Multiple Antennae

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Introduction

Orientation Problem

- Given a set S of sensors. Assume that each sensor has $k > 1$ directional antennae such that the sum is at most φ.

  What is the minimum range necessary to create a strongly connected network by appropiatly rotating the antennae?

- Two variants: Transmission angle (spread) is limited to φ, where φ is
  - either the **sum of angles** for antennae in the same node, or
  - the **maximum transmission angle** of the antennae.

The Setting

- Set of sensors represented as a set of points S in the 2D plane.
- Each sensor has k directional antennae.
- All antennae have the same transmission range r.
- Each antenna has a max transmission angle, forming a coverage sector up to distance r.
- Typically, we fix k and φ and try to minimize r for a given point set S.

Transmission Range

- $r_{(k,\varphi)-OPT}(S)$ denotes the optimal (shortest) range for which a solution exists.
- $r_{MST}(S)$ is the shortest range r such that UDG(S, r) is connected.
  - obviously, $r_{MST}(S) \leq r_{(k,\varphi)-OPT}(S)$
- As establishing $r_{(k,\varphi)-OPT}$ might be NP-hard, we will compare the radius r produced by a solution to r_{MST}.
  - for simplicity, we re-scale S to get $r_{MST} = 1$
  - later, we will discuss comparing to $r_{(k,\varphi)-OPT}$

Angle/Range Tradeoffs: Minimize Sum of Angles

Basic Observations

- Angle between (any two) incident edges of an MST is $\geq \pi/3$.
- For every point set there exists an MST of maximal degree 5.
- All angles incident to a vertex of degree 5 of the MST are between $\pi/3$ and $2\pi/3$ (included).
- **Observation:** with $k \geq 5$ antennae, each of spread 0, there exists a solution with range 1.
- **Main method:** Locally modify the MST, using various techniques when k is smaller than the degree of the node in the MST to (locally) ensure strong connectivity: Use
  1. antenna spread to cover several neighbors by one antenna,
  2. neighbour's antennae to locally ensure strong connectivity

Upper Bounds: Sum of Angles

#	Antennae Spread	Antennae Range	Paper
1	$0 \leq \varphi < \pi$	2	[4]
1	$\pi \leq \varphi < 8\pi/5$	$2\sin(\varphi/2)$	[2]
1	$8\pi/5 \leq \varphi$	1	[2]
2	$2\pi/3 \leq \varphi < \pi$	$2\cos(\varphi/4)$	[1]
2	$\pi \leq \varphi < 6\pi/5$	$2\sin(2\pi/9)$	[1]
2	$6\pi/5 \leq \varphi$	1	[1]
3	$4\pi/5 \leq \varphi$	1	[1]
4	$2\pi/5 \leq \varphi$	1	[1]

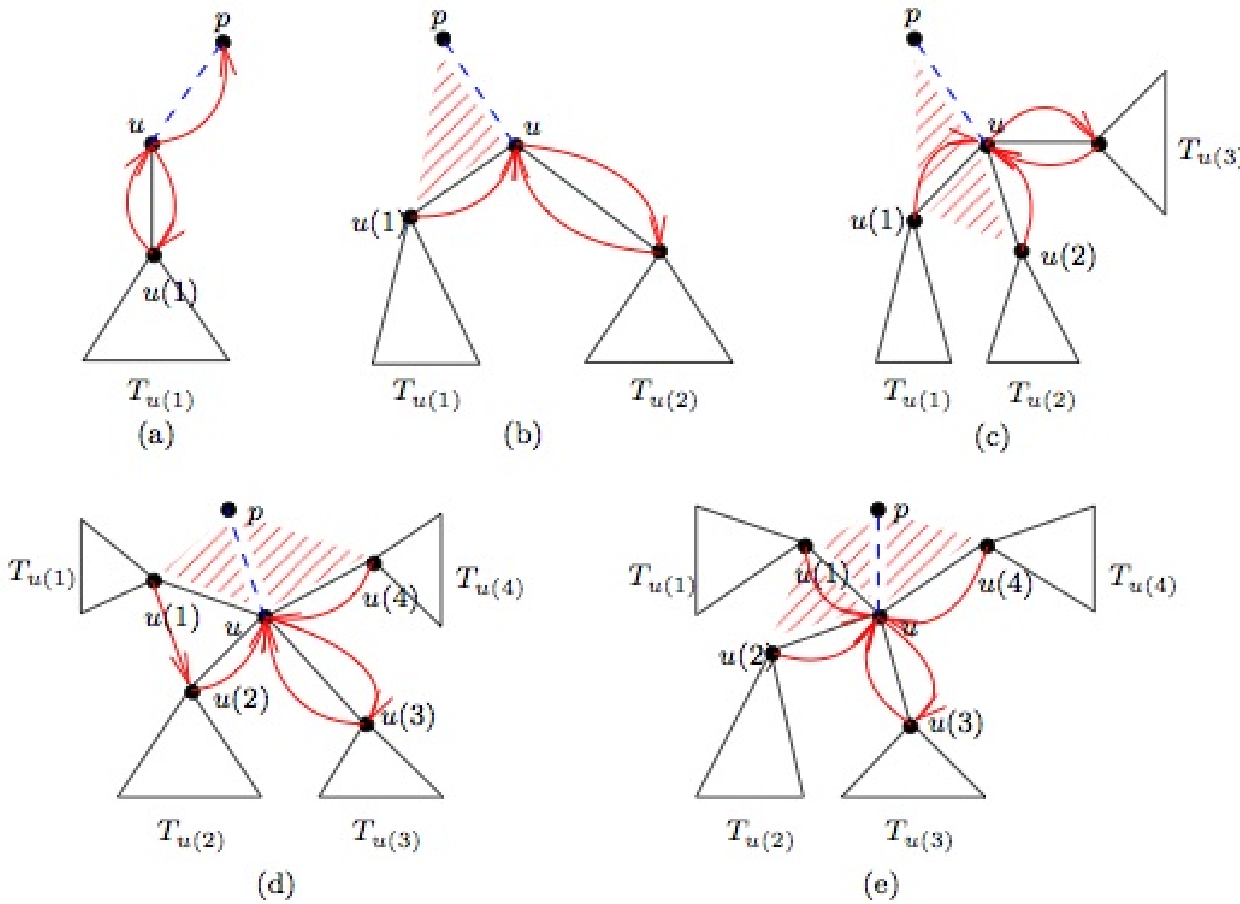
Antenna Range 1

- **Theorem.** For any $1 \leq k \leq 5$, there exists a solution with
  1. range 1, and
  2. sum of angles $\leq \frac{2(5-k)\pi}{5}$.
- Why?
- Here is the reason, briefly:
  - Consider the MST.
  - Take any vertex of degree 5 (other cases are similar).
  - Exclude k incident (consecutive) angles with sum $\leq 2k\pi/5$.
  - What is left can be covered with an antenna of angle $\leq \frac{2(5-k)\pi}{5}$ and $k-1$ antennae of angle 0 each.

Two Antennae, $\varphi \geq \pi$, Range $2\sin(2\pi/9)$

- A vertex p is a *nearby target vertex* to a vertex $v \in T$ if $d(v,p) \leq 2\sin(2\pi/9)$ and p is either a parent or a sibling of v in T.
- A subtree T_v of T is *nice* iff for any nearby target vertex p the antennae at vertices of T_v can be set up so that the resulting graph (over vertices of T_v) is strongly connected and p is covered by an antenna from v.
- **Theorem.** There is a way to set up 2 antennae per vertex, with antenna spread (i.e., sum of antenna angles) of π and range $2\sin(2\pi/9)$ in such a way that the resulting graph is strongly connected.
  - **Proof:** By proving that T_v is nice for all v, by induction on the depth of T_v.

Induction: Case Analvsis on the Number of Children of u



The length $2\sin(2\pi/9)$ arises from the fact that $\min\{\angle(u(1)uu(2)), \angle(u(2)uu(3)), \angle(u(3)u(4))\} \leq \frac{4\pi}{9}$

Angle/Range Tradeoffs: Minimize Max Range

Main Theorem (Upper Bound)

- Consider a set S of n sensors in the plane and suppose each sensor has k, $1 \leq k \leq 5$, directional antennae.
  - Then the antennae can be oriented at each sensor so that the resulting spanning graph is strongly connected and the range of each antenna is at most

    $$2 \cdot \sin\left(\frac{\pi}{k+1}\right)$$

    times the optimal.
  - Moreover, given a MST on the set of points the spanner can be constructed with additional $O(n)$ overhead.

Main Steps: Angle 0

- The more antennae per sensor the easier the proof.
- Algorithm is in three steps.
  1. 4 Antennae: Spread 0, Range $2\sin(\pi/5)$
  2. 3 Antennae: Spread 0, Range $2\sin(\pi/4)$
  3. 2 Antennae: Spread 0, Range $2\sin(\pi/3)$
- Details of complete algorithm too technical to present here!
- Lets outline the ideas for the proof of Item 1.

Main Idea: Angle 0

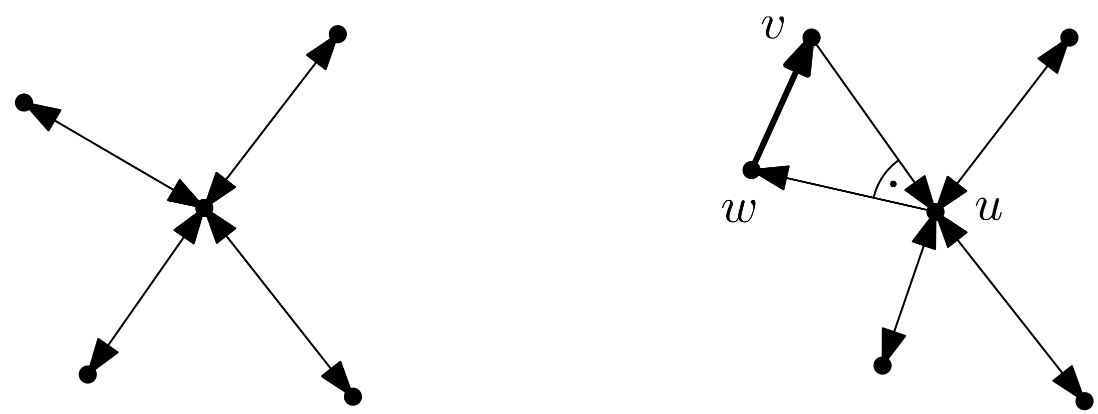
- **Idea:**
  The basic antenna orientation algorithm;
  - By induction on the depth of the MST T;
  - We avoid connecting child solutions to the parent, instead
    1. remove all leaves,
    2. apply induction hypothesis to the resulting tree,
    3. return back the leaves and show how to connect them to the original structure.
- **NB:**
  Since the spread is 0, a solution can be represented as a directed graph $\overrightarrow{G}$ with maximum out-degree k and edge lengths at most $2\sin(\frac{\pi}{k+1})$.

Example: 4 Antennae, Spread 0, Range $2\sin(\pi/5)$

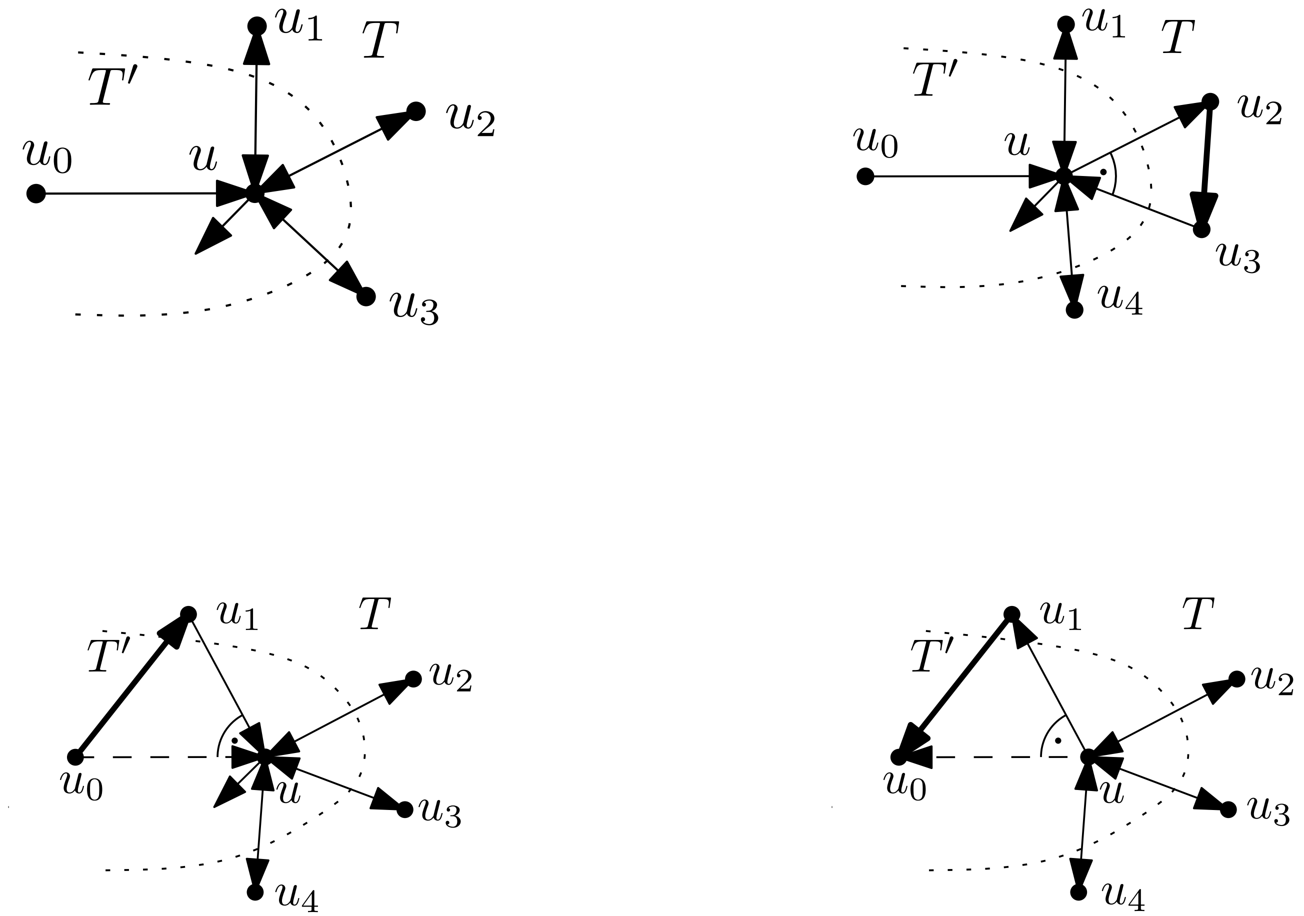
Induction hypothesis: Let T be an MST of a point set of radius at most x. Then, there exists a solution $\overrightarrow{G}$ for T such that:

- the out-degree of u in $\overrightarrow{G}$ is one for each leaf u of T
- every edge of T incident to a leaf is contained in $\overrightarrow{G}$

Base step



Inductive Step: 4 antennae, spread 0



Summary of Complete Picture: Upper Bounds

#	Antennae Spread	Antennae Range	Paper
1	$0 \leq \varphi < \pi$	2	[4]
1	$\pi \leq \varphi < 8\pi/5$	$2\sin(\pi - \varphi/2)$	[2]
1	$8\pi/5 \leq \varphi$	1	[2]
2	$0 \leq \varphi < 2\pi/3$	$\sqrt{3}$	[3]
2	$2\pi/3 \leq \varphi < \pi$	$2\sin(\pi/2 - \varphi/4)$	[1]
2	$\pi \leq \varphi < 6\pi/5$	$2\sin(2\pi/9)$	[1]
2	$6\pi/5 \leq \varphi$	1	[1]
3	$0 \leq \varphi < 4\pi/5$	$\sqrt{2}$	[3]
3	$4\pi/5 \leq \varphi$	1	[1]
4	$0 \leq \varphi < 2\pi/5$	$2\sin(\pi/5)$	[3]
4	$2\pi/5 \leq \varphi$	1	[1]

Lower Bounds

Is the Result Optimal?

- Consider a regular $k+1$-star.
  - With angle less then $\frac{2\pi}{k+1}$, the central vertex cannot reach all leaves using k antennae, hence a leaf must connect to another leaf, using range at least $2\sin(\frac{\pi}{k+1})$.
  - Hence results for spread 0 are optimal ...
  - ... with respect to r_{MST}.
- But what about $r_{(k,\varphi)-OPT}$?
- In regular $k+1$-star also $r_{(k,\varphi)-OPT}$ is large!

Main Theorem (Lower Bound)

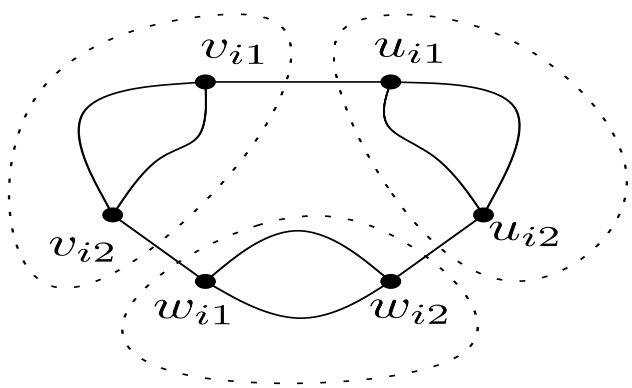
- For $k = 2$ antennae.
  - Let x and α be the solutions of equations

    $$x = 2\sin(\alpha) = 1 + 2\cos(2\alpha)$$

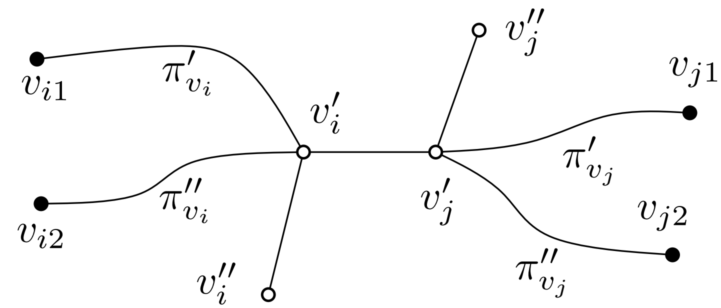
    (**Note:** $x \approx 1.30, \alpha \approx 0.45\pi$.)
  - If the angular sum of the antennae is less then α then it is NP-hard to approximate the optimal radius to within a factor of x.
- The proof is by reduction from the problem of finding Hamiltonian cycles in degree three planar graphs.

Key Gadgets

Take a degree three planar graph $G = (V, E)$ and replace each vertex v_i by a vertex-graph (meta-vertex) G_{v_i} shown in Figure 1a. Furthermore, replace each edge $e = \langle v_i, v_j \rangle$ of G by an edge-graph (meta-edge) G_e shown in Figure 1b.



(a) Vertex graph (The dotted ovals delimit the three parts.)



(b) Edge graph (The connecting vertices are black.)

Figure 1: Meta-vertex and meta-edge for the NP completeness proof

Embed Resulting Graph in the Plane:

1) Distance (in the embedding) between neighbours in G' is ≤ 1, 2) the distance between non-neighbours in G' is $\geq x$, and 3) the smallest angle between incident edges in G' is $\geq \alpha$.

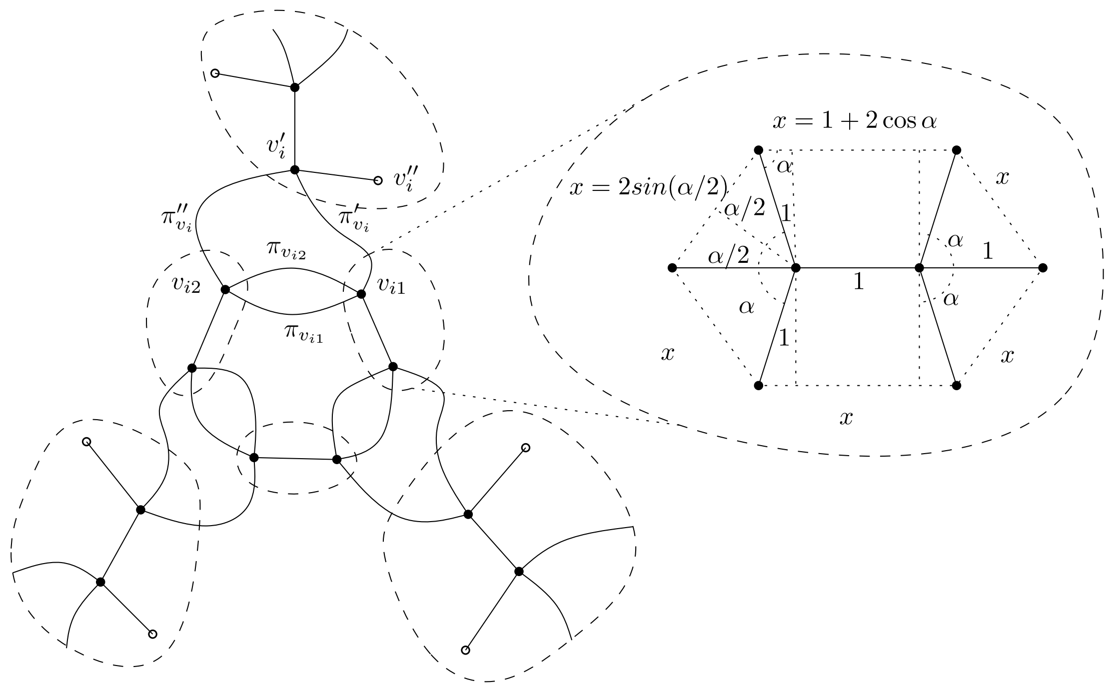


Figure 2: Connecting meta-edges with meta-vertices (The dashed ovals show the places where embedding is constrained.)

Key Observations

- Each meta-vertex must have at least incoming and one outgoing meta-edge
- Each meta-vertex can have at most one outgoing meta-edge
- Hence each meta-vertex has exactly one outgoing and one incoming meta-edge

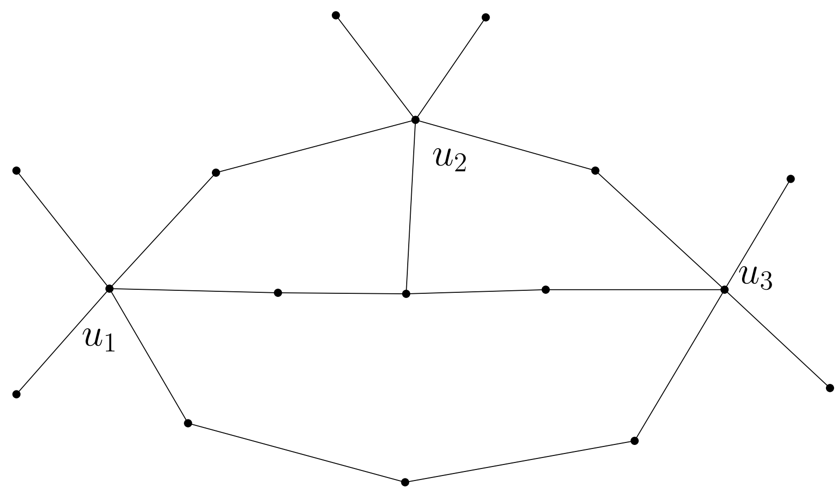
What we know so far

Out degree	Lower Bound	Upper Bound	Approx. Ratio	Complexity
4	r_{MST}	$2\sin(\pi/5)r_{MST}$	$2\sin(\pi/5)$	Polynomial
3	r_{MST}	$2\sin(\pi/4)r_{MST}$	$\sqrt{2}$	Polynomial
2	r_{MST}	$2\sin(\pi/3)r_{MST}$	$\sqrt{3}$	Polynomial
2	-	-	≤ 1.3	NP-Complete

Toughness of Antennae and Robust Range (Cases $k = 3, 4$)

Lower Bounds for $k = 3$ and $k = 4$: Main Idea

- For a pointset P: How robust is the radius r to point deletions?
- For $S \subseteq P$, let $r_k(S)$:= "smallest radius r s.t, $UDG(P \setminus S, r)$ does not contain a $(k+1)|S|$ connected components".
- Obviously, $r_k(S) \leq r_{(k,0)-OPT}(S)$. Is $r_k(S) = r_{(k,0)-OPT}(S)$?
- $r_3(S) < r_{(3,0)-OPT}(S)$! E.g., take $S = \{u_1, u_2, u_3\}$.



- How about $r_4(S) = r_{(4,0)-OPT}(S)$?

Tougness of UDGs

- The concept of toughness of a graph as a measure of graph connectivity has been extensively studied in the literature.
- Intuitively, graph toughness measures the resilience of the graph to fragmentation after subgraph removal.
- A graph G is t-tough if $|S| \geq t\omega(G \setminus S)$, for every subset S of the vertex set of G with $\omega(G \setminus S) > 1$.
- The toughness of G, denoted $\tau(G)$, is the maximum value of t for which G is t-tough (taking $\tau(K_n) = \infty$, for all $n \geq 1$).
- We are interested in the toughness of UDGs over a given point set P, and in particular how does the toughness of $U(P, r)$ depends on the radius r.

New Concept: Robust Range

Definition 1 *[Strong and Weak t-robustness for UDG radius] Let P be a set of points in the plane.*

1. *A subset $S \subseteq P$ is called t-tough if $\omega(U(P \setminus S; r)) \leq |S|/t$. Similarly, a point u is called t-tough if the singleton $\{u\}$ is t-tough.*
2. *The strong t-robustness of the set of points P, denoted by $\sigma_t(P)$, is the infimum taken over all radii $r > 0$ such that for all $S \subseteq P$, the set S is t-tough for the radius r.*
3. *The weak t-robustness of the set of points P, denoted by $\alpha_t(P)$, is the infimum taken over all radii $r > 0$ such that for all $u \in P$, the point u is t-tough for the radius r.*

Main Result

- **Theorem.** We have
  1. $\sigma_{1/k}(P) \leq r_k(P)$, for all k.
  2. For any set P of points, $\alpha_{1/4}(P) = \sigma_{1/4}(P)$.
  3. For every point of P, weak $1/i$-robustness, for $1 \leq i < 5$, can be computed in time $O(|P| \log |P|)$.
- In particular,
  1. the optimal range for the 4 antennae orientation problem (strong connectivity) can be solved in $O(n \log n)$ time,
  2. a $2 \sin(2\pi/9)$ approximation to the optimal range for the 3 antennae orientation problem can be solved in $O(n \log n)$ time.

Summary of Results

Out degree	Lower Bound	Upper Bound	Approx. Ratio	Complexity
4	$\sigma_{1/4}$	$\alpha_{1/4}$	1	$O(n \log n)$
3	$\sigma_{1/3}$	$2\sin(2\pi/9)\alpha_{1/3}$	$\leq 2\sin(2\pi/9)$	$O(n \log n)$

Conclusions/Open Problems

- There are still gaps between the lower and upper bounds, especially for non-zero φ
- The x and φ in the NP-hardness results might possibly be improved
- Consider different model variants
  - directional receivers
  - temporal aspects (antennae steering, ...)
- and different problems...

Minimum Number of Antennae

Antenna Orientation Problem

- Given a connected network formed by a set of sensors with omnidirectional antennae and an angle $\varphi \geq 0$.

  Compute the minimum number of arcs in the network in such a way that the resulting network is strongly connected and the stretch factor does not depend on the size of the network.

- Two variants:
  - Notice that you must respect the underlying network.
  - Can consider angle/range tradeoffs.

Orienting Edges of Undirected Graph with Original Range

- Orient every edge in both directions
  - stretch factor 1 but $2|E|$ arcs
- Orient edges along a Hamiltonian cycle (if it exists)
  - $|V|$ arcs but unbounded stretch factor
- **(Roberts, 1935)** Strong Orientation Procedure
  1. label vertices $1..n$ according to DFT T
  2. orient ij as $i \to j$ iff $ij \in T$ and $i < j$
  3. orient ij as $i \to j$ iff $ij \notin T$ and $i > j$
- **(Robbins, 1939)** G has a strong orientation iff it is connected and 2-edge connected.
- **(Nash-Williams, 1960)** Every G has an orientation D so that $\forall u, v \in V$, $\lambda_D(u,v) \geq \lfloor \frac{1}{2} \lambda_G(u,v) \rfloor$, where $\lambda(u,v)$ is the number of $u - v$ paths

Strong Orientation Algorithms

Can give algorithms to strongly orient a given (planr) graph $G = (V, E)$ for[a]

- More than $|E|$ edges
- Exactly $|E|$ edges
- Less than $|E|$ edges

[a]E. Kranakis, O. Morales Ponce, L. Stacho. Strong Orientations of Planar Graphs with Bounded Stretch Factor. In proceedings of 17th SIROCCO (Colloquium on Structural Information and Communication Complexity) 2010. LNCS 6058 Springer.

Orientation Algorithms (More than $|E|$ Edges)

- **Theorem.** Let $G = (V, E)$ be a plane 2-edge connected graph with a face λ-coloring. Then it has a strong orientation with at most
$$\left(2 - \frac{4\lambda - 6}{\lambda(\lambda - 1)}\right) \cdot |E|$$
arcs and stretch factor at most $\phi(G) - 1$, where $\phi(G) =$ max number of edges of a face of G.

Orientation Algorithms (Exactly $|E|$ Edges)

- **Theorem.** Let $G = (V, E)$ be a plane 2-edge connected graph with a face λ-coloring. Then it has a strong orientation with exactly $|E|$ arcs and stretch factor at most

$$(\phi(G) - 1)^{\lceil \frac{\lambda+1}{2} \rceil}.$$

Orientation Algorithms (Less than $|E|$ Edges)

- **Theorem.** Let $G = (V, E)$ be a plane 3-edge connected graph. Then it has a strong orientation with at most

$$\left(1 - \frac{k}{10(k+1)}\right) \cdot |E|$$

arcs and stretch factor at most $\phi(G)^2 \cdot (\phi(G) - 1)^{2k+4}$, for any $k \geq 1$.

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