Toughness in Graphs

A Survey

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Abstract. In this survey we have attempted to bring together most of the results and papers that deal with toughness related to cycle structure. We begin with a brief introduction and a section on terminology and notation, and then try to organize the work into a few self explanatory categories. These categories are circumference, the disproof of the 2-tough conjecture, factors, special graph classes, computational complexity, and miscellaneous results as they relate to toughness. We complete the survey with some tough open problems!

Key words. toughness, t-tough graph, Hamilton cycle, hamiltonian graph, traceable graph, circumference, factor, k-factor, chordal graph, triangle-free graph, planar graph, computational complexity

1. Introduction

More than 30 years ago Chvátal [67] introduced the concept of toughness. Since then a lot of research has been done, mainly relating toughness conditions to the existence of cycle structures. Historically, most of the research was based on a number of conjectures in [67]. The most challenging of these conjectures is still open: Is there a finite constant t_0 such that every t_0-tough graph contains a cycle through all of its vertices? For a long time it was believed that this conjecture should hold for $t_0 = 2$. This '2-tough conjecture' would then imply a number of related results and conjectures which we will present later. But in 2000, it was shown [13] that the 2-tough conjecture is false. On the other hand, we now know that the more general t_0-tough conjecture is true for a number of graph classes, including planar graphs, claw-free graphs, and chordal graphs, to name just a few. The early research in this area concentrated on sufficient degree conditions which, combined with a certain level of toughness, would yield the existence of long cycles. Another stream involved finding toughness conditions for the existence of certain factors in graphs. Research on toughness has also focused on computational complexity issues. In particular, we now know that it is NP-hard to compute the toughness of a graph [16].

For the last four Kalamazoo conferences [30, 32, 33, 10], we surveyed results on toughness and its relationship to cycle structure. In this extended survey we have attempted to bring together most of the results that deal with toughness related to cycle structure. As was true in our previous Kalamazoo surveys, the present survey is undoubtedly not comprehensive. To be fair to everyone, we will force ourselves to omit some of our own results.

We begin with a brief section on terminology and notation and then try to organize the work into a few self explanatory categories. Many of the results fit easily into more than one category. These categories are circumference, the disproof of the 2-tough conjecture, factors, special graph classes, computational complexity, miscellaneous results, and open problems as they relate to toughness.

2. Terminology

Much of the background for this survey can be found in [30, 32, 33, 10]. A good reference for any undefined terms in graph theory is [60] and in complexity theory is [93]. We consider only undirected graphs with no loops or multiple edges. The definitions and terminology presented below will appear often in the sequel. Other definitions will be given later as needed.

Let $\omega(G)$ denote the number of components of a graph G. A graph G is t-*tough* if $|S| \geq t\omega(G-S)$ for every subset S of the vertex set $V(G)$ with $\omega(G-S) > 1$. The *toughness* of G, denoted $\tau(G)$, is the maximum value of t for which G is t-tough (taking $\tau(K_n) = \infty$ for all $n \geq 1$). Hence if G is not complete, $\tau(G) = \min\{|S|/\omega(G-S)\}$, where the minimum is taken over all cutsets of vertices in G. In [145], Plummer defined a cutset $S \subseteq V(G)$ to be a *tough set* if $\tau(G) = |S|/\omega(G-S)$, i.e., a cutset $S \subseteq V(G)$ for which this minimum is achieved. We let $\alpha(G)$ denote the cardinality of a maximum set of independent vertices of G, and $c(G)$ denote the *circumference* of G, i.e., the length of a longest cycle in G. The *girth* of G is the length of a shortest cycle in G. We use $\kappa(G)$ for the *vertex connectivity* of G and $\gamma(G)$ to denote the *genus* of G. A graph G is *hamiltonian* if G contains a *Hamilton cycle*, i.e., a cycle containing every vertex of G; G is *traceable* if G contains a *Hamilton path*, i.e., a path containing every vertex of G; G is *pancyclic* if G contains cycles of every length between 3 and $|V(G)|$. A *dominating cycle* of G is a cycle C of G such that $G - V(C)$ is an independent set, i.e., such that every edge of G has at least one of its endvertices on C. A k-*factor* of a graph is a k-regular spanning subgraph. Of course, a Hamilton cycle is a (connected) 2-factor. We say G is *chordal* if it contains no chordless cycle of length at least four and k-*chordal* if a longest chordless cycle in G has length at most k. We use $N(v)$ to denote the set of neighbors of vertex v, $d(v) = |N(v)|$ to denote the degree of vertex v, and $\delta(G)$ for the minimum degree in G. For $k \leq \alpha(G)$, we use $\sigma_k(G)$ to denote the minimum degree sum taken over all independent sets of k vertices of G, and $NC_k(G)$ to denote the minimum cardinality of the union of the neighborhoods of any k such vertices. For $k > \alpha(G)$, we set $\sigma_k(G) = k(n - \alpha(G))$ and $NC_k(G) = n - \alpha(G)$, where $n = |V(G)|$. If G has a noncomplete component, we let $NC2(G)$ denote the cardinality of the minimum neighborhood union of any pair of vertices at distance two apart; otherwise $NC2(G) = n - 1$. We use dist(x, y) to denote the distance between two vertices x and y in a connected graph G, i.e., the length of a shortest path in G between x and y. If no ambiguity can arise we often omit the reference to the graph G, e.g., we use E for the edge set $E(G)$, etc. We use $\log$ for the logarithm with base 2, and $\ln$ for the natural logarithm.

3. Toughness and Circumference

In this section we survey results concerning the relationship between the toughness of a graph and its circumference. We begin our discussion with a well-known theorem of Dirac [77].

Theorem 1. *Let G be a graph on $n \geq 3$ vertices with $\delta \geq \frac{n}{2}$. Then G is hamiltonian.*

A long cycle version of Theorem 1 was also proved by Dirac.

Theorem 2. *Let G be a 2-connected graph on n vertices. Then $c(G) \geq \min\{n, 2\delta\}$.*

In 1960 Ore [141] generalized Theorem 1 as follows.

Theorem 3. *Let G be a graph on $n \geq 3$ vertices with $\sigma_2 \geq n$. Then G is hamiltonian.*

A long cycle version of Theorem 3 was later established independently by Bondy [42], Bermond [36], and Linial [130].

Theorem 4. *Let G be a 2-connected graph on n vertices. Then $c(G) \geq \min\{n, \sigma_2\}$.*

It is clear from the definition that being 1-tough is a necessary condition for a graph to be hamiltonian. A natural question, answered by Jung in 1978 [116], is how much the lower bound $\sigma_2 \geq n$ in Ore's Theorem can be weakened under the assumption that G is 1-tough.

Theorem 5. *Let G be a 1-tough graph on $n \geq 11$ vertices with $\sigma_2 \geq n - 4$. Then G is hamiltonian.*

The original proof of Theorem 5 in [116] is rather complicated. A much simpler proof for $n \geq 16$ appears in [23].

It is reasonable to consider a long cycle version of Jung's Theorem. The first step in this direction was taken by Ainouche and Christofides [1].

Theorem 6. *Let G be a 1-tough graph on $n \geq 3$ vertices. Then $c(G) \geq \min\{n, \sigma_2 + 1\}$.*

They also conjectured that $\sigma_2 + 1$ in Theorem 6 could be replaced by $\sigma_2 + 2$, and their conjecture was established by Bauer and Schmeichel [27].

Theorem 7. *Let G be a 1-tough graph on $n \geq 3$ vertices. Then $c(G) \geq \min\{n, \sigma_2 + 2\}$.*

In [21], this was strengthened by providing a characterization of the 2-connected graphs G with $c(G) < \sigma_2 + 2$, and noting they were not 1-tough.

In [14] it was shown that the degree bound in Jung's Theorem can be slightly lowered if $\tau(G) > 1$.

Theorem 8. *Let G be a graph on $n \geq 30$ vertices with $\tau > 1$. If $\sigma_2 \geq n - 7$, then G is hamiltonian.*

Theorem 8 is best possible with respect to the bound on σ_2. Later, the nonhamiltonian 1-tough graphs for which $\sigma_3 \geq (3n - 24)/2$ were characterized [117]. Theorem 8 also follows from this characterization.

While Theorem 7 is best possible, a stronger result may be obtained if $\sigma_2 \geq 2n/3$ (See Theorem 17). A useful intermediate result concerns the existence of a longest cycle which is also a dominating cycle. Since dominating cycles have played such a useful role in the early results on toughness and cycle structure, we digress to discuss them. They were first introduced by Nash-Williams in [139] and were later studied in detail by Veldman [155].

In [139], Nash-Williams proved the following.

Theorem 9. *Let G be a 2-connected graph on n vertices with $\delta \geq (n+2)/3$. Then every longest cycle in G is a dominating cycle.*

The next result follows easily [139].

Theorem 10. *Let G be a 2-connected graph with $\delta \geq \max\{(n+2)/3, \alpha\}$. Then G is hamiltonian.*

In 1980 Bondy [44] generalized Theorem 9.

Theorem 11. *Let G be a 2-connected graph on n vertices with $\sigma_3 \geq n+2$. Then every longest cycle in G is a dominating cycle.*

An analogous generalization of Theorem 10 occurs in [44].

Theorem 12. *Let G be a 2-connected graph on n vertices with $\sigma_3 \geq \max\{n+2, 3\alpha\}$. Then G is hamiltonian.*

Theorem 12 is an immediate consequence of the following result, established in [25].

Theorem 13. *Let G be a 2-connected graph on n vertices with $\sigma_3 \geq n+2$. Then $c(G) \geq \min\{n, n + \sigma_3/3 - \alpha\}$.*

In [25], Theorem 13 is proved by combining Theorem 11 with a technical lemma, which we state explicitly because of its central role in proofs of several results in this survey.

Lemma 1. *Let G be a graph on n vertices with $\delta \geq 2$ and $\sigma_3 \geq n$. Suppose G contains a longest cycle C which is a dominating cycle and v is a vertex in $V(G) - V(C)$. With respect to some orientation of C, let S be the set of immediate successors on C of the vertices adjacent to v. Then $(V(G) - V(C)) \cup S$ is an independent set of vertices.*

The above results on dominating cycles all assume that G is 2-connected. If instead G is assumed to be 1-tough, the bounds in Theorems 9 - 11 can be improved. The next two theorems are due to Bigalke and Jung [39].

Theorem 14. *Let G be a 1-tough graph on n vertices with $\delta \geq n/3$. Then every longest cycle in G is a dominating cycle.*

Theorem 15. *Let G be a 1-tough graph on $n \geq 3$ vertices with $\delta \geq \max\{n/3, \alpha - 1\}$. Then G is hamiltonian.*

We close this digression on dominating cycles with the following generalization of Theorem 14 appearing in [25].

Theorem 16. *Let G be a 1-tough graph on n vertices with $\sigma_3 \geq n$. Then every longest cycle in G is a dominating cycle.*

We now return to our discussion of toughness and circumference. By combining Lemma 1 with Theorem 16, a result similar to Theorem 13 was proved in [25].

Theorem 17. *Let G be a 1-tough graph on $n \geq 3$ vertices with $\sigma_3 \geq n$. Then $c(G) \geq \min\{n, n + \sigma_3/3 - \alpha\}$.*

It is easy to see that $\alpha \leq n/(\tau + 1)$. In particular, $\alpha \leq n/2$ for any 1-tough graph. Thus if G is a 1-tough graph on $n \geq 3$ vertices with $\delta \geq n/3$, then $c(G) \geq 5n/6$ by Theorem 17. Note that from Theorem 7 we could only conclude that $c(G) \geq 2n/3 + 2$. If G is 2-tough, then $\alpha \leq n/3$. So an immediate corollary of Theorem 17 is the following result from [25].

Corollary 1. *Let G be a 2-tough graph on $n \geq 3$ vertices. If $\sigma_3 \geq n$, then G is hamiltonian.*

Hoa [107] later showed that under the hypothesis of Theorem 17, $c(G) \geq \min\{n, n + \sigma_3/3 - \alpha + 1\}$. Since $\alpha \leq n/2$ for any 1-tough graph, we may conclude that under the hypothesis of Theorem 17, $c(G) \geq 5n/6 + 1$. Using a clever variation of Woodall's Hopping Lemma [160], Li [128] was able to improve on this result.

Theorem 18. *If G is a 1-tough graph on $n \geq 3$ vertices with $\delta \geq n/3$, then*

$$c(G) \geq \min\left\{n, \frac{2n+1+2\delta}{3}, \frac{3n+2\delta-2}{4}\right\} \geq \min\left\{\frac{8n+3}{9}, \frac{11n-6}{12}\right\}.$$

We do not believe, however, that this is best possible.

Conjecture 1. Let G be a 1-tough graph on $n \geq 3$ vertices with $\sigma_3 \geq n$. Then $c(G) \geq \min\{n, (3n+1)/4 + \sigma_3/6\}$.

Conjecture 1 if true is best possible, as can be seen by the examples given in [25]. We omit the details. Note that the truth of Conjecture 1 would allow us to conclude that under its hypothesis $c(G) \geq (11n+3)/12$. However the gap between $(8n+3)/9$ and $(11n+3)/12$ remains. The truth of Conjecture 1 would also imply the following generalization of Jung's Theorem, which was established by Faßbender [87].

Theorem 19. *Let G be a 1-tough graph on $n \geq 13$ vertices with $\sigma_3 \geq (3n-14)/2$. Then G is hamiltonian.*

Theorem 19 was conjectured in [25], and its proof relies on a result in [25]. This result has had a number of applications [11] and so we recall it now.

Theorem 20. *Let G be a 1-tough graph on n vertices with $\sigma_3 \geq n \geq 3$. Then every longest cycle in G is a dominating cycle. Moreover, if G is not hamiltonian, then G contains a longest cycle C such that* $\max\{d(v) \mid v \in V(G) - V(C)\} \geq \sigma_3/3$.

Examples in [25] show that the lower bound on σ_3 in Theorem 20 cannot be reduced. One important application of Theorem 20 is Theorem 17.

The next result of Li from [129] is also related to Theorem 17. It concerns long cycles through specified vertex sets in a 1-tough graph. Let G be a graph of order n and let $X \subseteq V(G)$. Denote by $G[X]$ the subgraph of G induced by X. Let $\alpha(X)$ be the number of vertices of a maximum independent set of $G[X]$, and $\sigma_k(X)$ the minimum degree sum in G of k independent vertices in X. A cycle C of G is called X*-longest* if no cycle of G contains more vertices of X than C, and C is called X*-dominating* if all neighbors of each vertex of $X - V(C)$ are on C.

The main result in [129] is the following extension of a result by Bauer et al. [25].

Theorem 21. *Let G be a 1-tough graph on n vertices and $X \subseteq V(G)$. If $\sigma_3(X) \geq n$, then G has an X-longest cycle C such that C is an X-dominating cycle and* $|V(C) \cap X| \geq \min\{|X|, |X| + \sigma_3(X)/3 - \alpha(X)\}$.

As we have seen, the use of dominating cycles to obtain long cycles has led to a number of interesting results. The results we have discussed so far all involved vertex degrees. By considering neighborhood unions (see [127] for early work in this area) it was possible to strengthen the conclusion of Theorem 17.

Define

$$\varepsilon(i) := \begin{cases} 0, & \text{if } i \equiv 0 \bmod 3 \\ 2, & \text{if } i \equiv 1 \bmod 3 \\ 1, & \text{if } i \equiv 2 \bmod 3. \end{cases}$$

The following appeared in [56].

Theorem 22. *Let G be a 1-tough graph on n vertices with $\sigma_3 \geq n + r \geq n \geq 3$. Then* $c(G) \geq \min\{n, n + NC_{r+5+\varepsilon(n+r)} - \alpha\}$.

In [56], it is shown that the lower bound on $c(G)$ and the subscript of NC in the conclusion of Theorem 22 cannot be increased in general. Since $NC_t(G)$ is a nondecreasing function of t and $NC_3 \geq \sigma_3/3$, Theorem 22 implies Theorem 17. Since also $NC_k \leq n - \alpha$, the following corollary, which slightly strengthens a result in [110], follows easily from Theorem 22. It also implies Theorem 19.

Corollary 2. *Let G be a 1-tough graph on n vertices with $\sigma_3 \geq n + r \geq n \geq 3$. Then* $c(G) \geq \min\{n, 2NC_{r+5+\varepsilon(n+r)}\}$.

It is also shown in [56] that the subscript of NC in the above corollary can be replaced by $\lfloor (n + 6r + 17)/8 \rfloor$, yielding an improvement if $r \leq n/2 - 19$.

A result closely related to Corollary 2 appeared in [15], where the conclusion is in terms of $NC2$ rather than NC_k.

Theorem 23. *Let G be a 1-tough graph on n vertices with $\sigma_3 \geq n \geq 3$. Then* $c(G) \geq \min\{n, 2NC2\}$.

In [15], it was conjectured that the conclusion of Theorem 23 can be replaced by $c(G) \geq \min\{n, 2NC2 + 4\}$.

We now continue with applications of Theorem 17 that do not involve neighborhood unions. Since clearly $\sigma_3 \geq 3\delta$ and $\alpha \leq n/(\tau + 1)$, we have the next result.

Theorem 24. *Let G be a t-tough graph on $n \geq 3$ vertices, where $1 \leq t \leq 2$. If $\delta > n/(t+1) - 1$, then G is hamiltonian.*

Notice that for this result to follow from Theorem 17 it is essential that $\tau \leq 2$. However this requirement can be removed, as shown in [6].

Theorem 25. *Let G be a t-tough graph on $n \geq 3$ vertices with $\delta > n/(t + 1) - 1$. Then G is hamiltonian.*

Thus Chvátal's Conjecture that there exists a finite constant t_0 such that all t_0-tough graphs are hamiltonian is true within the class of graphs having $\delta(G) \geq \epsilon n$, for any fixed $\epsilon > 0$.

Jung and Wittmann [118] established a long cycle analogue of Theorem 25 generalizing both Theorem 2 and Theorem 25.

Theorem 26. *Let G be a 2-connected t-tough graph on n vertices. Then* $c(G) \geq \min\{n, (t + 1)\delta + t\}$.

Another result in [6] related to Theorem 25 concerns the existence of a dominating cycle.

Theorem 27. *Let G be a t-tough graph $(t \geq 1)$ on $n \geq 3$ vertices with $\delta > n/(t + 2)$. Then G contains a dominating cycle.*

We now give a sufficient condition for a 1-tough graph G to be hamiltonian based on the vertex connectivity $\kappa(G)$ of G. The background for this result begins with a theorem of Häggkvist and Nicoghossian [104].

Theorem 28. *Let G be a 2-connected graph on n vertices with $\delta \geq (n+\kappa)/3$. Then G is hamiltonian.*

This obviously represents a great improvement over Dirac's Theorem for 2-connected graphs with small vertex connectivity. Theorem 28 was generalized in [8].

Theorem 29. *Let G be a 2-connected graph on n vertices with $\sigma_3 \geq n+\kappa$. Then G is hamiltonian.*

Both theorems are best possible, but not for 1-tough graphs. Bauer and Schmeichel [28] have established a result analogous to Theorem 28 for 1-tough graphs.

Theorem 30. *Let G be a 1-tough graph on $n \geq 3$ vertices with $\delta \geq (n+\kappa-2)/3$. Then G is hamiltonian.*

There are graphs to show that Theorem 30 is best possible when $\kappa = 2$, or when $\kappa = (n-5)/2 \geq 11$.

Wei [157] generalized Theorem 30 to the natural degree sum counterpart.

Theorem 31. *Let G be a 1-tough graph on $n \geq 3$ vertices with $\sigma_3 \geq n+\kappa-2$. Then G is hamiltonian.*

Hoa [108] was able to show that a 1-tough graph with $\sigma_2 \geq n+\kappa-\alpha$ is hamiltonian. He also established a number of results on the length of longest dominating cycles [109].

Many results involving long cycles rely on large degree sums of independent vertices. In [51], Brandt and Veldman showed that if a 1-tough graph G on $n \geq 2$ vertices satisfies $d(u)+d(v) \geq n$ for every edge $uv \in E(G)$, then G is pancyclic or $G = K_{n/2,n/2}$. The reader might find it interesting to compare this to a classical result of Bondy [43]. We omit the details.

We now present a result of Hoàng [111] that finds a Hamilton cycle in a t-tough graph based on the degree sequence of the graph. It generalizes the following well-known result of Chvátal [65].

Theorem 32. *Let G be a graph with degree sequence $d_1 \leq d_2 \leq \ldots \leq d_n$. If for all integers i with $1 \leq i < n/2$, $d_i \leq i$ implies $d_{n-i} \geq n-i$, then G is hamiltonian.*

Theorem 33. *Let $t \in \{1,2,3\}$ and let G be a t-tough graph with degree sequence $d_1 \leq d_2 \leq \ldots \leq d_n$. If for all integers i with $t \leq i < n/2$, $d_i \leq i$ implies $d_{n-i+t} \geq n-i$, then G is hamiltonian.*

In [32] we also discussed the notion of path-tough graphs. A graph G is *path-tough* if for every nonempty set S of vertices, the graph $G-S$ can be covered by at most $|S|$ vertex disjoint paths. Being path-tough is a necessary condition for a graph to be hamiltonian. In addition, every path-tough graph is 1-tough. A number of results on path-tough graphs appeared in [72]. In particular, it was shown that it is NP-complete to determine if a graph is path-tough. They also proved the following.

Theorem 34. *Let G be a path-tough graph on $n \geq 3$ vertices. If $\delta \geq \frac{3}{6+\sqrt{3}}n$, then G is hamiltonian.*

Schiermeyer [147] obtained the following counterpart of Jung's Theorem for path-tough graphs.

Theorem 35. *Let G be a path-tough graph on $n \geq 3$ vertices with $\sigma_2 > 4(n - 6/5)/5$. Then G is hamiltonian.*

This improved an earlier result of Häggkvist [103].

A new type of sufficient degree condition for a graph to be hamiltonian was introduced by Fan [86] in the following theorem. Note that in Theorem 3 it is necessary to examine the degrees of each pair of nonadjacent vertices. In Theorem 36 below it is only necessary to check the degrees of pairs of vertices at distance 2 apart. Theorem 36 has led to many new and interesting results in hamiltonian graph theory.

Theorem 36. *Let G be a 2-connected graph on n vertices. If for all vertices x, y, dist$(x, y) = 2$ implies $\max\{d(x), d(y)\} \geq n/2$, then G is hamiltonian.*

We can weaken the degree condition in Fan's Theorem when G is 1-tough. The following two theorems in [12] exemplify such results.

Theorem 37. *Let G be a 1-tough graph on $n \geq 3$ vertices such that $\sigma_3 \geq n$. If for all vertices x, y, dist$(x, y) = 2$ implies $\max\{d(x), d(y)\} \geq (n - 4)/2$, then G is hamiltonian.*

Theorem 37 is best possible in the sense that neither of the two degree conditions can be relaxed. Another result in [12] shows that the condition on σ_3 in Theorem 37 can be dropped completely if G is required to be 3-connected with enough vertices.

Theorem 38. *Let G be a 3-connected 1-tough graph on $n \geq 35$ vertices. If for all vertices x, y, dist$(x, y) = 2$ implies $\max\{d(x), d(y)\} \geq (n - 4)/2$, then G is hamiltonian.*

We do not believe that the requirement $n \geq 35$ in Theorem 38 is best possible.

Until now, the results in this section have all included an assumption concerning the vertex degrees or neighborhood unions. We now examine what is known about the circumference of a t-tough graph if no assumption is made regarding vertex degrees or neighborhood unions. First observe that if G is a k-connected graph on $n \geq 2k$ vertices, then $c(G) \geq 2k$. The graph $K_{k,n-k}$ $(n \geq 2k \geq 4)$ shows this is best possible, regardless of the size of n. However for t-tough graphs with $t > 0$, the situation is different. Let $\gamma_k(t, n) = \min\{c(G) \mid G$ is a k-connected, t-tough graph on n vertices$\}$. The following appears in [54].

Theorem 39. *Let $t > 0$ be fixed. Then $\gamma_2(t, n) \cdot \log(\gamma_2(t, n)) \geq (2 - o(1)) \log n \ \ (n \to \infty)$.*

Examples in [54] also show that for $0 < t \leq 1$, $\gamma_2(t, n) = O(\log n)$.
An important corollary of Theorem 39 is given below.

Corollary 3. *Let $t > 0$ be fixed. Then $\lim_{n \to \infty} \gamma_2(t, n) = \infty$.*

A stronger result than Theorem 39 can be obtained for 3-connected graphs [54].

Theorem 40. *Let $t > 0$ be fixed. Then* $\gamma_3(t, n) \geq \left(\dfrac{4}{5 \log((1/t) + 1)} - o(1) \right) \log n \ (n \to \infty)$.

It is shown in [54] that for $t \leq 1$, Theorem 40 is essentially best possible. This leads to the following conjecture, also in [54].

Conjecture 2. There is a positive constant A, depending only on t, such that for $t > 0$, $\gamma_2(t, n) \geq A \log n$.

While Conjecture 2 has still not been settled, progress has been made with respect to planar graphs. Let G be a planar graph of order n. Tutte [154] proved that if G is 4-connected, then G is hamiltonian, while Jackson and Wormald [115] showed that if G is 3-connected, then $c(G) \geq \rho n^{\theta}$ for certain positive constants ρ and θ. Now assume $\kappa = 2$. Regardless of the value of n, the circumference of G may be as small as 4 (consider $K_{2,n-2}$ for $n \geq 4$). However, by imposing a toughness condition a lower bound on the circumference of G can be derived which is logarithmic in n [40].

Theorem 41. *Let G be a planar graph of order n and connectivity 2 such that $\omega(G - S) \leq \xi$ for every subset S of $V(G)$ with $|S| = 2$. Then $c(G) \geq \psi \left(\frac{1}{\xi-1}\right)^{0.4}$ ln n, where $\psi \approx 0.10$.*

Corollary 4. *Let G be a planar graph of order n and connectivity 2. Then $c(G) \geq \psi(\frac{\tau}{2-\tau})^{0.4}$ln n.*

Examples in [40] show that under the hypotheses of Theorem 41 and Corollary 4 there is no hope for a super-logarithmic lower bound on $c(G)$. However Corollary 4 shows that Conjecture 2 above is true for planar graphs.

Another interesting problem, raised by Jackson in [112], is to determine whether $\gamma_3(t, n) \geq n^{\eta}$ for some positive constant η depending only on t. Bondy and Simonovits [46] have constructed examples to show that for $t = 3/2$, if such a constant η exists, $\eta \leq \log 8/\log 9$.

4. The disproof of the 2-tough conjecture

As noted earlier, being 1-tough is a necessary condition for a graph to be hamiltonian. In [67], Chvátal conjectured that there exists a finite constant t_0 such that every t_0-tough graph is hamiltonian. He showed in [67] that there exist $\frac{3}{2}$-tough nonhamiltonian graphs, and later Thomassen [[37], p. 132] found t-tough nonhamiltonian graphs with $t > \frac{3}{2}$. Later Enomoto et al. [84] have found $(2 - \epsilon)$-tough graphs having no 2-factor for arbitrary $\epsilon > 0$.

For many years, the focus was on determining whether all 2-tough graphs are hamiltonian. One reason for this is that if all 2-tough graphs were hamiltonian, a number of important consequences [5] would follow. In addition, the results of Enomoto et al. [84] below seemed to indicate that two might be the threshold for toughness that would imply hamiltonicity. The truth of the 2-tough conjecture would also imply the well-known result of Fleischner [92] that the square of any 2-connected graph is hamiltoninan. Moreover, it would imply the truth of two other conjectures that have been open for about twenty years: Every 4-connected line graph is hamiltonian [152], and every 4-connected claw-free graph is hamiltonian [135]. These conjectures have recently been shown to be equivalent [146]. However, it turns out that not all 2-tough graphs are hamiltonian. Indeed, we have the following result [13].

Theorem 42. *For every $\epsilon > 0$, there exists a $(\frac{9}{4} - \epsilon)$-tough nontraceable graph.*

We now give a brief outline of the construction of these counterexamples, which were inspired by constructions in [5] and [29].

For a given graph H and $x, y \in V(H)$ we define the graph $G(H, x, y, l, m)$ as follows. Take m disjoint copies $H_1, \ldots, H_m$ of H, with x_i, y_i the vertices in H_i corresponding to the vertices x and y in H ($i = 1, \ldots, m$). Let F_m be the graph obtained from $H_1 \cup \ldots \cup H_m$ by adding all possible edges between pairs of vertices in $\{x_1, \ldots, x_m, y_1, \ldots, y_m\}$. Let $T = K_l$ and let $G(H, x, y, l, m)$ be the join $T \vee F_m$ of T and F_m.

The proof of the following theorem occurred in [13] and almost literally also in [5].

Theorem 43. *Let H be a graph and x, y two vertices of H which are not connected by a Hamilton path of H. If $m \geq 2l+3$, then $G(H, x, y, l, m)$ is nontraceable.*

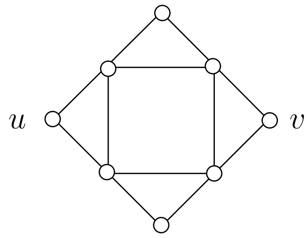

Figure 1. The graph L.

Consider the graph L of Figure 1. There is obviously no Hamilton path in L between u and v. Hence $G(L, u, v, l, m)$ is nontraceable for every $m \geq 2l+3$. The toughness of these graphs was established in [13].

Theorem 44. *For $l \geq 2$ and $m \geq 1$,*

$$\tau(G(L, u, v, l, m)) = \frac{l+4m}{2m+1}.$$

Combining Theorems 43 and 44 for sufficiently large values of m and l, one obtains the next result [13].

Corollary 5. *For every $\epsilon > 0$, there exists a $\left(\frac{9}{4} - \epsilon\right)$-tough nontraceable graph.*

It is easily seen from the proof in [13] that Theorem 43 remains valid if "$m \geq 2l+3$" and "nontraceable" are replaced by "$m \geq 2l+1$" and "nonhamiltonian", respectively. Thus the graph $G(L, u, v, 2, 5)$ is a nonhamiltonian graph, which by Theorem 44 has toughness 2. This graph is sketched in Figure 2. It follows that a smallest counterexample to the 2-tough conjecture has at most 42 vertices. Similarly, a smallest nontraceable 2-tough graph has at most $|V(G(L, u, v, 2, 7))| = 58$ vertices.

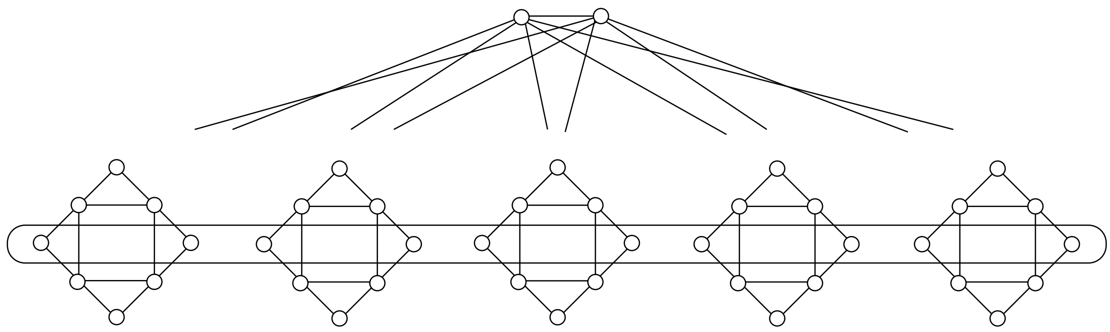

Figure 2. The graph $G(L, u, v, 2, 5)$.

A graph G is *neighborhood-connected* if the neighborhood of each vertex of G induces a connected subgraph of G. In [67], Chvátal also stated the following weaker version of the 2-tough conjecture: every 2-tough neighborhood-connected graph is hamiltonian. Since all counterexamples described above are neighborhood-connected, this weaker conjecture is also false.

Most of the ingredients used in the above counterexamples were already present in [5]. It only remained to observe that using the specific graph L as a "building block" produced a graph with toughness at least 2. We hope that other building blocks and/or smarter constructions will lead to counterexamples with a higher toughness. Constructions similar to those used to prove Theorem 42 have been used to establish other important results. Chvátal [67] obtained $\left(\frac{3}{2}-\epsilon\right)$-tough graphs without a 2-factor for arbitrary $\epsilon > 0$. These examples are all chordal. It was shown in [22] that every $\frac{3}{2}$-tough chordal graph has a 2-factor. Based on this, Kratsch [125] raised the question whether every $\frac{3}{2}$-tough chordal graph is hamiltonian. Using Theorem 43 in [13] it has been shown that this conjecture, too, is false.

Consider the graph M of Figure 3.

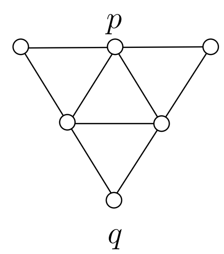

Figure 3. The graph M.

The graph M is chordal and has no Hamilton path with endvertices p and q. The graphs $G(M,p,q,l,m)$ are also chordal and, by Theorem 43, they are nontraceable whenever $m \geq 2l+3$. By arguments similar to those used in the proof of Theorem 44, the toughness of $G(M,p,q,l,m)$ is $\dfrac{l+3m}{2m+1}$ if $l \geq 2$. Hence for $l \geq 2$ the graph $G(M,p,q,l,2l+3)$ is a chordal nontraceable graph with toughness $\dfrac{7l+9}{4l+7}$. This gives the following result from [13].

Theorem 45. *For every $\epsilon > 0$, there exists a $\left(\frac{7}{4}-\epsilon\right)$-tough chordal nontraceable graph.*

We will return to questions on tough chordal graphs in Section 6.

A k-*walk* in a graph G is a closed spanning walk of G that visits every vertex of G at most k times. Of course a Hamilton cycle is then a 1-walk. In [80], Ellingham and Zha used the same construction as above to give an infinite class of graphs of relatively high toughness without a k-walk. They obtained the following results.

Theorem 46. *Every 4-tough graph has a 2-walk.*

Theorem 47. *For every $\epsilon > 0$ and every $k \geq 1$ there exists a $\left(\frac{8k+1}{4k(2k-1)}-\epsilon\right)$-tough graph with no k-walk.*

To prove the latter theorem they first modified the graph L from Figure 1 and then relied on the same basic construction that was used in [13].

5. Toughness and Factors

In [67], Chvátal conjectured that every k-tough graph on $n \geq k+1$ vertices and kn even contains a k-factor. Enomoto et al. [84] gave a decisive answer to Chvátal's conjecture in the following two theorems.

Theorem 48. *Let G be a k-tough graph on n vertices with $n \geq k+1$ and kn even. Then G has a k-factor.*

Theorem 49. *Let $k \geq 1$. For every $\epsilon > 0$, there exists a $(k-\epsilon)$-tough graph G on n vertices with $n \geq k+1$ and kn even which has no k-factor.*

In particular, every 2-tough graph contains a 2-factor, and for every $\epsilon > 0$, there exist infinitely many $(2-\epsilon)$-tough graphs with no 2-factor.

In [81], Enomoto strengthened Theorem 48.

Theorem 50. *Let k be a positive integer and G be a graph on n vertices with $n \geq k+1$ and kn even. Suppose $|S| \geq k \cdot \omega(G-S) - \frac{7k}{8}$ for all $S \subseteq V$ with $\omega(G-S) \geq 2$. Then G has a k-factor.*

In [82], Enomoto first improved Theorem 50 for $k=1$ and $k=2$. We need the following definition. For a graph G let

$$\begin{aligned} \tau^{'}(G) &= \max\{t \mid |S| \geq t \cdot \omega(G-S) - t \text{ for all } S \subset V(G)\} \\ &= \min\left\{\frac{|S|}{\omega(G-S)-1} \mid \omega(G-S) \geq 2\right\} \end{aligned}$$

if G is not complete. If G is complete, set $\tau^{'}(G) = \infty$.

Theorem 51. *Let G be a graph on n vertices, where n is even. If $\tau^{'} \geq 1$, then G has a 1-factor.*

Theorem 52. *Let G be a graph on $n \geq 3$ vertices. If $\tau^{'} \geq 2$, then G has a 2-factor.*

Both Theorem 51 and Theorem 52 were also shown to be sharp.

Finally, Enomoto and Hagita [83] were able to generalize Theorem 52 and strengthen Theorem 48 for graphs with a sufficiently large number of vertices.

Theorem 53. *Let k be a positive integer and G be a graph on $n \geq k^2 - 1$ vertices with kn even. If $\tau^{'} \geq k$, then G has a k-factor.*

For $1 \leq t < 2$, it is natural to ask how large the minimum vertex degree of a t-tough graph can be, if the graph contains no 2-factor. This problem was studied in [29].

Theorem 54. *Let G be a t-tough graph on $n \geq 3$ vertices, where $1 \leq t \leq 2$. If $\delta \geq \left(\frac{2-t}{1+t}\right) n$, then G has a 2-factor.*

It is also shown in [29] that for any $t \in [1, 3/2]$ there are infinitely many t-tough graphs with no 2-factor satisfying $\delta \geq \left(\frac{2-t}{1+t}\right) n - \frac{5}{2}$.

However one can improve the bound in Theorem 54 for $3/2 < t < 2$ [29].

Theorem 55. *Let G be a t-tough graph on $n \geq 3$ vertices, where $3/2 < t < 2$. If $\delta \geq \left(\frac{2-t}{1+t}\right)\left(\frac{t^2-1}{7t-7-t^2}\right) n$, then G has a 2-factor.*

Examples in [29] show that Theorem 55 is asymptotically tight if $t = (2r-1)/r$ for any integer $r \geq 2$.

More recently, minimum degree conditions for a t-tough graph $(1 \leq t < 3)$ to have a 3-factor have been established [26]. The results in [26] are similar to those given in [29] for 2-factors.

In [61], Chen improved on Theorem 48 by showing that under similar conditions it is possible to find a k-factor containing a specified edge and also to find a k-factor not containing a specified edge. Another improvement of Theorem 48 was obtained by Katerinis [119] . An $[a,b]$-*factor* of a graph G is a spanning subgraph F of G such that $a \leq d_F(x) \leq b$, for all $x \in V(G)$.

Theorem 56. *Let $a \leq b$ and G be a graph on n vertices such that $a < b$ or bn is even. Then G has an $[a,b]$-factor if $\tau \geq a + \frac{a}{b} - 1$.*

Chen [62] improved this for $a = 2 < b$.

Theorem 57. *Let $b > 2$ and G be a graph on $n \geq 3$ vertices. Then G has a $[2,b]$-factor if $\tau \geq 1 + \frac{1}{b}$.*

Ellingham et al. [79] have extended this result to connected factors.

Katerinis [120] has shown that a 1-tough bipartite graph on $n \geq 3$ vertices has a 2-factor.

We next give a minimum degree condition for a 1-tough graph to have a 2-factor with a specific number of cycles. First note that Jung's Theorem (Theorem 5) implies the following weaker theorem with a minimum degree condition.

Theorem 58. *Let G be a 1-tough graph on $n \geq 11$ vertices with $\delta \geq (n-4)/2$. Then G is hamiltonian.*

Faudree et al. [88] generalized Theorem 58 as follows.

Theorem 59. *There exists an integer n_0 such that every 1-tough graph on $n \geq n_0$ vertices with $\delta \geq (n-4)/2$ has a 2-factor with k cycles, for all k such that $1 \leq k \leq (n-10)/4$.*

A number of results on factors have appeared relating toughness to (r,k)-factor-critical graphs. A graph G is (r,k)-*factor-critical* if $G - X$ contains an r-factor for all $X \subseteq V$ with $|X| = k$. For $r \geq 2$, these graphs were studied by Liu and Yu [131] under the name (r,k)-extendable graphs. They proved the following.

Theorem 60. *Let G be a graph on n vertices with $\tau \geq 3$. Then G is $(2,k)$-factor-critical for every integer k such that $3 \leq k \leq \tau$ and $k \leq n-3$.*

They also conjectured that if G is a graph on n vertices with $\tau \geq q$ and $n \geq 2q+1$ for some integer $q \geq 1$, then G is $(2, 2q-2)$-factor-critical. Note that this conjecture is false for $q = 1$ by Theorem 49. However it was shown by Cai et al. [59], and independently by Enomoto [82], that the conjecture is true for all integers $q \geq 2$.

Theorem 61. *Let G be a graph on n vertices with $\tau \geq 2$. Then G is $(2,k)$-factor-critical for every non-negative integer k with $k \leq \min\{2\tau - 2, n-3\}$.*

It was also shown in [59] that the bound $2\tau - 2$ is sharp.

Progress has also been made on the relationship between toughness and (r,k)-factor-critical graphs for $r = 1$ and $r = 3$. In [89], Favaron considered $r = 1$.

Theorem 62. *Let G be a graph on n vertices and k be an integer with $2 \leq k < n$ and $n + k$ even. Then G is $(1,k)$-factor-critical if $\tau > k/2$.*

The value $k/2$ in Theorem 62 was also shown to be sharp.

In [149], Shi et al. considered $r = 3$.

Theorem 63. *Let G be a graph on n vertices with $\tau \geq 4$. Then G is $(3, k)$-factor-critical for every non-negative integer k such that $n + k$ is even, $k < 2\tau - 2$ and $k \leq n - 7$.*

This result is best possible with respect to each of the upper bounds on k.

In [121], Katona introduced the notion of "t-edge-toughness". The definition is rather involved, and we refer the reader to [121] for the precise definition. We note that it is easy to verify that a graph is not t-edge-tough in the same way one easily verifies that a graph is not t-tough. Edge-toughness is nicely related to both toughness and hamiltonicity, as the following results from [121] show.

Theorem 64. *If G is a hamiltonian graph, then G is 1-edge-tough.*

Theorem 65. *If G is a t-edge-tough graph, then G is t-tough.*

Theorem 66. *If G is a $2t$-tough graph, then G is t-edge-tough.*

We know, by Theorem 48, that 2-tough graphs have 2-factors. In light of Theorem 66, it would be interesting to know if 1-edge-tough graphs have 2-factors. This was answered by Katona [122] in the affirmative.

Theorem 67. *Let G be a 1-edge-tough graph on $n \geq 3$ vertices. Then G has a 2-factor.*

We close this section with a conjecture from [122].

Conjecture 3. Let t be a positive integer and G be a t-edge-tough graph on $n \geq 2t + 1$ vertices. Then G has a $2t$-factor.

6. Toughness and Special Graph Classes

Triangle-free graphs have received much attention in the literature. In particular, tough triangle-free graphs have a number of interesting properties. We begin by considering the problem of finding the best possible minimum degree condition to ensure that a 1-tough triangle-free graph on n vertices is hamiltonian. The degree condition for the existence of a 2-factor in the following theorem from [20] is best possible.

Theorem 68. *Let G be a 1-tough triangle-free graph on $n \geq 3$ vertices. If $\delta(G) \geq (n + 2)/4$, then G has a 2-factor.*

Define $C(G)$ to be the set of cycle lengths of a graph G. Brandt [47] has proven the following.

Theorem 69. *Let $G \neq C_5$ be a triangle-free, nonbipartite graph of order n. If $\delta > n/3$, then $C(G) = \{4, 5, ..., r\}$, where $r = \min\{n, 2(n - \alpha)\}$.*

On the other hand, Moon and Moser [136] have shown that in a balanced bipartite graph G on n vertices, if $\delta > n/4$, then G is hamiltonian. Since $\alpha \leq n/2$ in any 1-tough graph, and 1-tough bipartite graphs are balanced, we easily obtain the following result.

Theorem 70. *Let G be a 1-tough triangle-free graph on $n \geq 3$ vertices. If $\delta > n/3$, then G is hamiltonian.*

Combining Theorem 68 and Theorem 70, we see that the best minimum degree guaranteeing that a 1-tough triangle-free graph is hamiltonian is somewhere between $(n+2)/4$ and $(n+1)/3$.

In [67], it was conjectured that there exists a positive constant t_1 such that every t_1-tough graph is pancyclic. Later, Jackson and Katerinis [114] asked if there is a positive constant t_2 such that every t_2-tough graph contains a triangle. In [19], both of these questions were answered in the negative.

Theorem 71. *There exist arbitrarily tough, triangle-free graphs.*

This was accomplished by constructing a sequence of "layered graphs". If one begins with a triangle-free graph, a sequence of layered graphs can be constructed that remain triangle-free and whose toughness approaches infinity.

Subsequently, Alon [2] proved a stronger result.

Theorem 72. *For every t and g there exists a t-tough graph of girth greater than g.*

Alon's technique involved showing that regular graphs with well separated eigenvalues are tough. He was then able to use the Ramanujan graphs [133, 134] with appropriate parameters to get explicit examples.

Later, Brandt, Faudree, and Goddard [50] also demonstrated that Chvátal's pancyclic conjecture is false. A graph is called *weakly pancyclic* if it contains cycles of every length between its girth and its circumference. They show [50] there is no sufficiently large value of toughness that will ensure that a graph is weakly pancyclic. Their short clever argument is presented in [50] and originally appeared in Brandt [48]. For their construction, however, they need graphs with large connectivity whose girth exceeds the maximum degree. For these graphs they also rely on the Ramanujan graphs constructed by Lubotsky, Phillips, and Sarnak [133].

We now turn to some conjectures presented in [19]. It is easy to see that if some vertex in a t-tough graph G on n vertices has degree larger than $n/(t+1)$, then G must contain a triangle. This led to the natural question of whether there exists an $n/(t+1)$-regular t-tough triangle-free graph for arbitrarily large t. Since it appeared that the sequence of layered graphs constructed in [19] had this property, it was conjectured that such graphs exist, and in [19] this was proven for infinitely many t such that $1 \le t < 3$. The full conjecture was proven independently by Brandt [49] and Brouwer [58]. In fact, Brandt [49] proved slightly more.

Theorem 73. *For every $\epsilon > 0$ there exists a real number t_0 such that for every $t > t_0$ there is a triangle-free graph G on n vertices with toughness $\tau = n/\delta - 1$ and $t - \epsilon \le \tau \le t + \epsilon$.*

Theorem 71 is related to a number of other results on triangle-free graphs. It has been shown [74, 123, 137, 162] that there exist triangle-free graphs with arbitrarily large chromatic number. Let $\chi(G)$ denote the chromatic number of a graph G. It is easy to see that if G is a graph on n vertices,

$$\chi(G) \ge \frac{n}{\alpha(G)} \ge \tau(G) + 1. \tag{1}$$

In [85], Erdös used a clever probabilistic argument to show that there exist graphs with arbitrarily high girth and arbitrarily high chromatic number. In fact, he showed that these graphs have arbitrarily high n/α ratio. By (1) we see that Theorem 71 represents a strengthening of these previous results for triangle-free graphs. In fact, Brandt [49] has shown that for an appropriate sequence of layered graphs, (1) can be satisfied with equality.

Theorem 74. *For every positive integer k there exists a triangle-free graph G with $\chi = k = n/\alpha = \tau + 1$.*

It was also conjectured in [19] that a t-tough graph on n vertices with $\delta > n/(t+1)$ must be pancyclic. Such a graph clearly contains a triangle and by Theorem 25 it must also be hamiltonian. The following result from [50] demonstrates that this conjecture is true for a t-tough graph if $t < 3 - 4000/n$.

Theorem 75. *Let G be a graph of order n with minimum degree $\delta \geq n/4 + 250$ that contains a triangle and a hamiltonian cycle. Then G is pancyclic.*

A problem that has received much attention is that of determining the minimum level of toughness to ensure that a member from a special graph family is hamiltonian. We first consider chordal graphs, as well as a few other subclasses of perfect graphs (for definitions, see [52] or [100]). First recall that we have seen in Section 3 an infinite class of chordal graphs with toughness close to $7/4$ having no Hamilton path. Hence, 1-tough chordal graphs need not be hamiltonian. In fact, even 1-tough planar chordal graphs need not be hamiltonian [41]. The following result was established, however.

Theorem 76. *Let G be a chordal, planar graph with $\tau > 1$. Then G is hamiltonian.*

Gerlach [94] showed that the chordality assumption in the above theorem can be weakened to the assumption that separating cycles of length at least four have chords.

To see that being 1-tough will not suffice in Theorem 76, we must first define the "shortness exponent" of a class of graphs. This concept was first introduced in [102] as a way of measuring the size of longest cycles in *polyhedral*, i.e., 3-connected planar graphs.

Let Σ be a class of graphs. The *shortness exponent* of the class Σ is given by

$$\sigma(\Sigma) = \liminf_{H \in \Sigma} \frac{\log\ c(H_n)}{\log\ |V(H_n)|}.$$

The $\lim\ \inf$ is taken over all sequences of graphs H_n in Σ such that $|V(H_n)| \to \infty$ as $n \to \infty$.

In [41], it was also shown that the shortness exponent of the class of all 1-tough chordal planar graphs is at most $\log 8/\log 9$. Hence there exists a sequence $G_1, G_2, \ldots$ of 1-tough chordal planar graphs with $\dfrac{c(G_i)}{|V(G_i)|} \to 0$ as $i \to \infty$. On the other hand, all 1-tough $K_{1,3}$-free chordal graphs are hamiltonian. This follows from the well-known result of Matthews and Sumner [135] relating toughness and vertex connectivity in $K_{1,3}$-free graphs, and a result of Balakrishnan and Paulraja [3] showing that 2-connected $K_{1,3}$-free chordal graphs are hamiltonian.

While being 1-tough will not ensure hamiltonicity for chordal graphs, it will for other subclasses of perfect graphs. For example, in [124] it was shown (implicitly) that 1-tough interval graphs are hamiltonian, and in [73] it was shown that 1-tough cocomparability graphs are hamiltonian.

Let us now consider $3/2$-tough chordal graphs. We have already seen that such graphs need not be hamiltonian. However for a certain subclass of chordal graphs, namely split graphs, we have a different result. A graph G is called a *split graph* if $V(G)$ can be partitioned into an independent set and a clique. We mention the following two results from [126].

Theorem 77. *Every $3/2$-tough split graph is hamiltonian.*

Theorem 78. *There is a sequence $\{G_n\}_{n=1}^{\infty}$ of split graphs with no 2-factor and $\tau(G_n) \to 3/2$.*

Even though $3/2$-tough chordal graphs need not be hamiltonian, it was shown in [22] that they will have a 2-factor. In fact, we can say a bit more.

Theorem 79. *Let G be a $3/2$-tough 5-chordal graph. Then G has a 2-factor.*

Theorem 79 is best possible in two ways. Chvátal's examples in [67] show it is best possible with respect to toughness and examples in [29] contain 6-chordal graphs without a 2-factor whose toughness approaches 2 from below.

The previous results on tough chordal graphs lead to a very natural question. Does there exist a $t_1 > 0$ such that every t_1-tough chordal graph is hamiltonian? This was settled in the affirmative by Chen et al. [63], who gave a constructive proof of the following.

Theorem 80. *Every 18-tough chordal graph is hamiltonian.*

The authors did not claim that 18 is best possible. The natural question, in light of the disproof of the 2-tough conjecture for general graphs, is what minimum level of toughness will ensure that a chordal graph is hamiltonian. More specifically, are 2-tough chordal graphs hamiltonian?

What about triangle-free graphs? Are 2-tough triangle-free graphs hamiltonian? It is conjectured in [20] that for all $\epsilon > 0$, there exists a $(2-\epsilon)$-tough triangle-free graph that does not even contain a 2-factor. An infinite collection of triangle-free graphs are given that clearly have no 2-factor. It appears that the toughness of these graphs approaches 2 as the order $n \to \infty$; however establishing the toughness appears difficult. On the other hand, Ferland [90] has found an infinite class of nonhamiltonian triangle-free graphs whose toughness is at least $5/4$. Of course, the toughness of the Petersen graph is $4/3$; however the Petersen graph is not an infinite class.

In [57], toughness conditions are studied that guarantee the existence of a Hamilton cycle in k-*trees*. In this context, a k-tree is a graph that can be obtained from a K_k by repeatedly adding new vertices and joining them to a set of k mutually adjacent vertices. It is clear that a k-tree is a chordal graph. In [57], it is shown that every 1-tough 2-tree on at least three vertices is hamiltonian, a best possible result since 1-toughness is a necessary condition for hamiltonicity. This is generalized to a result on k-trees for $k \geq 2$ as follows: Let G be a k-tree. If G has toughness at least $(k+1)/3$, then G is hamiltonian. Moreover, infinite classes of nonhamiltonian 1-tough k-trees for each $k \geq 3$ are presented.

7. Computational Complexity of Toughness

The problem of determining the complexity of recognizing t-tough graphs was first raised by Chvátal [66] and later appeared in [151] and [[68], p. 429]. We refer the reader to [93] for the basic ideas of complexity theory.

Consider the following decision problem, where t is any positive rational number.

***t*-TOUGH**
INSTANCE : Graph G.
QUESTION : Is $\tau(G) \geq t$?

The following was established in [16].

Theorem 81. *For any positive rational number t, t-TOUGH is NP-hard.*

In [16], a well-known NP-hard variant of INDEPENDENT SET [[93], p. 194] is reduced to the problem of recognizing 1-tough graphs. Then the latter problem is reduced to recognizing t-tough graphs, for any fixed positive rational number t. In fact, it is easy to use an argument analogous to that used in [16] to reduce INDEPENDENT SET to the problem of recognizing 1-tough graphs, as shown in [30].

It is natural to inquire whether the problem of recognizing t-tough graphs remains NP-hard for various subclasses of graphs. For example, Matthews and Sumner [135] have shown that for $K_{1,3}$-free graphs, $\tau = \kappa/2$. Hence the toughness of $K_{1,3}$-free graphs, and consequently of line graphs, can be determined in polynomial time. Thus, while it is NP-complete to determine if a line graph is hamiltonian [38], it is polynomial to determine if a line graph is 1-tough. Another class of graphs for which this is the case is the class of split graphs. Recall that a graph G is called a split graph if $V(G)$ can be partitioned into an independent set and a clique. Determining if a split graph is hamiltonian was shown to be NP-complete in [70]. On the other hand, the following was shown in [126].

Theorem 82. *The class of 1-tough split graphs can be recognized in polynomial time.*

Noting that submodular functions can be minimized in polynomial time [71,101], Woeginger [159] then gave a short proof of the following result.

Theorem 83. *For any rational number $t \geq 0$, the class of t-tough split graphs can be recognized in polynomial time.*

For many subclasses of graphs, however, it is NP-hard to recognize t-tough graphs. For example, in [24] it was shown that it is NP-hard to recognize t-tough graphs, even within the class of graphs having minimum degree "almost" high enough to ensure that the graph is t-tough.

Theorem 84. *Let $t \geq 1$ be a rational number. If $\delta \geq (\frac{t}{t+1})n$, then G is t-tough. On the other hand, for any fixed $\epsilon > 0$, it is NP-hard to determine if G is t-tough for graphs G with $\delta \geq (\frac{t}{t+1} - \epsilon)n$.*

Häggkvist [103] has shown that if $\delta \geq n/2 - 2$, there is a polynomial time algorithm to determine whether G is hamiltonian. As a consequence of Jung's Theorem (Theorem 5), a graph G on $n \geq 11$ vertices satisfying $\delta \geq n/2 - 2$ is hamiltonian if and only if G is 1-tough. It follows that 1-tough graphs can be recognized in polynomial time when $\delta \geq n/2 - 2$.

Another interesting class of graphs is the class of bipartite graphs. Obviously $\tau \leq 1$ for any bipartite graph. The complexity of recognizing 1-tough bipartite graphs had been raised a number of times; see, e.g., [[55], p. 119]. In [126], Kratsch et al. were able to reduce 1-TOUGH for general graphs to 1-TOUGH for bipartite graphs by using the classical Nash-Williams construction [138] .

Theorem 85. *1-TOUGH remains NP-hard for bipartite graphs.*

Consequently, 1-TOUGH is also NP-hard for the larger class of triangle-free graphs.

An important class of graphs that has received considerable attention is the class of regular graphs. Note that the maximum possible toughness of an r-regular graph is $r/2$, since $\tau \leq \kappa/2 \leq r/2$.

Chvátal [67] asked for which values of r and $n > r + 1$ there exists an r-regular, $r/2$-tough graph on n vertices, and observed that this is always the case for r even. He also conjectured that for r odd and n sufficiently large, it would be necessary that $n \equiv 0 \bmod r$, and verified this for $r = 3$. But for all odd $r \geq 5$, Doty [78] and Jackson and Katerinis [114] independently constructed an infinite family of r-regular, $r/2$-tough graphs on n vertices with $n \not\equiv 0 \bmod r$.

Jackson and Katerinis [114] gave a characterization of cubic $3/2$-tough graphs which allowed such graphs to be recognized in polynomial time. Their characterization of these graphs uses the concept of *inflation*, introduced by Chvátal in [67].

Theorem 86. *Let G be a cubic graph. Then G is $3/2$-tough if and only if $G = K_4$, $G = K_2 \times K_3$, or G is the inflation of a 3-connected cubic graph.*

Goddard and Swart [99] conjectured an analogous characterization of r-regular, $r/2$-tough graphs for all $r \geq 1$, which would allow such graphs to be recognized in polynomial time.

In the opposite direction, it was established in [17] that it is NP-hard to recognize 1-tough cubic graphs. This was generalized in [18] as follows.

Theorem 87. *For any integer $t \geq 1$ and any fixed $r \geq 3t$, it is NP-hard to recognize r-regular, t-tough graphs.*

The complexity of recognizing r-regular, t-tough graphs remains completely open when $2t < r < 3t$, and the complexity when $r = 2t + 1$ seems especially intriguing.

There are still many interesting subclasses of graphs for which the complexity of recognizing t-tough graphs is unknown. A number of these classes are given in [17]. In particular, Dillencourt [75,76] has noted that we still do not know the complexity of recognizing 1-tough planar graphs or 1-tough maximal planar graphs.

The fact that it is NP-hard to recognize 1-tough graphs makes it desirable to strengthen some theorems by replacing the assumption that a graph G is 1-tough with the weaker assumption that G is 2-connected. The idea is to try to draw the same conclusion regarding the cycle structure of G under the weaker hypothesis that G is 2-connected by specifying an easily described family of exceptional graphs for which the new theorem does not hold.

To illustrate this type of improvement, consider again Theorem 7 (the long cycle version of Jung's Theorem). By Theorem 4, a 2-connected graph G on $n \geq 3$ vertices satisfies $c(G) \geq \min\{n, \sigma_2\}$. In [21], it was noted that the 2-connected graphs with $c(G) = \sigma_2$ or $\sigma_2 + 1$ constituted a family $\mathcal{H}$ of eight easily-recognized classes of graphs. This led to the following improvement of Theorem 7.

Theorem 88. *Let G be a 2-connected graph on $n \geq 3$ vertices. Then $c(G) \geq \min\{n, \sigma_2 + 2\}$ unless $G \in \mathcal{H}$.*

A similar improvement of Theorem 5 was found by Skupień [150], and an analogous improvement of both parts of Theorem 20 was found by Bauer et al. [31].

There are several results in hamiltonian graph theory of the form $\mathcal{P}_1$ implies $\mathcal{P}_2$, where $\mathcal{P}_1$ is an NP-hard property of graphs and $\mathcal{P}_2$ is an NP-hard cycle structure property, and one might wonder about the practical value of such theorems.

Two such theorems are the well-known theorems of Chvátal and Erdös [69] and Jung [116].

In [68], Chvátal gave a proof of the Chvátal-Erdös Theorem [69] which constructs in polynomial time either a Hamilton cycle in a graph G or an independent set of more than κ vertices in G. In [9], the authors provided a similar type of polynomial time constructive proof for Jung's Theorem [116] on graphs with at least 16 vertices.

Theorem 89. *Let G be a graph on $n \geq 16$ vertices with $\sigma_2 \geq n-4$. Then we can construct in polynomial time either a Hamilton cycle in G or a set $X \subseteq V(G)$ with $\omega(G-X) > |X|$.*

It is possible that other theorems in graph theory with an NP-hard hypothesis and an NP-hard conclusion also have polynomial time constructive proofs.

8. Other Toughness Results

In [144], Plummer investigated the relationship between the toughness of a graph and whether a given matching in a graph can be extended to a perfect matching. In [67], it was noted that every 1-tough graph on an even number of vertices has a perfect matching. Let m and n be positive integers with $m \leq n/2-1$ and let G be a graph on n vertices with a perfect matching. A graph G is *m-extendable* if every matching of size m extends to a perfect matching. In [144], Plummer proved the following result on m-extendable graphs.

Theorem 90. *Suppose G is a graph on n vertices, with n even. Let m be a positive integer with $m \leq n/2-1$. If $\tau > m$, then G is m-extendable. Moreover, the lower bound on τ is tight for all m.*

As just noted, every 1-tough graph on an even number of vertices has a perfect matching. In fact, more is true. In [132], a graph G is called *elementary* if G has a perfect matching, and if the edges of G which occur in a perfect matching induce a connected subgraph of G. The following was established in [7].

Theorem 91. *Let G be a 1-tough graph on an even number of vertices. Then G is elementary.*

Similarly, 1-tough graphs on an odd number of vertices have special matching properties. A graph G is called *factor-critical* [132] if $G-v$ has a perfect matching, for all $v \in V(G)$. The following was also established in [7].

Theorem 92. *Let G be a 1-tough graph on an odd number of vertices. Then G is factor-critical.*

Let $\omega_o(G)$ denote the number of odd components of the graph G. A set $T \subseteq V(G)$ is called a *Tutte set* for G if $\omega_o(G-T) - |T| = \max_{X \subseteq V(G)}\{\omega_o(G-X) - |X|\}$. The importance of Tutte sets rests on the fact that the size of a maximum matching in G is precisely $\frac{1}{2}(|V(G)| - (\omega_o(G-T) - |T|))$, by a well-known theorem of Berge [35]. Maximum Tutte sets in G seem especially interesting. In [7], it was established that finding maximum Tutte sets in a general graph is NP-hard. However Theorems 91 and 92, together with the special structure of Tutte sets in elementary and factor-critical graphs [132], yield the following result from [7].

Theorem 93. *Maximum Tutte sets can be found in polynomial time for the class of 1-tough graphs.*

Recall that a cutset $S \subseteq V(G)$ is called a *tough set* for G if $\tau(G) = |S|/\omega(G-S)$, and any component of $G-S$ is called a *tough component* of G. In [145], Plummer investigated the toughness of tough components. In particular, he showed that if G is not complete and $\tau \geq 1$, then any tough component C in G satisfies $\tau(C) \geq \dfrac{\lceil \tau(G) \rceil}{2}$.

We have mentioned that it is NP-hard to determine if a cubic graph is 1-tough. It is possible, however, to obtain an upper bound on the toughness of a cubic graph in terms of its independence number [96].

Theorem 94. *Let G be a noncomplete cubic graph on n vertices. Then*

$$\tau \leq \min\left\{\frac{2n-3\alpha}{n-\alpha}, \frac{2\alpha}{4\alpha-n}\right\}.$$

In [96], Goddard also considered the toughness of a special class of cubic graphs. A *cycle permutation graph* is a cubic graph on $2m$ vertices obtained by taking two vertex disjoint cycles on m vertices and adding a matching between the vertices of the two cycles. It was conjectured in [143] that the toughness of such a graph is at most 4/3. Goddard [96] came very close to proving this is true.

Theorem 95. *Let G be a cycle permutation graph on 2m vertices. Then*

$$\tau \begin{cases} \leq 4/3 & m \equiv 0, 1 \bmod 4, \\ < 4/3 & m \equiv 2 \bmod 4, \\ \leq 4/3 + 4/(9m-3) & m \equiv 3 \bmod 4. \end{cases}$$

In [158], Win considered the relationship between the toughness of a graph and the existence of a k-tree. In this context, a *k-tree* of a connected graph is a spanning tree with maximum degree at most k. Note that we used k-trees before for a different subclass of the class of chordal graphs.

Theorem 96. *Let G be a connected graph. Suppose $k \geq 2$, and that for any subset $S \subseteq V(G)$, $\omega(G-S) \leq 2 + (k-2)|S|$. Then G has a k-tree.*

For $k = 2$, this simply says that a connected graph with independence number at most 2 has a Hamilton path. For $k \geq 3$, Theorem 96 has the following corollary [158].

Corollary 6. *Let $k \geq 3$. If $\tau \geq 1/(k-2)$, then G has a k-tree.*

A graph is *polyhedral* if it is planar and 3-connected. Since a 4-connected planar graph is hamiltonian by a well-known theorem of Tutte, a nonhamiltonian planar graph is at most $3/2$-tough. In [105], Harant constructed nonhamiltonian regular polyhedral graphs of degree 3, 4, and 5 with maximum toughness $3/2$.

In [106], Harant and Owens constructed nonhamiltonian maximal planar graphs with toughness $5/4$. In [142], Owens improved this by constructing nonhamiltonian maximal planar graphs with toughness $3/2-\epsilon$, for any $\epsilon > 0$. In fact, Owens' graphs do not even contain a 2-factor. Since $(3/2+\epsilon)$-tough planar graphs are hamiltonian, it would be interesting to determine the cycle structure of $3/2$-tough planar and maximal planar graphs. In particular, it would be interesting to know if $3/2$-tough maximal planar graphs even contain a 2-factor.

A number of other results have considered the existence of tough nonhamiltonian maximal planar graphs. Let $\Gamma(t_0)$ denote the class of all t_0-tough maximal planar graphs. In [140], Nishizeki produced a nonhamiltonian graph on 19 vertices in $\Gamma(1)$, thus answering a question of Chvátal concerning the existence of such a graph. In [75], Dillencourt found such a graph with 15 vertices. Finally, Tkáč [153] was able to find a nonhamiltonian graph in $\Gamma(1)$ with just 13 vertices, and to show that no such graph can have fewer vertices.

In Section 6, we defined the shortness exponent, $\sigma(\Sigma)$, of a class of graphs Σ. Regarding $\sigma(\Gamma(1))$, Dillencourt [75] showed that $\sigma(\Gamma(1)) \leq \log_7 6$, and in [153], Tkáč improved this by showing $\sigma(\Gamma(1)) \leq \log_6 5$. In [105], it is shown that the shortness exponent of the class of 3-regular polyhedral graphs with toughness $3/2$ is less than 1. The same holds for the class of 4-regular such graphs. A recent result of Chen and Yu [64] shows that the shortness exponent of polyhedral graphs is $\log_3 2$. As a consequence, $\sigma(\Gamma(3/2)) \geq \log_3 2$.

More results on the shortness exponent can be found in [75, 102].

In [97], Goddard et al. considered bounds on the toughness of a graph G in terms of the graph's connectivity and genus. They made use of the following result of Schmeichel and Bloom [148], in which γ denotes the genus.

Theorem 97. *Let* G *be a graph with* $\kappa \geq 3$. *Then* $\omega(G-X) \leq \frac{2}{\kappa-2}(|X|-2+2\gamma)$ *for all* $X \subseteq V$ *with* $|X| \geq \kappa$.

After simplifying the proof of Theorem 97 they used the result to obtain lower bounds on τ.

Theorem 98. *Let* G *be a connected graph. Then*

(1) $\tau > \dfrac{\kappa}{2} - 1$, *if* $\gamma = 0$, *and*

(2) $\tau \geq \dfrac{\kappa(\kappa-2)}{2(\kappa-2+2\gamma)}$, *if* $\gamma \geq 1$.

They also discussed the quality of the bounds, and investigated upper bounds on τ. In particular, they showed that Theorem 98 (1) is sharp for $2 \leq \kappa \leq 5$, and that the bound in Theorem 98 (2) is attained by an infinite class of graphs, all of girth 4.

In [91], Ferland investigated the toughness of generalized Petersen graphs. These graphs were first defined by Watkins in [156]. For each $n \geq 3$ and $0 < k < n$, the *generalized Petersen graph* $G(n,k)$ has vertex set $V = \{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\}$ and edge set $E = \{(u_i, u_{i+1}) \mid 1 \leq i \leq n\} \cup \{(u_i, v_i) \mid 1 \leq i \leq n\} \cup \{(v_i, v_{i+k}) \mid 1 \leq i \leq n\}$, where all indices are modulo n. Of course, the Petersen graph is $G(5,2)$.

In [91], as well as in his earlier paper [90], Ferland was interested in bounds for $\tau(G(n,k))$, especially asymptotic bounds. He called a real number b an *asymptotic upper bound* for $\tau(G(n,k))$ if $\lim_{n\to\infty} \tau(G(n,k)) \leq b$. Asymptotic lower bounds are defined similarly. In [143], $\tau(G(n,1))$ is completely determined, and 1 is an asymptotic upper bound. In [90], it was found that for $\tau(G(n,2))$, $5/4$ is both a lower bound and an asymptotic upper bound. For $n \geq 3$ and $0 < k < n$, upper and lower (asymptotic) bounds for $\tau(G(n,k))$ were given in [91].

In [53], Broersma, Engbers and Trommel studied the relationship between the toughness of a graph and the toughness of its spanning subgraphs. In particular they proved the following.

Theorem 99. *Let* G *be a graph on* $n \geq 4$ *vertices with* $\tau > 1$. *Then there exists a spanning subgraph* H *of* G *with* $\tau(H) = 1$.

They also defined a graph G to be *minimally* t*-tough* if $\tau(G) = t$ and $\tau(H) < t$ for every proper spanning subgraph H of G. They also discussed conditions under which the square of a graph will be minimally 2-tough.

Other variants on the toughness concept are more related to vulnerability and reliability of graphs and networks than to their cycle structure. We refer the interested reader to some survey papers [4, 34, 95] and will not treat them here.

9. Conclusions and Open Problems

Since Chvátal [67] introduced toughness in 1973, much research has been done that relates toughness conditions to the existence of cycle structures. In this survey, we have gathered many of the important results in this area. Historically, the motivation for this research was based on a number of conjectures in [67]. The most challenging of these conjectures is still open: Is there a finite constant t_0 such that every t_0-tough graph is hamiltonian? If so, what is the smallest such t_0?

We now know that if the conjecture is true, then $t_0 \geq 9/4$. Although the conjecture is still open for general graphs, we know that it is true for a number of well-studied graph classes, e.g., planar graphs, claw-free graphs and chordal graphs. Since all 4-connected planar graphs are hamiltonian by a well-known theorem of Tutte, we have $t_0 > 3/2$ for planar graphs, and this result is best possible. For claw-free graphs we know $\tau = \kappa/2$; consequently $t_0 \leq 7/2$ by a result of Ryjacek [146], combined with a result of Zhan [161] and Jackson [113] stating that all 7-connected line graphs are hamiltonian. However Matthews and Sumner [135] have conjectured that 4-connected (2-tough) claw-free graphs are hamiltonian. Finally, we know $t_0 \leq 18$ for chordal graphs by Theorem 80. Examples in [13] show that this cannot be improved to a value below $7/4$. Are all $7/4$-tough chordal graphs hamiltonian? It would be interesting to know if 2-tough chordal graphs are hamiltonian. The gaps in our knowledge for claw-free and chordal graphs imply a number of challenging open problems. The same is true for the class of triangle-free graphs. It is known that there exists an infinite class of $5/4$-tough triangle-free nonhamiltonian graphs [90], and it even appears that a class of triangle-free graphs with no 2-factor constructed in [20] has toughness approaching 2 from below. These examples suggest the intriguing possibility that every 2-tough triangle-free graph is hamiltonian, though it remains completely open whether the t_0-tough conjecture holds for the class of triangle-free graphs. By contrast, we know by Theorem 25 that the conjecture is true within the class of graphs on n vertices satisfying $\delta \geq \epsilon n$, for any fixed $\epsilon > 0$.

Suppose we also impose a minimum degree condition. The examples in [13] that disproved the 2-tough conjecture all have $\delta = 4$. We know from Corollary 1 that if G is a 2-tough graphs on n vertices with $\delta \geq n/3$, then G is hamiltonian. What if $5 \leq \delta < n/3$? The early research on toughness and cycle structure concentrated on sufficient degree conditions which, combined with a certain level of toughness, would yield the existence of long cycles. This survey contains a wealth of results in this direction. One of the major open problems in this area is the conjecture that every 1-tough graph on n vertices with $\sigma_3 \geq n \geq 3$ has a cycle of length at least $\min\{n, (3n+1)/4 + \sigma_3/6\}$. Another interesting problem is to find the best possible minimum degree condition to ensure that a 1-tough triangle-free graph is hamiltonian. By Theorems 68 and 70, we know the answer lies somewhere between $(n+2)/4$ and $(n+1)/3$.

If we do not impose a degree condition, toughness conditions can still guarantee cycles of length proportional to a function of the number of vertices of the graph. Two of the most challenging open problems in this area are whether there exist positive constants A and B, depending only on t, such that every 2-connected, respectively 3-connected, t-tough graph on n vertices has a cycle of length at least $A \log n$, respectively n^B. Both problems have affirmative solutions for planar graphs.

Another area of research has involved finding toughness conditions for the existence of certain factors in graphs. One of the challenging open problems in this area is to determine whether every $3/2$-tough maximal planar graph has a 2-factor. If so, are they all hamiltonian? We also do not know if a $3/2$-tough planar graph has a 2-factor.

Research on toughness has also focused on computational complexity issues. In particular, we now know that recognizing t-tough graphs is NP-hard in general, whereas it is polynomial within the class of claw-free graphs and within the class of split graphs. For many other interesting classes, this complexity question is still open, e.g., for (maximal) planar graphs and for chordal graphs. Within the class of r-regular graphs with $r \geq 3t$, recognizing t-tough graphs has been shown to be NP-hard. The problem is trivial if $r < 2t$, but its complexity is open for values of r with $2t \leq r < 3t$. It was conjectured by Goddard and Swart [99] to be polynomial for $r = 2t$, and seems especially interesting when $r = 2t + 1$.

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