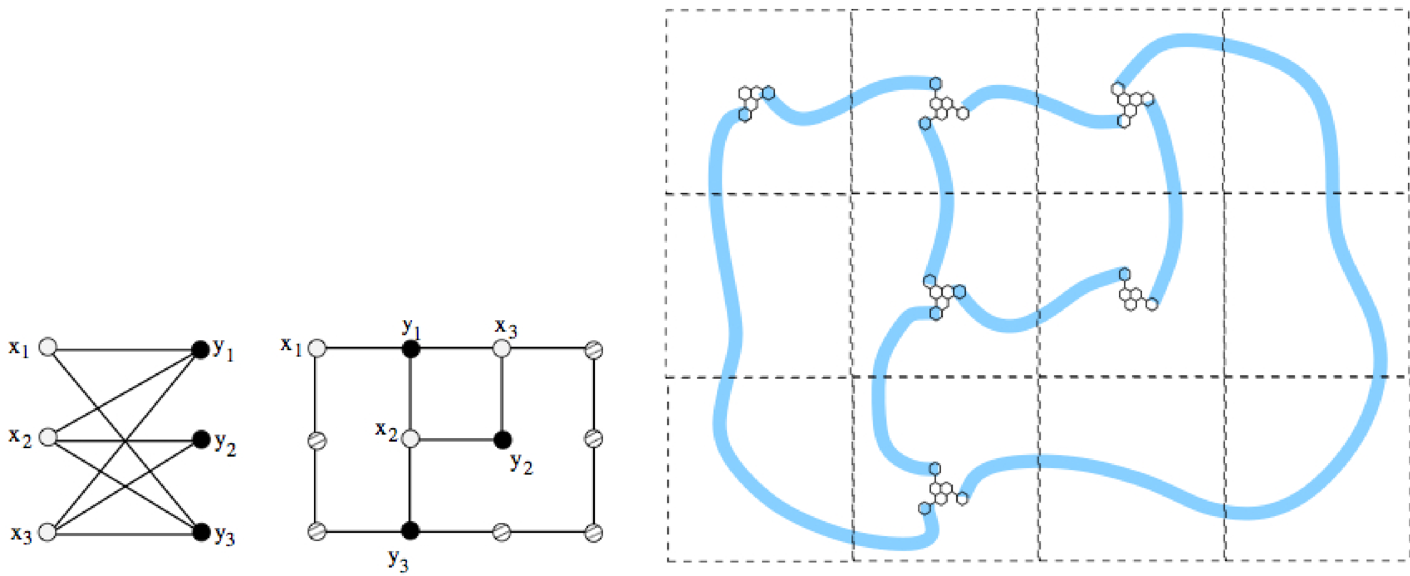
Diamonds and Necklaces

- Left to Right: A diamond (left) and its connection to necklaces when it corresponds to a node of V_0 (middle) or V_1 (right).



Embedding

- A bipartite planar graph of maximum degree 3, its embedding on the rectangular grid, and corresponding ϵ-hexagon graph.



Summary

- We can summarize known antenna angle/range tradeoffs as follows:

Angle	Approximation	Complexity	Reference
$\phi < \frac{2\pi}{3}$	$\sqrt{3} - \epsilon$	NP-C	This talk
$\frac{\pi}{2} \leq \phi \leq \frac{2\pi}{3}$	$4\cos(\phi/2) + 3$	Polynomial	To appear
$\frac{2\pi}{3} \leq \phi \leq \pi$	$2\cos(\phi/2) + 2$	Polynomial	To appear
$\frac{2\pi}{3} \leq \phi \leq \frac{4\pi}{3}$	$2\sin(\phi/2)$	Polynomial	This talk
$\frac{4\pi}{3} \leq \phi$	1 (optimal)	Polynomial	To appear

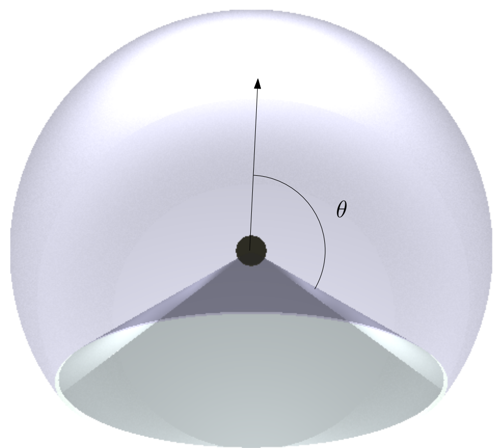
Antenna Orientation Problem in 3D Space

Sensors in 3D Space

- Due to the fact that sensors may lie in distinct altitudes, the previous algorithms do not work correctly in 3D space.
- We model an antenna in 3D space with solid angle Ω as a spherical sector of radius one.
- An omnidirectional antenna has solid angle 4π.

Sensors in 3D Space

- The **apex angle** θ of a spherical sector (with **solid angle** Ω) is the maximum planar angle between any two generatrices of the spherical sector.



- Their relation is given by Archimedes formula

$$\Omega = 2\pi(1 - \cos\theta)$$

Complexity of the Antenna Orientation Problem in 3D Space

- **Theorem 4.** Deciding whether there exists a strong orientation when each sensor has one directional antenna with solid angle less than π and optimal range is NP-Complete.[a]

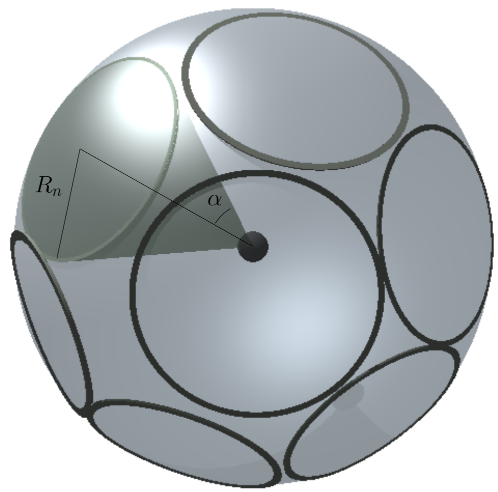
[a] E. Kranakis, D. Krizanc, A. Modi, O. Morales Ponce. Connectivity Trade-offs in 3D Wireless Sensor Networks Using Directional Antennae. In proceedings of IPDPS 2011, May 16-20, 2011.

Proof

- Consider a set S of n points in the plane.
- From Archimedes relation, any plane that cuts the coverage area of any 3D directional antennae through the apex with angle Ω has plane angle that satisfies $\cos(\theta) \leq 1 - \frac{\Omega}{2\pi}$.
- Therefore $\theta < 2\pi/3$ if and only if $\Omega < \pi$.
- A strong orientation of the directional antennae with angle less than $2\pi/3$ in 2D implies a strong orientation of directional antennae with angle less than π in 3D.
- The opposite is also true.

Tammes' Radius

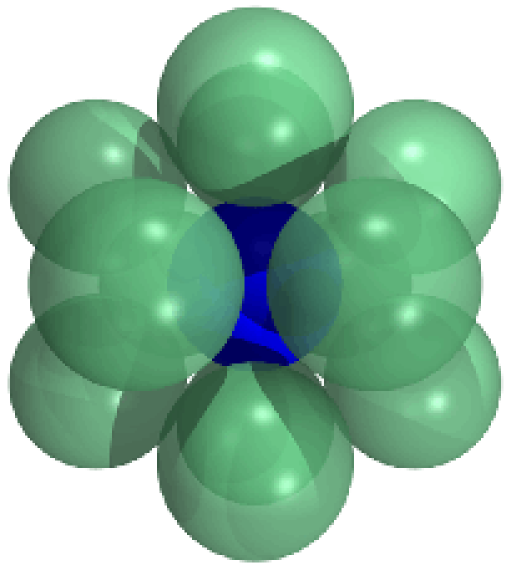
- The Tammes radius is the maximum radius of n equal non-overlapping circles on the surface of the sphere.



- We denote it by R_n.

Kissing Number and Tammes' Radius

- The Kissing number is the number of balls of equal radius that can touch an equivalent ball without any intersection,



Kissing Number and Tammes' Radius

- In particular, the Tammes' Radius is equivalent to the kissing number when all the balls have the same radius.
- The maximum degree of an MST is equal to the kissing number.
- In 3D it is 12.

Optimal Range Orientation in the Space

- **Theorem 5.** There exists an orientation of the directional antennae in 3D with optimal range when the solid angles of the antennae are at least $18\pi/5$.

Proof (1/4)

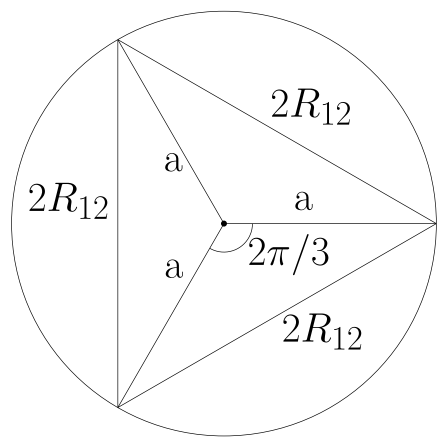
- Let T be an MST on the points.
- Let B_p be the sphere centered at p of minimum radius that covers all the neighbors of p in T.
- For each neighbor u of p in T, let u' be the intersection point of B_p with the ray emanating from p toward u

Proof (2/4)

- Thus, we have an unit sphere with at most 12 points.
- Compute the Delauney Triangulation on the points of the sphere.
- Orient the antenna in opposite direction of the center of largest triangle.

Proof (3/4)

- Observe that every edge of the Delaunay Triangulation has length at least twice the Tammes' Radius $R_{12} = \sin \frac{63^o 26}{2}$.
- Thus, every triangle is greater than the equilateral triangle of side $2R_{12}$.



Proof (4/4)

- It follows that

$$\begin{aligned} R_{12} &= \sin\left(\frac{63^o 26'}{2}\right) \\ a &= R_{12}/\sqrt{3} \\ \alpha &\leq \arcsin(a) \end{aligned}$$

- and therefore

$$\begin{aligned} \Omega &\geq 4\pi - 2\pi(1 - \cos(\alpha)) \\ &= 2\pi(1 + \cos(\alpha)) \\ &= 2\pi\left(1 + \cos\left(\arcsin\left(\frac{2R_{12}}{\sqrt{3}}\right)\right)\right) \\ &\geq \frac{18\pi}{5} \end{aligned}$$

Antenna Orientation With Approximation Range

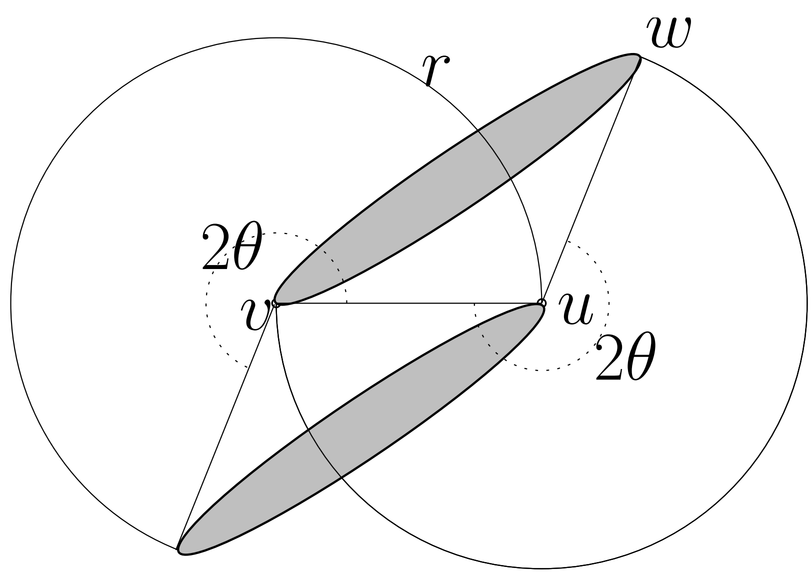
- **Theorem 6.** Given a solid angle φ with $2\pi \leq \varphi < 18\pi/5$ and a set of points in the space, there exists a polynomial time algorithm that computes a strong orientation with radius bounded by $\frac{\sqrt{\Omega(4\pi-\Omega)}}{\pi}$ times the optimal range.

Proof (1/3)

- Let T be the MST on the set of points.
- Consider a maximal matching such that each internal vertex is matched.
- Orient unmatched leaves to their immediate neighbors.
- Let $\{u, v\}$ be an edge in the matching. Consider the smallest sphere of same radius centered at u and v that contain all the neighbors of u and v in the MST.

Proof (2/3)

- Orient the directional antennae at u and v with plane angle 2θ in such a way that both spheres are covered.



Proof (3/3)

- From the law of cosine we can determine r.
- Let θ be the apex angle of Ω.
- Observe that

$$
\begin{aligned}
r &= \sqrt{|uv|^2 + |uw|^2 - 2|uv||uw|\cos(2\theta)} \\
&\leq \sqrt{2 - 2\cos(2\theta)} \\
&= 2\sin(\theta) \\
&= 2\sqrt{1 - \cos^2(\theta)} \\
&= 2\sqrt{1 - (1 - \frac{\Omega}{2\pi})^2} \\
&= \frac{\sqrt{\Omega(4\pi - \Omega)}}{\pi}
\end{aligned}
$$

Summary of the Antenna Orientation Problem

2D		3D	
Angle	Range	Solid Angle	Range
$\varphi < \frac{2\pi}{3}$	NP-C	$\Omega < \pi$	NP-C
$\frac{2\pi}{3} \leq \varphi < \pi$	Open	$\pi \leq \Omega < 2\pi$	Open
$\pi \leq \varphi < \frac{8\pi}{5}$	$2\sin(\varphi/2)$	$2\pi \leq \Omega < \frac{18\pi}{5}$	$\frac{\sqrt{\Omega(4\pi-\Omega)}}{\pi}$
$\varphi \geq \frac{8\pi}{5}$	1	$\Omega \geq \frac{18\pi}{5}$	1