Neighbor Discovery in a Sensor Network with Directional Antennae

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Abstract. Consider a network of n directional antennae in the plane. We consider the problem of efficient neighbor discovery in a (synchronous) network of sensors employing directional antennae. In this setting sensors send messages and listen for messages by directing their antennae towards a specific direction (which is not necessarily known in advance). In our model the directional antennae can be rotated by the sensors as required so as to discover all neighbors in their vicinity. In this paper we will limit ourselves to the (D, D) communication model whereby sensors employ directional antennae with identical transmission/reception beam widths. Our methodology is based on techniques for symmetry breaking so as to enable sender/receiver communication. We provide 1) deterministic algorithms that introduce delay in the rotation of the antennae and exploit knowledge of the existence of a vertex coloring of the network, and 2) randomized algorithms that require knowledge only of an upper bound on the size of the network so as to accomplish neighbor discovery. In both instances we study tradeoffs on the efficiency of the algorithms proposed.

Key words and phrases: Deterministic, Randomized algorithms, Neighbor discovery, Rotating directional antennae, Sensor network

1 Introduction

Directional antennae are known to reduce energy consumption because they can reach further for the same amount of energy consumed. However, unlike sensors with omnidirectional antennae sensors with directional antennae take longer to discover their neighbors. This is due to the fact that although sensors may be within transmission range the sender (respectively, receiver) sensor may not necessarily be located within the given sector determined by the beaming antenna of the transmitting sensor. This raises the question of what algorithms to employ so as to attain efficient communication (e.g., routing, broadcasting, etc.) using only directional antennae. This approach can be particularly beneficial in delay tolerant sensor networks, for example, whereby sensors may be able to take advantage of opportunistic appearances of sensors due to mobility and other factors.

For a given radius $r > 0$, assume that a given sensor, say S, can reach all other sensors within the disc having centre S and radius r. There are several directional antenna models, but for our study it will suffice to consider the following directional antenna model. We assume that either 1) the sensors are standing on a swivel and can rotate in any desired direction or 2) the sensors' coverage area can be divided into non-overlapping sectors that can be activated by an antenna switch so as to reach other sensors within a particular region. It is clear that in the former mode of operation the rotation of the antenna is continuous around the circle while in the latter the circular sectors are in discrete predefined sectors around the circle. We will not elaborate further in this paper the differences and similarities between these two modes of operation for directional antennae.

1.1 Preliminaries and notation

In this subsection we discuss several related antenna models that are related to our study.

Communication Models with Directional Antennae. Several communication models are possible for a pair of sensors with omnidirectional and directional antennae. Consider the pair (X, Y), where the first parameter X indicates the capability of the sender sensor and the second parameter Y the capability of the receiver sensor. To be more precise, X, Y may take either of the values O, D, where O means omnidirectional and D directional antenna. Thus, the (X, Y) *communication model* for a pair of communicating sensors means that the sender uses antenna of type X and the receiver of type Y. We also assume a *duplex* communication model whereby sensors can send and receive messages at the same time ignoring collisions. It is clear from the previous discussion that

– in the (O, O) model two sensors can communicate if they are within transmission range of each other,
– in the (D, O) (respectively, (O, D)) model, the sender (respectively, receiver) must turn its antenna so as to reach its neighbor, and
– in the (D, D) model both sender and receiver must direct their antennae towards each other at the same time.

More specifically, in all four models the sensors must be within range of each other so as to communicate. However, in the (D, O) and (O, D) models the sensor with the directional antenna must also turn its antenna toward the other sensor, while in the (D, D) model both sensors' antennae must face against each other. Therefore it follows that (D, D) is the weakest and (O, O) is the strongest among the four communication models.

More general models are also possible whereby a sensor's transmission beam width is not necessarily the same with its reception beam width. To simplify notation and terminology, in this paper we will limit ourselves to the (D, D) communication model with identical transmission/reception beam widths. Our results generalize without much difficulty to this more general setting.

The *neighbor discovery* process usually entails the exchange of identities (e.g., MAC addresses) between two adjacent nodes. It will not be necessary to go into the details of such an exchange and for our purposes it will be sufficient to assume that this is a one step process whereby one sensor sends its identity and the other acknowledges by sending back its own. Throughout this paper we will assume that the sensors have distinct identities but their corresponding locations (i.e., (x, y)-coordinates) in the plane are not known to each other.

Antenna Models. The transmission area of an omnidirectional antennae is modelled by a circular disk in the plane while the transmission area of a directional antennae is modelled by a circular sector in the disk. We assume that sensors have the capability to rotate their directional antenna and change sectors. so as to establish communication.

Consider a set of n sensors in the plane. Each sensor u is equipped with a directional antenna having beam width ϕ_u. Further we will assume that $\phi_u = \frac{2\pi}{k_u}$, for some integer k_u.[5] In particular, if $k_u = 1$ then we have an omnidirectional antenna at u. The sensors are synchronous and can rotate their antennae counter-clockwise (see Figure 1). Assume that the UDG formed by the sensors is connected

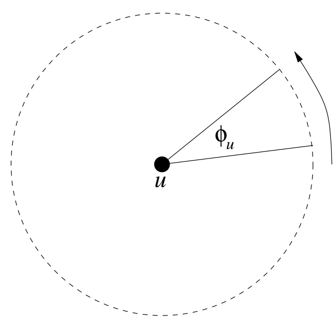


Fig. 1: An antenna at u rotating counter-clockwise.

and c-colorable, i.e., there is a coloring of its vertices $\chi : V \to \{0, 1, \ldots, c-1\}$ such that if sensors u, v are adjacent in the UDG then u and v have different colors, i.e., $\chi(u) \neq \chi(v)$. Observe any "integer based" identity scheme, e.g., the n sensors are numbered $0, 1, 2, \ldots, n-1$, that provides different numbering to different sensors satisfies this property (albeit it is not efficient).

1.2 Related Work

There are protocols using directional antennas in neighbor discovery processes. In [4], the authors proposed the gradual increase of directional communication range levels for neighbor discovery purposes. Nearby neighbors are discovered first and faraway neighbors will be discovered at later stages. Directional transmission and reception are used in this work. In [5], a direct discovery protocol and a gossip based neighbor discovery protocol using directional antennas in a static

[5] It turns out that this assumption is not required for the subsequent results; we use it because it makes the proofs simpler.

wireless network were proposed. During direct discovery process, a node discovers a neighbor node only when information is received from this neighbor, while nodes exchange their neighbors' location information to enable faster discovery in gossip based algorithm. The protocol tries to optimize the discovery probability in a randomized neighbor discovery process using directional transmission and reception. In [1], a neighbor discovery protocol which considers node movements was proposed where directions with less possibility of discovering new nodes will be bypassed during neighbor scanning and neighbor discovery frequency is adjusted according to node mobility. It uses directional antenna for transmissions and omnidirectional antenna for receptions. In [6], two Scan Based Algorithms (SBA-D, SBA-R) and one Completely Random Algorithm (CRA-DD) were proposed, which use only directional antennae. In SBA-D, a node decides whether to scan or listen depending on node ID, while a node transmits at one direction or receives at the opposite direction with probability $\frac{1}{2}$ in SBA-R. SBA-D and SBA-R algorithms require perfectly synchronized antenna rotation direction, time and instantaneous antenna rotation to any direction, which are very strong assumptions. In CRA-DD, at each time slot, nodes decide whether to transmit/receive and which direction to transmit/receive completely randomly, which is the simplest algorithm one can imagine and it also requires instantaneous antenna rotation to any direction. In [3], an analytical model was proposed for synchronized 2D neighbor discovery protocols. The model is based on directional transmission and directional reception and a node transmits in one direction and receives in the opposite direction simultaneously.

1.3 Outline and results of the paper

In this paper, we propose novel neighbor discovery algorithms in a (D, D) communication model whereby sensors employ directional antennae with identical transmission/reception beam widths and each sensor has only one directional antenna. Our methodology is based on techniques for symmetry breaking so as to enable sender/receiver communication. We provide 1) deterministic algorithms that introduce delay in the rotation of the antennae and exploit knowledge of the existence of a vertex coloring of the network, and 2) randomized algorithms that require knowledge only of an upper bound on the size of the network so as to accomplish neighbor discovery. In both instances we study tradeoffs on the efficiency of the algorithms proposed. Through experimentation, we also show that the algorithms achieve desirable neighbor discovery delays with efficiency in energy consumption. Details can be found in the full version of the paper [2].

The rest of the paper is organized as follows. Deterministic algorithms on neighbor discovery are presented in Section 2. As an alternative scenario, Section 3 gives out the randomized algorithm and its analysis. We conclude with possible future directions in Section 4.

2 Deterministic Algorithms for Neighbor Discovery

In this section we give algorithms for neighbor discovery in the (D, D) communication model and analyze their complexity. First we give a simple lower bound that indicates the complexity of the neighbor discovery problem.

In all the results below as measure of complexity for neighbor discovery we will use the time required for sensors to discover each other and we will ignore collisions during simultaneous transmissions. For two sensors, this is the number of steps until the first successful send/receive exchange. For a sensor network, this is the minimum for any algorithm taken over the maximum time required for any two adjacent sensors in the network to communicate.

2.1 Lower bound

In a setting whereby two adjacent sensors know each other's location all they need to do is turn their antennae towards each other in the specified locations. Therefore the observation below is useful when sensors do not know each other's location.

Theorem 1. *Consider two sensors u, v within communication range of each other and respective antenna beam widths $\frac{2\pi}{k_u}$ and $\frac{2\pi}{k_v}$, respectively. If the sensors do not know each other's location then any algorithm for solving the neighbor discovery problem in the (D, D) communication model requires at least $\Omega(k_u k_v)$ time steps.*

Proof. For a successful communication to occur each sensor must be within the beam of the other sensor's antenna at the same time. Since the sensors do not know each other's location they must attempt transmissions in all their respective sectors. This completes the proof of Theorem 1.

2.2 Antenna rotation algorithms

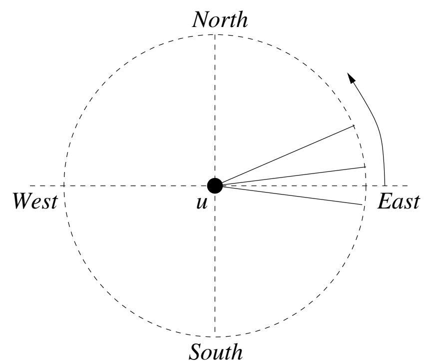
Given these preliminary definitions we consider the following class of *antenna rotation* algorithms. For each sensor u, let d_u be an integer delay parameter and k_u be defined so that $\phi_u = \frac{2\pi}{k_u}$. Given u, d_u, k_u the sensor executes the following algorithm.

Algorithm 1: Antenna Rotation Algorithm $ARA(d_u, k_u)$

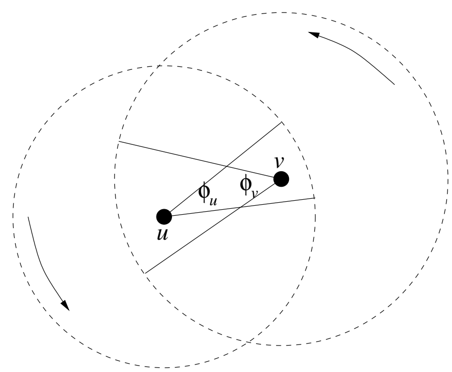
Start at a given orientation;
while *true* **do**
    for $i \leftarrow 0$ **to** $d_u - 1$ **do**
        `//For` d_u `steps stay in chosen sector`
        Send message to neighbor(s);
        Listen for messages from neighbor(s) (if any);
    Rotate antenna beam one sector counter-clockwise;
    `//rotate by an angle equal to` ϕ_u

Remarks and Observations on the *ARA* Algorithm. There are several issues concerning interpretations of the execution of the rotation algorithm which are worth discussing.

– In Step 1 the initial antenna orientation is selected. There are many consistent ways to define this but for simplicity in this paper it is taken to be the bisector of the angle which defines the antenna beam. Also, if the sensors are equipped with a compass then we may assume that they all start with identical orientations, say East (see Figure 2a). Otherwise, the initial orientation may



(a) An antenna at u with sectors counted counter-clockwise.



(b) Neighbor discovery for sensors u, v.

Fig. 2: Directional antennae.

be chosen in an arbitrary manner. It turns out that our analysis is valid in this more general setting.

– The main neighbor discovery algorithm is executed in Step 2. We are interested in measuring the number of steps until all (available) neighbors are discovered. For the duplex communication model being considered here, it is clear that two sensors u, v will be able to discover each other if (see Figure 2b)
  1. each sensor is within each other's range, and
  2. the corresponding antennae of the two sensors are oriented so that each sensor is within the other sensor's beam at the same time.

  These are the basic requirements we employ in order to prove the correctness and running time of our algorithm.

– In Step 3, the algorithm imposes a *rotation delay*, i.e., for d_u (equal to the delay imposed) steps the sensor sends messages and also listens for messages from neighbors. The delay imposed in Step 3 is required so as to break symmetry and ensure that neighboring sensors' antennae are within each other's beam range and will eventually communicate using the (D, D) communication model. There are several possibilities here. The sensor may elect to send/receive messages 1) at each step during the delay interval $[0, d_u - 1]$, 2) select a time within the delay interval $[0, d_u - 1]$ at random. In our analysis we will assume the former.

– Step 6 involves rotation of the antenna by ϕ_u which is also equal to the beam width of the antenna. This ensures that after each rotation a new region (located counter-clockwise from the old region) is covered. Several possibilities exist, for example 1) allow overlap between the new and old antenna beaming location, 2) select the new antenna beaming location at a sector chosen at random among the k_u possible sectors in the disk.[6]

2.3 Complexity of deterministic antenna orientation algorithm

Now we consider the complexity of the various antenna orientation algorithms. Assume the sensor network is synchronous. Recall our basic assumption that there is a coloring $\chi : V \to \{0, 1, \ldots, c-1\}$ of the vertices of the sensor network using c colors. Table 1 summarizes the results of this section.

Antenna at u	Knowledge	Running Time	Theorems
$2\pi/k$	Identical	$O(k^{c-1})$	Theorem 2
$2\pi/k$	Identical	$O(k(c \ln c)^3)$	Theorem 3

Table 1: List of theorems and running times of deterministic algorithms.

The simplest possible delay model is for a sensor to wait "sufficient amount of time" so as to send to (receive from) the desired node.

However, there are choices of delay under which sensors with directional antennae will never be able to communicate as illustrated in Figure 3.

Example 1. Assume the antenna beam width is $\frac{2\pi}{4} = \frac{\pi}{2}$ and the four sectors are labelled $0, 1, 2, 3$. Both sensors depicted in Figure 3 start beaming East. Sensor u employs delay $d_u = 2$ and sensor v delay $d_v = 1$. Sensors can communicate only if u's antenna faces East and v's antenna faces West at the same time. Observe that sensor u faces East only at time $t = 0, 1, 8, 9, 16, 17, \ldots$ while sensor v faces West only when $t = 2, 6, 10, \ldots$. Therefore u, v can never communicate.

The previous example indicates that sensor delays must be chosen judiciously so as to enable communication. The first theorem considers the simplest model whereby a sensor delays the rotation of its antenna sufficient time so as to allow all its neighbors' antennae to perform a complete rotation.

[6] The point of these assumptions is to consider collision models. In this paper we assume that the sensors send/receive messages at each step during the delay interval. Further, if we were to analyze a collision model we would have to assume that the corresponding intervals of adjacent nodes are disjoint.

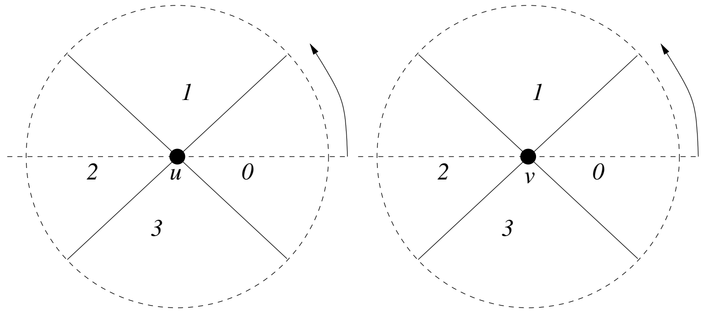


Fig. 3: Neighbor discovery for sensors u, v is not possible.

Theorem 2. *Consider a set of sensors in the plane with identical antenna beam widths equal to $\phi = \frac{2\pi}{k}$. For each sensor u let the delay be defined by $d_u := k^{\chi(u)}$. If each sensor u executes algorithm $ARA(d_u, k)$ then every sensor in the network will discover all its neighbors in at most k^{c-1} time steps.*

Proof. Consider two adjacent sensors u, v. Clearly, $\chi(u) \neq \chi(v)$ since they must have different colors. By assumption, $d_u = k^{\chi(u)}$ and $d_v = k^{\chi(v)}$. Without loss of generality assume that $\chi(u) < \chi(v)$. Observe that for each chosen sector the sensor v beams its antenna in this sector for $k^{\chi(v)}$ steps. But $k^{\chi(v)} = k^{\chi(v)-\chi(u)} k^{\chi(u)}$ and hence $k^{\chi(v)}$ is a multiple of $k^{\chi(u)}$. In particular, while sensor v waits in a given sector the other sensor u will execute $k^{\chi(v)-\chi(u)}$ rotations around the circle before returning to its original sector. It follows that sensors u, v will discover each other within the specified number of steps. This completes the proof of Theorem 2.

The running time of the algorithm depends on the coloring being used in Theorem 2. If no knowledge on the network is available then any integer identity scheme will work, however this will typically be of size $\Omega(n)$ thus giving an exponential running time $k^{\Omega(n)}$. If the sensor network is bipartite (e.g., tree) then it is easy to see that $c = 2$ is sufficient. For random UDGs with range at the connectivity threshold the number of colors required is $c = \Theta(\log n)$ in which case the running time of the algorithm is about $k^{\log n} = n^{\log_2 k}$, which is polynomial in n with exponent $\log_2 k$ (In many applications a typical value of k is 6.)

Nevertheless we would be interested to provide algorithms with running time not dependent on the size n of the network but rather on the number of colors of a vertex coloring. Indeed, this is the case as shown by the next theorem.

Theorem 3. *Consider a set of sensors in the plane with identical antenna beam widths equal to $\phi = \frac{2\pi}{k}$. Assume the sensor network is synchronous. Suppose that the delays d_u at the nodes are chosen so that*

1. $\gcd(k, d_u) = 1$, *and* $d_u > k$, *for all* u, *and*
2. *if* u, v *are adjacent then* $\gcd(d_u, d_v) = 1$.

If each sensor u executes algorithm $ARA(d_u, k)$ then every sensor in the network will discover all its neighbors in at most $O\left(k(\max_u d_u)^3\right)$ time steps. In addition, *the delays d_u can be chosen so that every sensor in the network will discover all its neighbors in at most $O(k(c \ln c)^3)$ time steps. In particular, this is at most $O\left((c \ln c)^3\right)$ time steps provided that $k \in O(1)$.*

Proof. Without loss of generality, in the proofs below we assume that the sensors can determine a fixed starting antenna sector facing East, say (see Figure 2a). Proofs carry over to the more general case and the necessary modifications are omitted. Consider two adjacent sensors u, v. Without loss of generality assume that

1. sensor u is to the left of sensor v, and
2. that both antennae orientations are initially set to *East*, say.

First we consider the case when the line segment connecting u to v is horizontal. Observe that u, v can communicate when v's antenna is facing *West* which is sector $\lfloor \frac{k}{2} \rfloor$. Since $\gcd(d_u, d_v) = 1$, by Euclid's algorithm there exist integers $0 < a_u < d_u, 0 < a_v < d_v$ such that

$$a_u d_u = a_v d_v + 1. \tag{1}$$

Lets look at sensor u first. Recall that because of the delay constrains of the algorithm, the sensor stays in the same sector for d_u steps before it rotates its antenna. After $d_u k$ steps sensor u will be in its starting position and, clearly, the same applies for any time duration that is a multiple of $d_u k$. Thus sensor u is in its initial position (facing East) at time $j a_u d_u k$, for any $j > 0$. If we multiply both sides of Equation $a_u d_u = a_v d_v + 1$ by jk we have that

$$j a_u d_u k = j a_v d_v k + jk$$

It follows that at time $t = j a_u d_u k$ the sensor at u is facing *East*. If there is a j such that $jk = \lfloor \frac{k}{2} \rfloor d_v + r$ for $0 \le r < d_v$, then sensor v is facing West and therefore the sensors u, v can discover each other. Starting from $j = 1$, with $k \le \lfloor \frac{k}{2} \rfloor d_v$, we can find a j such that,

$$jk \le \lfloor \frac{k}{2} \rfloor d_v < jk + k \tag{2}$$

which means that $jk + k = \lfloor \frac{k}{2} \rfloor d_v + r$, with $r \le k < d_v$. A simple modification of the proof will prove the result when the two sensors are not necessarily on a horizontal line.

The number of rotations required is $j a_u d_u k$, where j satisfies Inequality (2). Since $j a_u d_u k \le k(\max_u d_u)^3$ it follows that $k(\max_u d_u)^3$ is an upper bound on the time required by all pairs of sensors to discover each other.

If $k \in O(1)$ (this is a reasonable assumption since in practice $k = 6$) then we can satisfy the conditions of Theorem 3 by choosing the d_us to be prime numbers. Since the number of colors is c, we will need c prime numbers (one for each color class of vertices of the graph). Hence by the prime number theorem the largest prime needed in order to define the delays $\{d_u : u \in V\}$ will be in the order of the c-th prime number, which is in $O(c \ln c)$. Therefore every sensor in the network will discover all its neighbors in at most $O\left((c \ln c)^3\right)$ time steps. This completes the proof of Theorem 3. $\square$

Theorem 3 can be improved further with only slight modifications in the proof even in the case where $\frac{2\pi}{\phi}$ is not necessarily an integer. To this end define $k := \lfloor \frac{2\pi}{\phi} \rfloor$. We can modify algorithm $ARA(d_u, k)$ to a new algorithm $ARA'(d_u, \phi)$ as follows: we still have k sectors and we can modify Step 6 in algorithm $ARA(d_u, k)$ so that the antenna at u rotates along the corresponding sectors $0, 1, \ldots, k-1$ (thus there is overlap between the new and the old sector). It is easy to prove the following generalization of Theorem 3.

Theorem 4. *Consider a set of sensors in the plane such that the antenna beam width of sensor u is equal to ϕ. Define $k := \lfloor \frac{2\pi}{\phi} \rfloor$ Assume the sensor network is synchronous. Suppose that the delays d_u at the nodes are chosen so that*

1. *$gcd(d_u, k) = 1$ and $d_u > k$, for all u, and*
2. *if u, v are adjacent then $\gcd(d_u, d_v) = 1$.*

If each sensor u executes algorithm $ARA'(d_u, \phi)$ then every sensor in the network will discover all its neighbors in at most $k \left(\max_u d_u\right)^3$ time steps. In addition, the delays d_u can be chosen so that every sensor in the network will discover all its neighbors in at most $O(k(c \ln c)^3)$ time steps. In particular, this is at most $O\left((c \ln c)^3\right)$ time steps provided that $k \in O(1)$.

Proof. With some simple modifications, this is identical to the proof of Theorem 3. Details are left to the reader.

Observe that for a random UDG at the connectivity threshold we have that $c = \Theta(\ln n)$ and therefore the running time of the algorithms in Theorems 3 and 4 will be $O((\ln n \ln \ln n)^3)$.

3 Randomized Neighbor Discovery Algorithms

In this section we consider several randomized algorithms. The main advantage of the algorithms in Theorems 5 and 6 is that no a priori knowledge of coloring or of any proper identity scheme is required; just an upper bound n on the size of the network. Moreover, the algorithm in Theorem 7 requires only a bound on the antennae beam widths. Table 2 summarizes the results of this section.

Antenna at u	Knowledge	Running Time	Theorems
$2\pi/k$	Identical	$kn^{O(1)}$	Theorem 5
$2\pi/k$	Identical	$O(k^2 \log n)$	Theorem 6
$2\pi/k_u$	$\max_u k_u \leq k$	$O(k^4 \log n)$	Theorem 7

Table 2: List of theorems and running times of randomized algorithms.

3.1 Deterministic algorithm with selection of random delay

In this algorithm each sensor u selects a random prime number as delay d_u (in a range $k..R$ to be specified) and runs the deterministic algorithm $ARA(d_u, k)$.

Algorithm 2: Randomized Antenna Rotation Algorithm $RARA(d_u, k)$
1 Select $d_u \leftarrow RANDOMPRIME(k..R)$; **2** Execute $ARA(d_u, k)$;

Theorem 5. *Consider a set of sensors in the plane such that the antenna beam width of sensor u is equal to $\phi = \frac{2\pi}{k}$. Assume the sensor network is synchronous. If each sensor u executes algorithm $RARA(k; R)$, where $R = n^{O(1)}$ and n is an upper bound on the number of sensors, then every sensor in the network will discover all its neighbors in at most $kn^{O(1)}$ expected time steps, with high probability.*

Proof. For every node u, let $N(u)$ denote the neighborhood of u and $\deg(u)$ the degree of u. Further, let $D = \max_u \deg(u)$ denote the maximum degree of a node of the sensor network. By the prime number theorem, the number of primes $\leq R$ and $> k$ is approximately equal to $\frac{R}{\ln R} - \frac{k}{\ln k}$ and therefore the probability that the primes chosen by two adjacent nodes, say u and v, are different is $1 - \frac{1}{\frac{R}{\ln R} - \frac{k}{\ln k}}$.

Let E_u be the event that the prime chosen at u is different from all the primes chosen by its neighbors. It is easily seen that

$$\begin{aligned}
\Pr[E_u] &= 1 - \Pr[\neg E_u] \\
&= 1 - \Pr\left[\exists v \in N(u)(d_u = d_v)\right] \\
&\geq 1 - \sum_{v \in N(u)} \Pr\left[d_u = d_v\right] \\
&\approx 1 - \deg(u)\frac{1}{\frac{R}{\ln R} - \frac{k}{\ln k}} \\
&\geq 1 - D\frac{1}{\frac{R}{\ln R} - \frac{k}{\ln k}}.
\end{aligned}$$

Similarly, we can prove that

$$\begin{aligned}
\Pr\left[\bigcap_u E_u\right] &= 1 - \Pr\left[\bigcup_u \neg E_u\right] \\
&\geq 1 - \sum_u \Pr[\neg E_u] \\
&\geq 1 - nD\frac{1}{\frac{R}{\ln R} - \frac{k}{\ln k}} \\
&\geq 1 - \frac{1}{n}.
\end{aligned}$$

By choosing R in $n^{O(1)}$ and recalling that $D \leq n$ we see that all the primes chosen by all the nodes in the network are pairwise distinct, with high probability. The claim concerning the expected number of time steps follows immediately from the analysis of the antenna rotation algorithm in Theorem 3. This completes the proof of Theorem 5.

3.2 Algorithm with random selection of rotation mechanism

In the algorithms below we assume that the antenna beam width of u is equal to $\frac{2\pi}{k}$. In the main algorithm a sensor chooses a "rotation mechanism" between two given rotation mechanisms independently at random. In the first mechanism, the antenna cycles k rounds with no sector delay, while in the second the antenna cycles only one round but with delay k per sector. The two rotation mechanisms can be described formally as follows.

Algorithm 3: Rotate with no Sector Delay $Mech_0(k, d)$

`//Cycle` k `rounds with no sector delay`
for $j \leftarrow 1$ **to** d **do**
    for $i \leftarrow 0$ **to** $k-1$ **do**
        Send message to neighbor(s) in sector i;
        Listen for messages from neighbor(s) (if any) in sector i;
        Rotate antenna one sector;

Algorithm 4: Rotate with Delay k per Sector $Mech_1(k, d)$

`//Cycle one round with delay` k `per sector`
for $i \leftarrow 0$ **to** $k-1$ **do**
    for $j \leftarrow 0$ **to** d **do**
        Send message to neighbor(s) in sector i;
        Listen for messages from neighbor(s) (if any) in sector i;
    Rotate antenna one sector;

Algorithm 5: Random Selection Rotation Mechanism Algorithm $RSRMA(k)$

`//Choose rotation mechanism at random.`
1 Select $bit \leftarrow RANDOM(\{0, 1\})$;
2 **if** $bit = 0$ **then** Execute $Mech_0(k, k)$;
3 **if** $bit = 1$ **then** Execute $Mech_1(k, k)$;

Thus algorithm $RSRMA(u, k)$ selects the rotation mechanism at random. We can prove the following theorem.

Theorem 6. *Consider a set of n sensors in the plane with identical antenna beam width equal to $\phi = \frac{2\pi}{k}$. Assume the sensor network is synchronous. If each sensor u executes algorithm $RSRMA(u; k)$ for $O(\log n)$ times then every sensor in the network will discover all its neighbors in at most $O(k^2 \log n)$ expected time steps, with high probability.*

Proof. The proof of correctness is not difficult. The sensor flips a coin. If the outcome is $bit = 0$ (Step 2) then it rotates the antenna k rounds around the circle; in each round it rotates the antenna with no delay and sends messages and listens for messages. However, if the outcome is $bit = 1$ (Step 3) then it rotates the antenna once around the circle; in each sector it sends messages and listens for messages k times and then rotates the antenna one sector. Now consider two sensors u, v within range of each other and assume, without loss of generality, that u is to the left of v (The same proof will work regardless of the direction of the line segment uv connecting u to v). Both sensors start beaming *East*. We know that a necessary and sufficient condition to establish communication is for u's antenna to beam *East* and v's antenna to beam *West* at the same time. If both sensors' coin-flips give the same bit then the sensors will select the same rotation mechanism and their antennae will not face "against" each other. However, if their coin-flips give different bits then it is clear that their corresponding antennae will face *East* and *West*, respectively, at the same time.

Let $m = 3 \log n$ and suppose that all sensors run algorithm $RSRMA(k)$ for m times. The only case that two adjacent sensors u, v cannot communicate in m steps is that the coin flips yield identical outcomes m times. In particular we have two random binary strings of length m each one drawn from u and another from v. The probability that the strings are identical is equal to $2^{-m} = n^{-3}$ since $m = 3 \log n$.

Finally, we can prove the main result of the theorem. Let $E_{u,v}$ denote the event that sensors u, v can communicate (at some time). Consequently, from the discussion above we conclude that

$$\Pr[\neg E_{u,v}] \leq n^{-3}, \text{ for any pair } u, v \text{ of sensors.} \tag{3}$$

Therefore we obtain that the probability that any two adjacent sensors communicate is at least

$$\begin{aligned}
\Pr[\forall u, v E_{u,v}] &= 1 - \Pr[\neg(\forall u, v E_{u,v})] \\
&= 1 - \Pr[\exists u, v \neg E_{u,v}] \\
&= 1 - \Pr\left[\bigcup_{u,v} \neg E_{u,v}\right] \\
&\geq 1 - \sum_{u,v} \Pr[\neg E_{u,v}] \\
&\geq 1 - n^2 \frac{1}{n^3} \\
&= 1 - \frac{1}{n}.
\end{aligned}$$

This proves our assertion and completes the proof of Theorem 6.

3.3 Algorithm if bound on antenna beam widths is known

We now indicate how to extend Theorem 6 to the case of sensors with arbitrary antenna beam widths. First of all, we modify the rotation mechanisms by introducing the delay as a parameter.

Following the proof of Theorem 6, observe that if two adjacent sensors u, v execute the following algorithm for $m = 3 \ln n$ times then they will discover each other with high probability.

Algorithm 6: Random Selection Rotation Mechanism Algorithm $RSRMA'(k_u, d)$

```
//Choose rotation mechanism at random
```
Select $bit \leftarrow RANDOM(\{0, 1\})$;
if $bit = 0$ **then** Execute $Mech_0(k_u, d)$;
if $bit = 1$ **then** Execute $Mech_1(k_u, d)$;

This idea is for each sensor to use the neighbor sensor's antenna beam width to determine an appropriate delay. However, this will not work because sensor u (respectively, v) does not necessarily know the beam width of v's (respectively, u's) antenna. However, this difficulty is easy to resolve if an upper bound, say k, on $\max\{k_u, k_v\}$ is known by both u and v. Namely, sensor u executes algorithm $RSRMA'(k'_u, k'_v)$ and sensor v executes algorithm $RSRMA'(k'_v, k'_u)$, for all pairs (k'_u, k'_v) such $k'_u, k'_v \leq k$. To maintain synchronicity all k^2 pairs of algorithms are executed in the same lexicographic order by all pairs of sensors each algorithm for $m = 3 \ln n$ times. Clearly, the running time of the algorithm is $O(k^4 \log n)$ with high probability.

Putting these ideas together and repeating the proof of Theorem 6 it is easy to prove the following theorem.

Theorem 7. *Consider a set of n sensors in the plane such that sensor u has antenna beam width equal to $\phi_u = \frac{2\pi}{k_u}$. Assume the sensor network is synchronous and that an upper bound k is known to all sensors so that $\max_u k_u \leq k$. If each sensor u executes algorithm $RSRMA'(a, b)$, for each pair (a, b), with $a, b \leq k$, for $O(\log n)$ times then every sensor in the network will discover all its neighbors in at most $O(k^4 \log n)$ expected time steps, with high probability.* □

4 Conclusion and Open Problems

An interesting class of problems arises in considering the efficiency of broadcasting in the single channel UDG model, i.e., if first there is a single send/receive channel and multiple transmissions on the same node produce packet collisions, and second a link between two sensors u, v exists if and only if $d(u, v) \leq 1$. In general, broadcasting with omnidirectional antennae requires scheduling of transmissions (typically using *group testing* techniques) so as to avoid collisions. Clearly, if broadcasting time with omnidirectional antennae without collisions is B then the result of Theorem 3 indicates that broadcasting in the directional antennae model can be accomplished in time $O(B(c \ln c)^3)$, where c is the number of colors of a vertex coloring of the sensor network. The main question arising is whether we can improve on this time bound when using directional antennae.

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