STRONG CONNECTIVITY IN SENSOR NETWORKS WITH GIVEN NUMBER OF DIRECTIONAL ANTENNAE OF BOUNDED ANGLE*

STEFAN DOBREV†

Institute of Mathematics, Slovak Academy of Sciences,
Bratislava, Slovak Republic.
Stefan.Dobrev@savba.sk

EVANGELOS KRANAKIS‡

School of Computer Science, Carleton University,
Ottawa, Ontario, K1S 5B6, Canada.
kranakis@scs.carleton.ca

DANNY KRIZANC

Department of Mathematics and Computer Science, Wesleyan University,
Middletown CT 06459, USA.
dkrizanc@wesleyan.edu

JAROSLAV OPATRNY§

Department of Computer Science and Software Engineering, Concordia University,
Montréal, Quebuec, H3G 1M8, Canada.
opatrny@cs.concordia.ca

OSCAR MORALES PONCE¶

School of Computer Science, Carleton University,
Ottawa, Ontario, K1S 5B6, Canada.
oscarmponce@gmail.com

LADISLAV STACHO‖

Department of Mathematics, Simon Fraser University,
8888 University Drive, Burnaby, British Columbia, V5A 1S6, Canada.
lstacho@math.sfu.ca

Traditional approaches to connectivity in sensor networks are based on the omnidirectional antenna model which relies on the assumption that the sensors send and receive in all directions. Current

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technologies make possible the utilization of sensors with directional antenna capabilities whereby the sensors send and/or receive along a sector of a predefined angle (or beam-width). Although several researchers in the scientific literature have investigated the impact of directional antennae on network throughput, energy consumption, as well as security very little is known concerning the effect of directional antennae on its connectivity. In this paper, we introduce for the first time a new sensor model with each sensor being able to transmit in any one of k directions, for some fixed k, and explore the algorithmic limits and potential of such a directional antenna model.

More specifically, given a set of n sensors in the plane, we consider the problem of establishing a strongly connected ad hoc network from these sensors using directional antennae. In particular, we prove that given such set of sensors, each equipped with k, $1 \leq k \leq 5$, directional antennae with any angle of transmission, these antennae can be oriented in such a way that the resulting communication structure is a strongly connected digraph spanning all n sensors. Moreover, the transmission range of the antennae is at most $2 \cdot \sin(\frac{\pi}{k+1})$ times the optimal range (a range necessary to establish a connected network on the same set of sensors using omnidirectional antennae). The algorithm which constructs this orientation runs in $O(n)$ time provided a minimum spanning tree on the set of sensors is given. We show that the previous solution can be used to give a tradeoff on the range and angle when each sensor has one antenna. Further, we also prove that for two antennae it is NP-hard to decide whether such an orientation exists if both the transmission angle and range are small for each antennae.

Keywords: Antenna; Directional Antenna; Minimum Spanning Tree; Sensors; Spanning Graphs; Strongly Connected.

1. Introduction

The sensors of a wireless network can be connected using either omnidirectional antennae that transmit in all directions around the sensor, or directional antennae that transmit only within a limited predefined angle. The energy usage of an antenna is proportional to its coverage area (for a directional antenna, this is usually taken as the area delimited by the angle of transmission and the range of the antenna). Therefore, directional antennae can often perform more efficiently than omnidirectional ones in order to attain overall network connectivity.

Given a set of sensors S, a necessary transmission range of an antenna can be determined as the smallest length of a longest edge over all minimum spanning trees constructed on the set of sensors S. In this paper we will refer to this length as an optimal range for the set of sensors S. A reasonable way to lower energy consumption is by reducing the transmission angle of the antenna being used. However, by reducing antenna angles the graph may be disconnected, since direct communication between sensors can be lost. Therefore an interesting question is how to maintain network connectivity when antenna angles are being reduced while at the same time the transmission range of antennae is being kept as low as possible.

Formally, we consider a set S of n sensors in the plane. Let k, $1 \leq k \leq 5$ be an integer, and φ, $0 \leq \varphi \leq 2\pi$, an angle. Each sensor is equipped with k directional antennae of transmission angle φ and a given transmission range. The reception of each sensor is assumed to be omnidirectional. This network gives rise to a directed graph that models communication in the network as follows: The vertices are the sensors, and there is a directed edge (u,v) from sensor u to sensor v if v is within the transmission range of u, and it lies inside the sector of angle φ formed by an antenna at u.

We are interested in the problem of providing an algorithm for orienting the antennae at each sensor, and estimating the value of transmission range so that we obtain a strongly connected graph which spans all the sensors.

1.1. *Preliminaries and Notation*

Given k antennae of transmission angle φ in each sensor, let $r_k(S,\varphi)$ denote the minimum range of these antennae with which it is possible to direct the antennae at each sensor so that a strongly connected network (or spanning graph) on S is formed. A special case of this is when the angle $\varphi = 0$, i.e. there is a direct line connection, in which case we use the simpler notation $r_k(S) = r_k(S,0)$. Let $\mathcal{D}_k(S)$ be the set of all strongly connected digraphs on S which have out-degree at most k. For any digraph $G \in \mathcal{D}_k(S)$ let $r_k(G)$ be the length of a longest edge of G. It is easy to see that $r_k(S) = \min_{G \in \mathcal{D}_k(S)} r_k(G)$.

It is useful to relate $r_k(S)$ to another quantity which arises from a Minimum Spanning Tree (MST) on S. Let $MST(S)$ denote the set of all MSTs on S. For $T \in MST(S)$ let $r(T)$ denote the length of longest edge of T, and let $r_{MST}(S) = \min\{r(T) : T \in MST(S)\}$. Clearly, for any angle $\varphi \geq 0$ we have that $r_{MST}(S) \leq r_k(S,\varphi)$, since every strongly connected, directed graph on S has an underlying spanning tree.

1.2. *Related work*

The first paper to address this problem in the case when each sensor is equipped with one directional antenna is [4]. In that paper the authors present polynomial time algorithms for the case when the transmission angle of antennae is at least $8\pi/5$. For smaller angles they present approximation algorithms for the minimum range. When the angle is smaller than $2\pi/3$, they show that the problem of determining the minimum range which achieves strong connectivity is NP-hard.

A different problem is considered in a subsequent paper [2]. In this paper, each sensor has a fixed number of directional antennae, and the strong connectivity problem is considered under the assumption that the maximum (taken over all sensors) sum of antennae angles is minimized. The authors present trade-offs between antennae range and specified sums of antennae angles per sensor.

When each sensor has one antenna of transmission angle $\varphi = 0$, then our problem is equivalent to finding a Hamiltonian cycle that minimizes the length of its longest edge. This is a special case of the following well-known problem. For a set of n points $1,2,\ldots,n$ with associated edge weights $c(i,j)$ satisfying the triangle inequality the *Bottleneck Traveling Salesman Problem (BTSP)* is the problem of finding a Hamiltonian cycle on these points which minimizes the maximum weight of an edge, i.e., $\min\{\max_{(i,j)\in H} c(i,j) : H$ is a permutation of [n]$\}$. Paper [10] shows that no polynomial time $(2-\varepsilon)$-approximation algorithm is possible for BTSP unless $P = NP$, and it also gives a 2-approximation algorithm for this problem.

No results are known in the literature on the connection between the MST of a set of points and strongly connected spanning digraph with given out-degree on the same set of points, except for the following two papers somehow relating these two concepts: In [5] it is shown that to decide for a given set S of n points in the plane and a given real k, whether S

admits a spanning tree of maximum degree four whose sum of edge lengths does not exceed k is NP-hard. A simple algorithm to find a spanning tree that simultaneously approximates a shortest-path tree and a minimum spanning tree is given in [7].

Directional antennae can reduce the total energy consumption in the network in comparison with omnidirectional antennae. Furthermore, they are known to enhance ad hoc network capacity and performance. A theoretical model presented in [6] shows that when n omnidirectional antennae are optimally placed and assigned optimally chosen traffic patterns, the transport capacity is $\Theta(\sqrt{W/n})$, where W is the number of bits each antenna can transmit per second over the common channel(s). When both transmission and reception are directional, [14] proves $\sqrt{2\pi/(\varphi\beta)}$ capacity gain as well as corresponding throughput improvement, where φ is the transmission angle and $\beta/2\pi$ is the average proportion of the number of receivers inside the transmission zone that will get interfered with. Additional experimental studies confirm the importance of using directional antennae in ad hoc networking (see, for example, [1,9,8,11,12,13]).

1.3. *Results of the paper*

We are interested in estimating the value of $r_k(S,\varphi)$. The optimality of antennae ranges will be compared to $r_{MST}(S)$ called here the *optimal*, and without loss of generality $r_{MST}(S)$ will be normalized, i.e., $r_{MST}(S) = 1$. The two main results in this paper are the following.

Theorem 1.1. *Consider a set S of n sensors in the plane and suppose each sensor has k, $1 \le k \le 5$, directional antennae with transmission angle $\varphi \ge 0$. If the range of each antenna is at least $2 \cdot \sin(\frac{\pi}{k+1})$ times the optimal, then the antennae can be oriented at each sensor so that the resulting spanning digraph is strongly connected. Moreover, given an MST on the set of points S, such orientation can be constructed with additional $O(n)$ overhead.*

Note that the case $k = 1$ was derived in [10], and that the case $k = 5$ follows from the comment after Definition 2.1 .

Theorem 1.2. *For two antennae and angular sum of the antennae at most α, it is NP-hard to approximate the optimal range to within a factor of x, where x and α are the solutions of equations $x = 2\sin(\alpha)$ and $x = 1 + 2\cos(2\alpha)$.*

Using the identity $\cos(2\alpha) = 1 - 2\sin^2\alpha$ and solving the resulting quadratic equation we obtain numerical values $x \approx 1.30, \alpha \approx 0.45\pi$.

The proof of the first theorem is given in Section 2, and due to its length we split it into three parts. In Subsections 2.1,2.2, and 2.3 we deal with the case $k = 4$, $k = 3$ and $k = 2$ as Theorems 2.4, 2.5, and 2.6, respectively.

The pseudocode of Algorithm 1 that constructs a strongly connected spanning graph with max out-degree $2 \le k \le 5$, and range bounded by $2 \cdot sin(\pi/(k+1)$ times the optimal, is presented in Subsection 2.4.

Section 3 contains a tradeoff when each sensor has one antenna and Section 4 contains the proof of Theorem 1.2.

2. Upper Bound Result on Strongly Connected Spanners

We begin by introducing some notation which is required for the subsequent proofs.

$D(u;r)$ denotes the open disk with radius r, centered at u, and $C(u,r)$ is the circle with radius r and centered at u. We use $d(u,v)$ to denote the usual Euclidean distance between points u and v. We say that two neighbours of a vertex u are *consecutive* if the smaller sector they form with u does not contain any other neighbour of u. In addition, we define below the concept of *Antenna-Tree* (*A-Tree* for short) which isolates the particular properties of an MST that we need in the course of the proofs.

Definition 2.1. *An A-Tree is a tree T embedded in the plane satisfying the following three conditions:*

(1) Its maximum degree is five.
(2) The minimum angle among nodes with a common parent is at least $\pi/3$.
(3) For any point u and any edge $\{u,v\}$ of T, the disk $D(v;d(u,v))$ does not contain a point $w \neq v$ which is also a neighbor of u in T.

It is well known and easy to prove that for any set of points in the plane there is an MST on these points which is also an A-Tree. Recall also that we consider normalized ranges i.e., we assume $r(T)=1$).

Definition 2.2. *For any real $r>0$, we define the geometric r-th power of an A-Tree T, denoted by T^r, as the graph obtained from T by adding all edges between vertices of (Euclidean) distance at most r.*

In the sequel we refer to *geometric r-th power* as *r-th power*, for simplicity.

Definition 2.3. *Let G be a graph. An orientation $\overrightarrow{G}$ of G is a digraph obtained from G by orienting every edge of G in at least one direction.*

As usual, (u,v) denotes a directed edge from u to v, whereas $\{u,v\}$ denotes an undirected edge between u and v. Furthemore, $d^{+}_{\overrightarrow{G}}(u)$ denotes the out-degree of u in $\overrightarrow{G}$ and $\Delta^{+}(\overrightarrow{G})$ denotes the maximum over out-degrees of vertices in $\overrightarrow{G}$.

2.1. *Maximum Out-Degree 4*

Theorem 2.4. *Let T be an A-Tree. Then there exists a spanning graph $G \subseteq T^{2\sin(\pi/5)}$ and its orientation $\overrightarrow{G}$ so that $\overrightarrow{G}$ is strongly connected and $\Delta^{+}(\overrightarrow{G}) \leq 4$. Moreover, $d^{+}_{\overrightarrow{G}}(u) \leq 1$ for each leaf u of T and every edge of T incident to a leaf is contained in G.*

Proof. We first introduce a definition used in this proof. We say that two consecutive neighbors of a vertex are *close* if the smaller angle they form with their common vertex is at most $2\pi/5$. Observe that if v and w are close, then $d(v,w) \leq 2\sin\pi/5$.

Let l be the diameter of T. The proof is done by induction on the diameter of the tree. First, we do the base case for $l \leq 2$. If $l \leq 1$, let $G=T$ and the result follows trivially. If $l=2$, then T is an A-Tree which is a star with $2 \leq d \leq 5$ leaves. Two cases can occur:

(1) $d < 5$. Let $G = T$ and orient every edge in both directions. This results in a strongly connected digraph which trivially satisfies the hypothesis of the theorem.
(2) $d = 5$. Let u be the center of T. Two consecutive neighbors of u, say, v and w must be close. Let $G = T \cup \{\{v,w\}\}$ and orient edges of G as depicted in Figure 1. It is easy to check that G satisfies the hypothesis of the theorem.

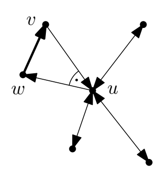


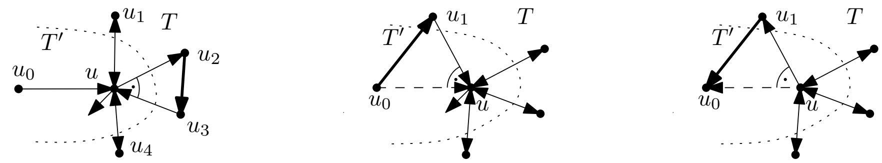
Fig. 1: T is a tree with five leaves and diameter $l = 2$ (The angular sign with a dot depicts an angle of size at most $2\pi/5$.)

Next we continue with the inductive step. Assume $l \geq 3$ and that the theorem is valid for any A-Tree of diameter $< l$. Let T be an A-Tree of diameter l. Consider T', the tree obtained from T by removing all leaves. Since the removal of leaves does not violate the property of being an A-Tree, T' is also an A-Tree and has diameter less than the diameter of T. Thus, by inductive hypothesis, there exists $G' \subseteq T'^{2\sin\pi/5}$ and its orientation $\overrightarrow{G'}$ which is strongly connected, and $\Delta^+(\overrightarrow{G'}) \leq 4$. Moreover, $d^+_{\overrightarrow{G'}}(u) \leq 1$ for each leaf u of T' and every edge of T' incident to a leaf is contained in G'.

Now we add all the removed leaves back to T and construct G from G' as well as corresponding orientation $\overrightarrow{G}$. We will add all removed vertices at once for each leaf u of T'. We describe this process only for fixed u. By the way we modify G' and since the diameter of T is at least three, all these modifications are independent so well defined. After we add all removed vertices the resulting graph G will be a spanning subgraph of $T^{2\sin(\pi/5)}$ and its orientation $\overrightarrow{G}$ will have all the required properties. Following is the required modification for a fixed leaf u of T'. Let u_0 be the neighbor of u in T' and $u_1,..,u_c$ be the c neighbors of u in $T \setminus T'$ in clockwise order around u starting from u_0. Two cases can occur:

(1) $c \leq 3$. Let $G = G' \cup \{\{u,u_1\},..,\{u,u_c\}\}$ and orient these c edges in both directions thus obtaining $\overrightarrow{G}$. The graph $G \subseteq T^{2\sin\pi/5}$, $\Delta^+(\overrightarrow{G}) \leq 4$, $d^+_{\overrightarrow{G}}(x) \leq 1$ for each leaf x adjacent to u in T, and every edge of T joining u and a leaf is contained in G.
(2) $c = 4$. We consider two cases. In the first case suppose that two consecutive neighbors of u in $T \setminus T'$ are close. Consider that u_j and u_{j+1} are close; where $1 \leq j < 4$. Define $G = G' \cup \{\{u,u_1\},\{u,u_2\},\{u,u_3\},\{u,u_4\},\{u_j,u_{j+1}\}\}$ and orient edges of G as depicted in Figure 2a.

In the second case, either u_0 and u_1 are close, or u_0 and u_4 are close. Without loss of generality assume that u_0 and u_1 are close. Let $G = \{G' \setminus \{u,u_0\}\} \cup \{\{u,u_1\},\{u,u_2\},\{u,u_3\},\{u,u_4\},\{u_0,u_1\}\}$, but now the orientation of G will de-



(a) $u_{j=2}$ and $u_{j+1=3}$ are close

(b) u_0 and u_1 are close and (u_0, u) is in the orientation of G'

(c) u_0 and u_1 are close and (u, u_0) is in the orientation of G'

Fig. 2: Depicting the inductive step when u has four neighbors in $T' \setminus T$ (The dashed edge $\{u_0, u\}$ indicates that it does not exist in G but exists in G', the angular sign with a dot depicts an angle of size at most $2\pi/5$, and the dotted curve is used to separate T' from T.)

pend on the orientation of $\{u, u_0\}$ in G'. Thus, if (u_0, u) is in $\overrightarrow{G'}$, then orient edges of G as depicted in Figure 2b, otherwise orient edges of G as depicted in Figure 2c. The graph $G \subseteq T^{2\sin\pi/5}$, $\Delta^+(\overrightarrow{G}) \leq 4$, $d^+_{\overrightarrow{G}}(x) \leq 1$ for each leaf x adjacent to u in T and every edge of T incident to u and a leaf is contained in G.

This completes the proof of the theorem. □

2.2. *Maximum Out-Degree 3*

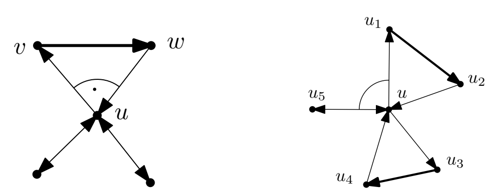
Theorem 2.5. *Let T be an A-Tree. Then there exists a spanning graph $G \subseteq T^{\sqrt{2}}$ and its orientation $\overrightarrow{G}$ which is strongly connected and $\Delta^+(\overrightarrow{G}) \leq 3$. Moreover, $d^+_{\overrightarrow{G}}(u) \leq 1$ for each leaf u of T and every edge of T incident to a leaf is contained in G.*

Proof. In this proof we say that two consecutive neighbors of a vertex are *close* if the smaller angle they form with their common vertex is at most $\pi/2$. Otherwise we say that they are *far*. Observe that if v and w are close, then $d(v, w) \leq \sqrt{2}$.

The proof is by induction on the diameter l of T. First, we do the base case for $l \leq 2$. If $l \leq 1$, let $G = T$ and the result follows trivially. If $l = 2$, then T is an A-Tree which is a star with $2 \leq d \leq 5$ leaves, respectively. Three cases can occur:

(1) $d < 4$. Let $G = T$ and orient every edge in both directions. This results in a strongly connected digraph which trivially satisfies the hypothesis of the theorem.
(2) $d = 4$. Let u be the center of T. Since T is a star, two consecutive neighbors of u, say, u_1 and u_2 are close. Let $G = T \cup \{\{u_1, u_2\}\}$ and orient edges of G as depicted in Figure 3a. It is easy to check that $\overrightarrow{G}$ satisfies the hypothesis of the theorem.
(3) $d = 5$. Let u be the center of T and u_1, u_2, u_3, u_4, u_5 be the five consecutive neighbors of u in clockwise order around u starting at any arbitrary neighbor of u. Observe that at most two consecutive neighbors of u are far since T is a star and the angle between two nodes with a common parent is at least $\pi/3$. Assume without

loss of generality that u_5 and u_1 are far. Let $G = T \cup \{\{u_1,u_2\},\{u_3,u_4\}\}$ and orient edges of G as depicted in Figure 3b. Thus, $\overrightarrow{G}$ satisfies trivially the hypothesis of the theorem.



(a) T has four leaves (b) T has five leaves

Fig. 3: T is a tree of diameter $l = 2$ (The angular sign with a dot depicts an angle of size at most $\pi/2$ and the angular sign depicts an angle of size greater than $\pi/2$.)

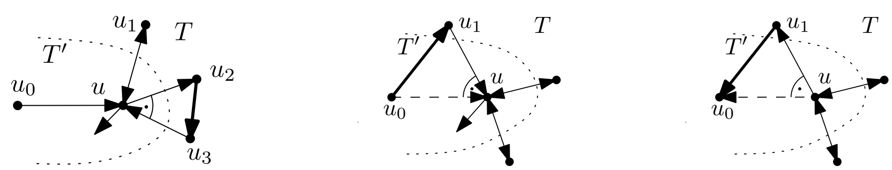
Next we continue with the inductive step. Assume $l \geq 3$ and that the theorem is valid for any A-Tree of diameter $< l$. Let T be an A-Tree of diameter l. Consider T', the tree obtained from T by removing all leaves. Since removal of leaves does not violate the property of being an A-Tree, T' is also an A-Tree and has diameter less than l. Thus, by inductive hypothesis there exists $G' \subseteq T'^{\sqrt{2}}$ and its orientation $\overrightarrow{G'}$ which is strongly connected, $\Delta^+(\overrightarrow{G'}) \leq 3$. Moreover, $d^{\pm}_{\overrightarrow{G'}}(u) \leq 1$ for each leaf u of T' and every edge of T' incident to a leaf is contained in G'.

Now we add all the removed leaves back to T and construct G from G' as well as corresponding orientation $\overrightarrow{G}$. We will add all removed vertices at once for each leaf u of T'. As before, we describe this process only for fixed u. By the way how we modify G' and since the diameter of T is at least three, all these modifications are independent so well defined. After we add all removed vertices the resulting graph G will be a spanning subgraph of $T^{\sqrt{2}}$ and its orientation $\overrightarrow{G}$ will have all the required properties. Following is the required modification for a fixed leaf u of T'. Let u be a leaf of T', u_0 be the neighbor of u in T' and $u_1, \ldots u_c$ be the c neighbors of u in $T \setminus T'$ in clockwise order around u starting from u_0. Three cases can occur:

(1) u has at most two neighbors in $T \setminus T'$. Let $G = G' \cup \{\{u,u_1\},\{u,u_2\}\}$ and orient these c edges in both directions. The graph $G \subseteq T^{\sqrt{2}}$, $\Delta^+(\overrightarrow{G}) \leq 3$, $d^{\pm}_{\overrightarrow{G}}(x) \leq 1$ for each leaf x adjacent to u in T, and every edge of T joining u and a leaf is contained in G.

(2) u has three neighbors in $T \setminus T'$. We consider two cases. In the first case suppose that two consecutive neighbors of u in $T \setminus T'$ are close. Assume that u_j and u_{j+1} are close; where $1 \leq j < 3$. Let $G = G' \cup \{\{u,u_1\},\{u,u_2\},\{u,u_3\},\{u_j,u_{j+1}\}\}$ and orient edges of G as depicted in Figure 4a.

In the second case, either u_0 and u_1 are close or u_0 and u_3 are close. Without loss of generality assume that u_0 and u_1 are close. Thus, let $G = \{G' \setminus \{u,u_0\}\} \cup \{\{u,u_1\},\{u,u_2\},\{u,u_3\},\{u_0,u_1\}\}$. Now the orientation of G will depend on the orientation of $\{u,u_0\}$ in G'. Thus, if (u_0,u) is in $\overrightarrow{G'}$, then orient edges of G as depicted in Figure 4b. Otherwise orient edges of G as depicted in Figure 4c. The graph $G \subseteq T^{\sqrt{2}}$, $\Delta^+(\overrightarrow{G}) \leq 3$, $d^+_{\overrightarrow{G}}(x) \leq 1$ for each leaf x of T incident to u, and every edge of T joining u and a leaf is contained in G.



(a) $u_{j=2}$ and $u_{j+1=3}$ are close

(b) u_0 and u_1 are close and (u_0,u) is in the orientation of G'

(c) u_0 and u_1 are close (u,u_0) is in the orientation of G'

Fig. 4: Depicting the inductive step when u has three neighbors in $T \setminus T'$ (The dashed edge $\{u,u_0\}$ indicates that it does not exist in G but exists in G', the angular sign with a dot depicts an angle of size at most $\pi/2$ and the dotted curve is used to separate T' from T.)

(3) u has four neighbors in $T \setminus T'$. We consider two cases. In the first case suppose that either u_0 and u_1 are far or u_2 and u_3 are far or u_4 and u_0 are far. Assume without loss of generality that u_0 and u_1 are far. Let

$$G = G' \cup \{\{u,u_1\},\{u,u_2\},\{u,u_3\},\{u,u_4\},\{u_1,u_2\},\{u_3,u_4\}\}$$

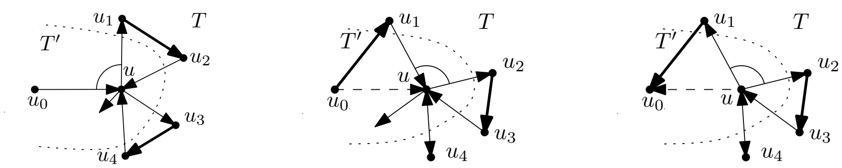
and orient edges of G as depicted in Figure 5a.

In the second case, assume either u_1 and u_2 are far or u_3 and u_4 are far. Assume without loss of generality that u_1 and u_2 are far. Let $G = \{G' \setminus \{u,u_0\}\} \cup \{\{u,u_1\},\{u,u_2\},\{u,u_3\},\{u,u_4\}\{u_0,u_1\},\{u_2,u_3\}\}$. The orientation $\overrightarrow{G}$ will depend on the orientation of $\{u,u_0\}$ in G'. Thus, if (u_0,u) is in $\overrightarrow{G'}$, then orient edges of G a s depicted in Figure 5b. Otherwise orient edges of G as depicted in Figure 5c. The graph $G \subseteq T^{\sqrt{2}}$, $\Delta^+(\overrightarrow{G}) \leq 3$, $d^+_{\overrightarrow{G}}(x) \leq 1$ for each leaf x of T adjacent to u, and every edge of T joining u and a leaf is contained in G.

This completes the proof of the theorem. □

2.3. *Maximum Out-Degree 2*

Theorem 2.6. *Given an A-Tree T, there exists a spanning graph $G \subseteq T^{\sqrt{3}}$ and its orientation $\overrightarrow{G}$ which is strongly connected and $\Delta^+(\overrightarrow{G}) \leq 2$. Moreover, for each leaf u of T, $d^+_{\overrightarrow{G}}(u) \leq 1$, and either the edge incident to u is in G or u has two other siblings (one immediately preceding it and other immediately following it in the embedding of T) and u is adjacent to both in G.*



(a) u_0 and u_1 are far

(b) u_1 and u_2 are far and (u_0, u) is in the orientation of G'

(c) u_1 and u_2 are far and (u, u_0) is in the orientation of G'

Fig. 5: Depicting the inductive step when u has four neighbors in $T \setminus T'$ (The dashed edge $\{u_0, u\}$ indicates that it does not exist in G but exits in G', the angular sign depicts an angle of size greater than $\pi/2$ and the dotted curve is used to separate the tree T' from T.)

Before proving Theorem 2.6, we need to introduce a definition and two lemmas which provide information on the proximity of two vertices with a common parent.

In the rest of this section we say that two neighbors of a vertex are *close* if the distance between them is at most $\sqrt{3}$. Otherwise we say that they are *far*.

Lemma 2.7. *Let u and v be two consecutive siblings in an A-Tree with common parent p such that $\alpha = \angle(upv) \le 2\pi/3$ and v is at distance one from p. Then, a child v' of v with angle $\angle(pvv') \le \gamma$ is close to u; where:*

$$\gamma = \begin{cases} \frac{5\pi}{3} - 2\alpha & \text{if } \frac{\pi}{3} \le \alpha \le \frac{\pi}{2} \\ \frac{2\pi}{3} & \text{if } \frac{\pi}{2} < \alpha \le \frac{\pi}{6} + \arccos\left(\frac{1}{2\sqrt{3}}\right) \\ \frac{5\pi}{9} & \text{if } \frac{\pi}{6} + \arccos\left(\frac{1}{2\sqrt{3}}\right) < \alpha \le \frac{2\pi}{3} \end{cases}$$

Proof. We prove each case separately:

(1) Consider a fixed angle $\frac{\pi}{3} \le \alpha \le \frac{\pi}{2}$. Observe that $2\cos(\alpha) \le d(u,p) \le 1$, since from definition of A-Tree, $u \notin D(v; d(v,p))$. Consider the intersection area I among all the disk of radius $\sqrt{3}$ centered at each point u with angle $\angle(upv) = \alpha$ and $2\cos(\alpha) \le d(u,p) \le 1$ as depicted in Figure 6. Observe that each neighbor of v inside I is close to u. It is sufficient to calculate the minimum angle with apex at v that covers I. Observe that it is determined by $D(u; \sqrt{3})$ where $d(u,p) = 2\cos(\alpha)$. Fix u at distance $2\cos(\alpha)$ from p and angle $\angle(upv) = \alpha$. Let $y \in C(u; \sqrt{3}) \cap C(v; 1)$ be the intersection point in I. Let $\angle(pvy) = \angle(pvu) + \angle(uvy)$. It is easy to see that $\angle(pvu) = \pi - 2\alpha$ and from the Law of cosine in the triangle uvy, $\angle(uvy) = 2\pi/3$ since $d(u,y) = \sqrt{3}$, and $d(u,v) = d(v,y) = 1$. Therefore, $\angle(pvy) \le \gamma = \frac{5\pi}{3} - 2\alpha$.

(2) Consider a fixed angle $\frac{\pi}{2} < \alpha \le \pi/6 + \arccos(\frac{1}{2\sqrt{3}})$. Since $\alpha > \frac{\pi}{2}$, $0 < d(u,p) \le 1$. Consider the intersection area I among all the disk of radius $\sqrt{3}$ centered at each point u with angle $\angle(upv) = \alpha$ and $0 < d(u,p) \le 1$ as depicted in Figure 7. Observe that each neighbor of v inside I is close to u. It is sufficient to calculate the minimum angle with apex at v that covers I. Consider $y \in C(p; \sqrt{3}) \cap C(u, \sqrt{3}))$

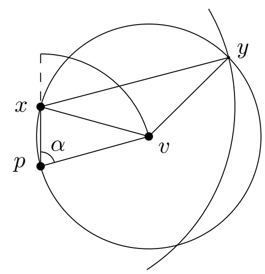


Fig. 6: Depicting the case when $\pi/3 \leq \alpha \leq \pi/2$

be the intersection near v where $d(p,u) = 1$ and $v' \in C(p,1) \cap C(y,1)$ be the intersection point furthest from u. If $\alpha \leq \angle(upv')$, then the minimum angle is determined by $D(p,\sqrt{3})$. Using the Law of cosine in upy and $pv'y$, $\angle(upv') = \pi/6 + \arccos\left(\frac{1}{2\sqrt{3}}\right)$ since $d(u,p) = d(p,v') = d(v'y) = 1$ and $d(u,u) = d(p,y) = \sqrt{3}$. Let $y \in C(p;\sqrt{3}) \cap C(v;1)$ be the intersection point in I. Hence, $\angle(pvy) \leq \gamma = 2\pi/3$.

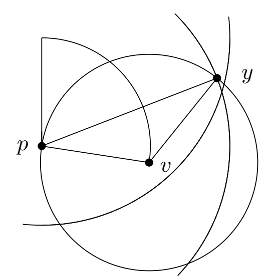


Fig. 7: Depicting the case when $\pi/2 < \alpha \leq \pi/6 + \arccos(1/2\sqrt{3})$

(3) Consider a fixed angle $\frac{\pi}{6} + \arccos\left(\frac{1}{2\sqrt{3}}\right) < \alpha \leq \frac{2\pi}{3}$. Since $\alpha > \frac{\pi}{2}$, $0 < d(u,p) \leq 1$. Consider the intersection area I among all the disk of radius $\sqrt{3}$ centered at each point u with angle $\angle upv = \alpha$ at distance in the interval $(0,1]$ from p as depicted in Figure 7. Observe that each neighbor of v inside I is close to u. It is sufficient to calculate the minimum angle with apex at v that covers I. However, from the previous case, it is determined by $D(u,\sqrt{3})$ where $d(u,p) = 1$. Moreover, the angle decreases when α increases. Therefore, the minimum angle is reached when the $\alpha = 2\pi/3$. Thus, fix $\alpha = 2\pi/3$. Let $y \in C(v,1) \cap C(u,\sqrt{3})$ be the intersection point in I as depicted in Figure 8. From the law of cosine in the triangle upv, $\angle(pvu) = \pi/6$ since $d(u,p) = d(p,v) = 1$. Similarly, by law of cosine in the triangle uvy, $\angle(uvy) = \arccos(\frac{1}{2\sqrt{3}})$ since $d(v,y) = 1$ and $d(u,y) = d(u,v) = \sqrt{3}$. Therefore, $\angle(pvy) \leq \frac{\pi}{6} + \arccos(\frac{1}{2\sqrt{3}}) > \frac{5\pi}{9}$.

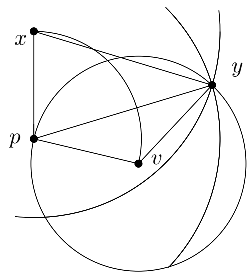


Fig. 8: Depicting the case when $\pi/6 + \arccos(1/2\sqrt{3}) < \alpha \leq 2\pi/3$

□

Lemma 2.8. *Let u, v and w be three consecutive siblings with parent p in an A-Tree T such that $\angle(upv) + \angle(vpw) \leq \pi$.*

(1) If $d(v) = 3$ and the only two children of v are far, then at least one of them is close to either u or w.
(2) If $d(v) = 4$ and each pair of consecutive children of v are close, then at least one of them is close to either u or w.
(3) If $d(v) = 4$, two consecutive children of v are far and all children of v are at distance at least $\sqrt{3}-1$ of v, then one child of v is close to u and another child of v is close to w.
(4) If $d(v) = 4$, two consecutive children of v are far and one child x of v is at distance at most $\sqrt{3}-1$ of v, then at most one child of v different from x are far from u and w.
(5) If $d(v) = 5$, then at least one child of v is close to either u or w.

Proof. Let $\alpha = \angle(upv)$ and $\beta = \angle(vpw)$. We first prove the particular cases when $\alpha + \beta = \pi$ and $d(p, v) = 1$. After that, we prove the general case when $d(p, v) < 1$ and/or $\alpha + \beta < \pi$.

Without loss of generality, consider $\pi/3 \leq \alpha \leq \pi/2$. Let $\beta = \pi - \alpha$. Using Lemma 2.7 we divide the circle into three different regions: $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ as depicted in Figure 9 in such a way that: $\angle\mathcal{C} \geq \frac{5\pi}{3} - 2\alpha$ and if $\alpha \leq 5\pi/6 - \arccos(\frac{1}{2\sqrt{3}})$, then $\angle\mathcal{D} \geq \frac{5\pi}{9}$. Otherwise, $\angle\mathcal{D} = 2\pi/3$. Let $\angle(\mathcal{E}) < 2\pi - (\angle(\mathcal{C}) + \angle(\mathcal{D}))$. i.e., if $\alpha \leq 5\pi/6 - \arccos(\frac{1}{2\sqrt{3}}) \leq \frac{13\pi}{30}$, then $\angle(\mathcal{E}) < 2\alpha - \frac{2\pi}{9} \leq 29\pi/45$. Otherwise, $\angle(\mathcal{E}) < \frac{2\pi}{3}$. Observe that the neighbors of v inside $\mathcal{C}$ are close to u and the neighbors of v inside $\mathcal{D}$ are close to w and the neighbors of v inside $\mathcal{E}$ are (possibly) far from u or w.

Let $v_0 = p, v_1, \cdots, v_c$ the neighbors of v in clockwise order. Now, we prove each case of Lemma 2.8.

(1) $d(v) = 3$ and v_1 is far from v_2. At most one child of v can be in $\mathcal{E}$, since $\angle(\mathcal{E})$ is less than $2\pi/3$ and $\angle(v_1 v v_2) \geq 2\pi/3$.
(2) If $d(v) = 4$ and each pair of consecutive children of v are close. Since the mini-

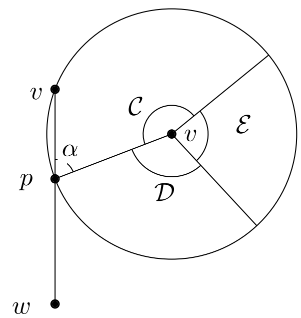


Fig. 9: Cones $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$ with apex at v

mum distance among children is $\pi/3$ and $\angle(\mathcal{E}) < 2\pi/3$, at most two children of v can be in $\mathcal{E}$.

(3) If $d(v) = 4$, two consecutive children of v are far and all children of v are at distance at least $\sqrt{3}-1$ from v. Since two children are far, $\angle(v_1 v v_3) > \pi$. Hence, when $\alpha \geq 5\pi/6 - \arccos(1/2\sqrt{3})$, $\angle(\mathcal{C}) \geq 2\pi/3$ and $\angle(\mathcal{D}) = 2\pi/3$. Therefore, $v_1 \in \mathcal{C}$ and $v_3 \in \mathcal{D}$. It remains to prove the case when $\alpha < 5\pi/6 + \arccos(1/2\sqrt{3})$. Assume without loss of generality that v_1 is close to u. ¿From Definition of A-Tree and the hypothesis, $u \notin D(v_1;1) \cup D(v;1)$ and $v_1 \notin D(v;\sqrt{3}-1)$. Let $y \in C(u;1) \cap C(v;\sqrt{3}-1)$ be the intersection point farthest from p as depicted in Figure 10. Therefore, $\angle(pvv_1) \geq \angle(pvy) = \angle(pvu) + \angle(uvy)$. We will prove that $\angle(pvy) \geq 4\pi/9$ and since two consecutive children are far, $\angle(wvv_3) \leq 2\pi - (\angle(pvv_1) + \pi) \leq 5\pi/9$. In consequence, $v_3 \in \mathcal{D}$. ¿From the Law of cosine in uvy, $\angle(uvy) \geq 17\pi/45$ since $d(u,y) \geq 1$, $d(u,v) \geq 1$ and $d(v,y) = \sqrt{3}-1$. Further, $\angle(pvu) \geq \pi - 2\alpha \geq 2\pi/15$ since $\alpha < 5\pi/6 + \arccos(1/2\sqrt{3})$ Therefore, $\angle(pvy) > 4\pi/9$.

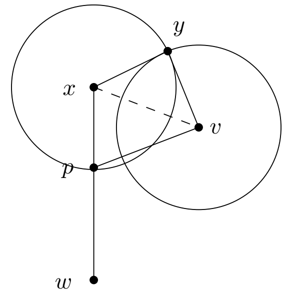


Fig. 10: Depicting when $d(v) = 4$, two consecutive children of v are far and all the children are at distance at least $\sqrt{3}-1$ from v

(4) If $d(v) = 4$, two consecutive children of v are far and one child of v is at distance at most $\sqrt{3}-1$ from v. Notice that if v_2 is at distance at most $\sqrt{3}-1$ from v, then

v_1 is close to v_2 and v_2 is close to v_3. Therefore, v_2 is at distance at least $\sqrt{3}-1$ from v and either v_1 or v_3 is at distance $\sqrt{3}-1$ from v. Assume without loss of generality that v_1 is at distance at most $\sqrt{3}-1$ from v. Therefore, v_2 is far from v_3 and only one of them can be inside $\mathcal{E}$ since $\angle(\mathcal{E})$ is less than $2\pi/3$.

(5) $d(v) = 5$. At most two children of v can be in $\mathcal{E}$, because $\angle(\mathcal{E})$ is less than $2\pi/3$ and two children are at distance at least $\pi/3$.

This proves the case when $d(p,v) = 1$. To prove the case when $d(p,v) < 1$, consider the intersection point v' with $C(p,1)$ and the ray emanating from p toward v. Therefore, $d(p,v) < d(p,v')$ and $d(u,v) < d(u,v')$. If we move all children of v toward v', the distance from u to them will increase. Hence, the solution for $d(p,v) = 1$ covers all cases in line segment $\overline{p,v'}$.

Now we prove the case when $\alpha+\beta < \pi$. Consider the line segment $\overline{u,w}$ and its intersection point p' with the edge $\{p,v\}$. Notice that by replacing p with p' we get $\alpha'+\beta' = \pi$ such that $\alpha < \alpha'$ and $\beta < \beta'$. Hence, the solution in the case $\alpha+\beta = \pi$ is also a solution when $\alpha+\beta < \pi$. This completes the proof of Lemma 2.7. □

Proof of Theorem 2.6. The proof is by induction on the diameter l of T. First, we do the base case $l \le 2$. If $l \le 1$, let $G = T$ and the result follows trivially.

If $l = 2$, then T is an A-Tree which is a star with $2 \le d \le 5$ leaves, respectively. Four cases can occur:

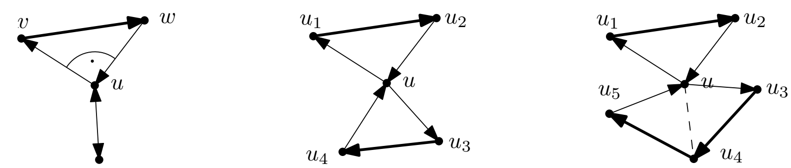
(1) $d = 2$. Let $G = T$ and orient every edge in both directions. This results in a strongly connected digraph which trivially satisfies the hypothesis of the theorem.

(2) $d = 3$. Let u be the center of T. Since T is a star, two consecutive neighbors, say u_1 and u_2 are close. Let $G = T \cup \{\{u_1,u_2\}\}$ and orient edges of G as depicted in Figure 11a. It is easy to check that $\overrightarrow{G}$ satisfies the hypothesis of the Theorem.

(3) $d = 4$. Let u be the center of T and u_1,u_2,u_3,u_4 be the four neighbors of u in clockwise order around u starting at any arbitrary neighbor of u. Observe that at most two consecutive neighbors of u are far since T is a star and the angle between two nodes with a common parent is at least $\pi/3$. Assume without loss of generality that u_4 and u_1 are far. Let $G = T \cup \{\{u_1,u_2\},\{u_3,u_4\}\}$ and orient edges of G as depicted in Figure 11b. Thus, $\overrightarrow{G}$ satisfies trivially the hypothesis of the Theorem.

(4) $d = 5$. Let u be the center of T and u_1,u_2,u_3,u_4,u_5 be the five neighbors of u in clockwise order around u starting at any arbitrary neighbor of u. Observe that all consecutive neighbors are close since T is a star and the angle between two nodes with a common parent is at least $\pi/3$. Let $G = T \setminus \{u,u_4\} \cup \{\{u_1,u_2\},\{u_3,u_4\},\{u_4,u_5\}\}$ and orient edges of G as depicted in Figure 11c. Observe that $\angle(u_3uu_5) \le \pi$. Orientation $\overrightarrow{G}$ is strongly connected and $\Delta^+(\overrightarrow{G}) \le 2$. Moreover, $d^+_{\overrightarrow{G}}(u) \le 1$, all edges of T except $\{u,u_4\}$ are contained in G and $\{u_3,u_4\}$ and $\{u_4,u_5\}$ are contained in G.

Next we continue with the inductive step. Assume $l \ge 3$ and that the theorem is valid for any A-Tree of diameter $< l$. Let T be an A-Tree of diameter l. Consider T', the



(a) T has three leaves (b) T has four leaves (c) T has five leaves

Fig. 11: T is a tree with diameter $l = 2$ (The heavy arrows represent the newly added edges, the angular sign with a dot depicts an angle of size at most $2\pi/3$ and dashed edge indicates that it exists in T but not in G.)

tree obtained from T by removing all leaves. Since removal of leaves does not violate the property of being an A-Tree, T' is also an A-Tree and has diameter less than l. Thus, by inductive hypothesis there exists $G' \subseteq T'^{\sqrt{3}}$ and its orientation $\overrightarrow{G'}$ which is strongly connected, $\Delta^+(\overrightarrow{G'}) \leq 2$. Moreover, for each leaf u of T', $d^+_{\overrightarrow{G'}}(u) \leq 1$, and either the edge incident to u is in G' or u has two other siblings (one immediately preceding it and other immediately following it in the embedding of T') and u is adjacent to both in G'.

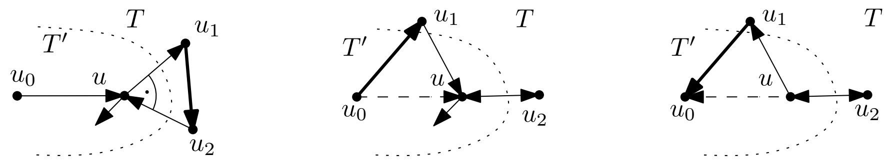
Now we add all the removed leaves back to T and construct G from G' as well as corresponding orientation $\overrightarrow{G}$. We will add all removed vertices at once for each leaf u of T'. As before, we describe this process only for fixed u. By the way how we modify G' and since the diameter of T is at least three, all these modifications are independent so well defined. After we add all removed vertices the resulting graph G will be a spanning subgraph of $T^{\sqrt{3}}$ and its orientation $\overrightarrow{G}$ will have all the required properties. Following is the required modification for a fixed leaf u of T'. Let u_0 be the neighbor of u in T' and $u_1, \ldots, u_c$ be the c neighbors of u in $T \setminus T'$ in clockwise order around u starting from u_0. Four cases can occur:

(1) u has one neighbor in $T \setminus T'$. Let $G = G' \cup \{\{u, u_1\}\}$ and orient it in both directions. It is easy to see that $\overrightarrow{G}$ satisfies the inductive hypothesis.

(2) u has two neighbors in $T \setminus T'$. We consider two cases. In the first case suppose that u_1 and u_2 are close. Let $G = G' \cup \{\{u, u_1\}, \{u, u_2\}, \{u_1, u_2\}\}$ and orient edges of G as depicted in Figure 12a. In the second case, u_1 and u_2 are far. Again we need to consider two cases:

   (a) $\{u_0, u\}$ is in G'. Either u_0 and u_1 are close or u_2 and u_0 are close. Without loss of generality assume that u_1 and u_0 are close. Let $G = \{G' \setminus \{u_0, u\}\} \cup \{\{u, u_1\}, \{u, u_2\}, \{u_0, u_1\}\}$. If (u_0, u) is in $\overrightarrow{G'}$, then orient edges of G as depicted in Figure 12b. Otherwise orient edges of G as depicted in Figure 12c. Thus, $\overrightarrow{G}$ is strongly connected and $\Delta^+(\overrightarrow{G}) \leq 2$. Moreover, the leaves u_1 and u_2 of T have degree one and the edges of T incident to them are contained in G.

   (b) $\{u_0, u\}$ is not in G' By inductive hypothesis, u is connected to its two sib-



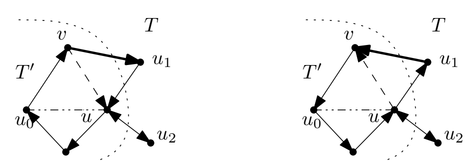
(a) u_1 and u_2 are close $\{u_0, u\}$

(b) u_0 and u_1 are far and (u, u_0) is in the orientation of G'

(c) u_0 and u_1 are far and (u, u_0) is in the orientation of G'

Fig. 12: Depicting the inductive step when u has two neighbors in $T \setminus T'$ (The dashed edge $\{u_0, u\}$ indicates that it does not exist in G but exists in G' and the dotted curve is used to separate T' from T.)

lings v and w in G'. Thus, by Lemma 2.8, either u_1 or u_2 are close to v or w. Without loss of generality assume that u_1 and v are close. Let $G = (G' \setminus \{v, u\}) \cup \{\{u_1, u\}, \{u_2, u\}, \{v, u_1\}\}$. If (v, u) is in $\overrightarrow{G'}$, then orient edges of G as depicted in Figure 13a. Otherwise orient edges of G as depicted in Figure 13b. Thus, $\overrightarrow{G}$ is strongly connected and $\Delta^+(\overrightarrow{G}) \leq 2$. Moreover, the leaves u_1 and u_2 of T have degree one and the edges of T incident to them are contained in G.



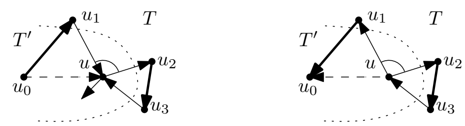
(a) (u_0, u) is in the orientation of G'

(b) (u, u_0) is in the orientation of G'

Fig. 13: Depicting the inductive step when u has two neighbors in $T \setminus T'$, u_0 and u_1 are far and $\{u_0, u\}$ is not in G' (The dashed edge $\{v, u\}$ indicates that it does not exist in G but exists in G', the dash dotted edge $\{u_0, u\}$ indicates that it exists in T' but not in G' and the dotted curve is used to separate T' from T.)

(3) u has three neighbors in $T \setminus T'$. Two cases can occur:

(a) $\{u_0, u\}$ is in G'. At most two neighbors of u are far. First, suppose that u_3 and u_0 are far (This case is equivalent to the case when u_1 and u_2 are far.) Let $G = \{G' \setminus \{u_0, u\}\} \cup \{\{u_1, u\}, \{u_2, u\}, \{u_3, u\}, \{u_1, u_0\}, \{u_2, u_3\}\}$. If (u_0, u) is in $\overrightarrow{G'}$, then orient edges of G as depicted in Figure 14a. Otherwise orient edges of G as depicted in Figure 14b. Thus, $\overrightarrow{G}$ is strongly connected and $\Delta^+(\overrightarrow{G}) \leq 2$. Moreover, the leaves u_1, u_2 and u_3 of T have degree one and the edges of T incident to them are contained in G. By symmetry, we can prove the case when u_1 and u_0 are far or u_2 and u_3 are far.



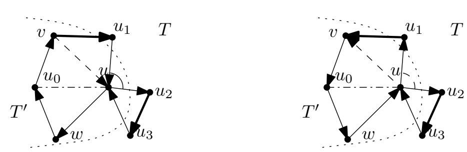
(a) (u_0, u) is in the orientation of G'

(b) (u, u_0) is in the orientation of G'

Fig. 14: Depicting the inductive step when u has three neighbors in $T \setminus T'$, u_1 and u_2 are far and $\{u_0, u\}$ is in G' (The dashed edge $\{u_0, u\}$ indicates that it does not exist in G but exists in G' and the dotted curve is used to separate T' from T.)

(b) $\{u_0, u\}$ is not in G'. By inductive hypothesis u is connected to its two siblings v and w in G'. Three cases can occur.

i. u_1 is close to u_2 and u_2 is close to u_3. By Lemma 2.8, either u_1 or u_3 is close to either v or w. Assume that v and u_1 are close. Let $G = \{G' \setminus \{v, u\}\} \cup \{\{u_1, u\}, \{u_2, u\}, \{u_3, u\}, \{v, u_1\}, \{u_2, u_3\}\}$. If (v, u) is in $\overrightarrow{G'}$, then orient edges of G as depicted in Figure 15a. Otherwise orient edges of G as depicted in Figure 15b. Thus, $\overrightarrow{G}$ is strongly connected and $\Delta^+(\overrightarrow{G}) \leq 2$. Moreover, the leaves u_1, u_2 and u_3 of T have degree one and the edges of T incident to them are contained in G.



(a) (v, u) is in the orientation of G'

(b) (u, v) is in the orientation of G'

Fig. 15: Depicting the inductive step when u has three neighbors in $T \setminus T'$, u_1 and u_2 are far and $\{u_0, u\}$ is not in G' (The dashed edge $\{v, u\}$ indicates that it does not exist in G but exists in G', the dash dotted edge $\{u_0, u\}$ indicates that it exists in T' but not in G' and the dotted curve is used to separate T' from T.)

ii. Either u_1 is far from u_2 or u_2 is far from u_3 and u_1, u_2 and u_3 are at distance greater than $\sqrt{3} - 1$ from u. By Lemma 2.8 u_1 is close to one sibling of u, say v and u_3 is close to another sibling of u, say w. Without loss of generality assume that u_2, u_3 are close and u_0, u_1 are close. Observe that this case is identical to the case i.

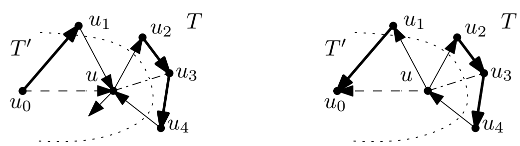
iii. Either u_1 is far from u_2 or u_2 is far from u_3 and at least one child of u is at distance less than $\sqrt{3}-1$. Without loss of generality assume that u_1 is far from u_2. Therefore, $d(u,u_1) > \sqrt{3}-1$ and $d(u,u_3) \leq \sqrt{3}-1$. Observe that u_3 is close to u_1 and u_2. By Lemma 2.8 either u_1 or u_2 are close to v or w. Thus, if v is close to u_1, then we can apply case i. If w is close to u_2, then let $u'_1 = u_2$, $u'_2 = u_1$ and $u'_3 = u_3$ and we can apply case i again.

(4) u has four neighbors in $T \setminus T'$. Two cases can occur:

(a) $\{u_0,u\}$ is in G'. Let

$$G = \{G' \setminus \{u_0,u\}\} \cup \{\{u_1,u\},\{u_2,u\},\{u_4,u\},\{u_1,u_0\},\{u_2,u_3\},\{u_3,u_4\}\}.$$

If (u_0,u) is in $\overrightarrow{G'}$, then orient edges of G as depicted in Figure 16a. Otherwise orient edges of G as depicted in Figure 16b. Thus, $\overrightarrow{G}$ is strongly connected and $\Delta^+(\overrightarrow{G}) \leq 2$. Moreover, the leaves u_1, u_2, u_3 and u_4 of T have degree one, the edges of T incident to u_1,u_2 and u_4 are contained in G and u_3 is adjacent to u_2 and u_4 in G. Observe that $\angle(u_2uu_4) \leq \pi/2$.



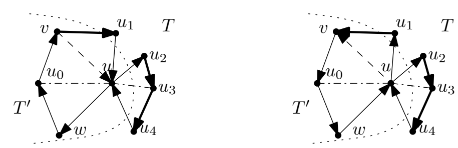
(a) (u_0,u) is in the orientation of G'

(b) (u,u_0) is in the orientation of G'

Fig. 16: Depicting the inductive step when u has four neighbors in $T \setminus T'$, $\{u_0,u\}$ is in G' (The dashed edge $\{u_0,u\}$ indicates that it does not exist in G but exists in G', the dotted curve is used to separate T' from T and the dash dotted edge $\{u,u_3\}$ indicates that it exists in T but not in G.)

(b) $\{u_0,u\}$ is not in G'. By inductive hypothesis u is connected to its two siblings v and w in G'. By Lemma 2.8 either u_1 or u_4 is close to v or w. Without loss of generality assume that u_1 and v are close. Let $G = \{G' \setminus \{v,u\}\} \cup \{\{u_1,u\},\{u_2,u\},\{u_4,u\},\{v,u_1\},\{u_2,u_3\},\{u_3,u_4\}\}$. If (v,u) is in $\overrightarrow{G'}$, then orient edges of G as depicted in Figure 17a. Otherwise orient edges of G as depicted in Figure 17b. Thus, $\overrightarrow{G}$ is strongly connected and $\Delta^+(\overrightarrow{G}) \leq 2$. Moreover, u_1, u_2, u_3 and u_4 have degree one, the edges of T incident to u_1,u_2 and u_4 are contained in G and u_3 is adjacent to u_2 and u_4 in G.

This completes the proof of the theorem. $\square$



(a) (v,u) is in the orientation of G'

(b) (u,v) is in the orientation of G'

Fig. 17: Depicting the inductive step when u has four neighbors in $T \setminus T'$ and $\{u_0, u\}$ is not in G' (The dashed edge $\{v, u\}$ indicates that it does not exist in G but exists in T', the dotted curve is used to separate T' from T, the dash dotted edge $\{u_0, u\}$ indicates that it exists in T' but not in G' and the dash dotted edge $\{u, u_3\}$ indicates that it exist in T but not in G.)

2.4. *Main Algorithm*

In this section we present Algorithm 1 that constructs a strongly connected spanning graph with max out-degree $2 \le k \le 5$ and range bounded by $2 \cdot \sin\left(\frac{\pi}{k+1}\right)$ times the optimal. It uses the recursive Procedure kAntennae when $3 \le k \le 5$ and the recursive Procedure TwoAntennae when $k = 2$. See the detailed algorithms for these two procedures further below.

It is not difficult to see that Algorithm 1 runs in $O(n)$ time. The correctness of the algorithm is derived from Theorems 2.4, 2.5 and 2.6.

Algorithm 1: Strongly connected spanning graph with max out-degree $2 \le k \le 5$ and edge length bounded by $2 \cdot \sin\left(\frac{\pi}{k+1}\right)$

input : T, k; where T is an MST with max length 1 and k an integer in $[2,5]$.
output: Strongly connected spanning graph G with max out-degree k and range bounded by $2 \cdot \sin\left(\frac{\pi}{k+1}\right)$

Let u be any leaf of T and v its neighbor in T;
Let $G \leftarrow \{(v,u),(u,v)\}$;
if $k = 2$ **then** TwoAntennae(G, T, v, u);
if $3 \le k < 5$ **then** kAntennae(G, T, v, u, k);

3. Treadoffs on the Range and Angle

In this section we show how to use Theorem 1.1 to derive a treadoff on the range and angle when each sensor has only one antenna.

Theorem 3.1. *Given a set of sensors in the plane with one directional antenna each and an angle $\phi \ge \pi$, there exists an orientation of the antennae of angle ϕ and range r that*

Procedure kAntennae(G,T,u,w,k)

```
Let u_0 = w, u_1, ..., u_{d(u)-1} be the neighbors of u ∈ T in clockwise order around u;
if d(u) ≤ k then  Add to G a bidirectional arc for each u_i such that i > 0;
else if d(u) = k+1 then
    Let u_i, u_{i+1} be the consecutive neighbor of u with smallest angle;
    if i = 0 or i+1 = 0 then
        if i = 0 then  Let i ← 1 ;
        if (u,u_0) ∈ G then  Let G ← {G \ {(u,u_0)}} ∪ {(u,u_i),(u_i,u_0)};
        else  Let G ← {G \ {(u_0,u)}} ∪ {(u_0,u_i),(u_i,u)};
    end
    else  Let G ← G ∪ {(u,u_i),(u_i,u_{i+1}),(u_{i+1},u)};
    Add to G a bidirectional arc for each u_j such that j ∉ {0,i,i+1};
end
else if d(u) = k+2 then
    Let u_i, u_{i+1} be the consecutive neighbors of u with longest angle;
    if i = 0 or  i = 2 or  i = 4 then  Let
    G ← G ∪ {(u,u_1),(u_1,u_2),(u_2,u),(u,u_3),(u_3,u_4),(u_4,u)};
    else
        if (u,u_0) ∈ G then  Let G ← {G \ {(u,u_0)}} ∪ {(u,u_1),(u_1,u_0)} ;
        else  Let G ← {G \ {(u_0,u)}} ∪ {(u_0,u_1),(u_1,u)} ;
        Let G ← G ∪ {(u,u_2),(u_2,u_3),(u_3,u),(u,u_4),(u_4,u)};
    end
end
for i ← 1 to d(u)-1 do if d(u_i) > 1 then  G ←kAntennae(G,T,u_i,u,k)  ;
```

results in a strongly connected network; where

$$r \leq \begin{cases} 1 & \textit{if } \phi \in [8\pi/5, 2\pi] \\ 2\sin(\pi/5) & \textit{if } \phi \in [3\pi/2, 8\pi/5) \\ \sqrt{2} & \textit{if } \phi \in [4\pi/3, 3\pi/2) \\ \sqrt{3} & \textit{if } \phi \in [\pi, 4\pi/3) \end{cases}$$

Proof. Let G be the strongly connected digraph with out-degree $k \in [2,5]$ and range bounded by $2\sin(\frac{\pi}{k+1})$ times the longest edge of the MST obtained from Theorem 1.1. Since G is strongly connected, it is sufficient for every vertex to cover its out-going edges. By the pigeon hole principle, every vertex v of degree $d^+(v)$ has a circular sector between two consecutive outgoing edges of angle at least $2\pi/d^+(v)$. Therefore, an antenna of angle at most $2\pi - 2\pi/k$ is always sufficient to cover all outgoing edges of v since $d^+(v) \leq k$. Clearly, for $k \in [2,5]$ an antenna of angle of $2\pi - 2\pi/k$ with range $2\sin(\frac{\pi}{k+1})$ times the longest edge of the MST is always sufficient. The theorem easily follows. □

Procedure TwoAntennae(G, T, u, w)

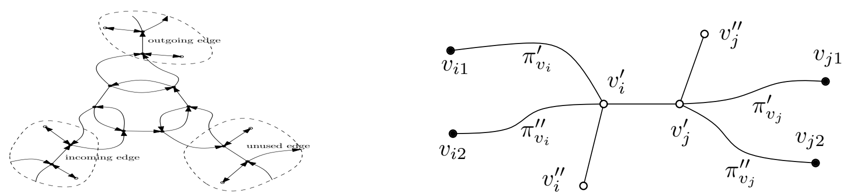
```
Let u_0 = w, u_1, ···, u_{d(u)-1} be the neighbors of u ∈ T in clockwise order around u;
if d(u) = 2 then Let G ← G ∪ {(u, u_1), (u_1, u)};
if d(u) = 3 then
    if u_1 is close to u_2 then Let G ← G ∪ {(u, u_1), (u_1, u_2), (u_2, u)} ;
    else
        if (u, u_0) ∈ G or (u_0, u) ∈ G then Let v ← u_0 and v' be the closest neighbor
        to u_0 and x be the neighbor of u ∈ T different to v' and v;
        else Let v be the sibling of u in T closest to a neighbor v' ≠ u_0 of u and x be
        the neighbor of u ∈ T different to v' and u_0;
        if (v, u) ∈ G then Let G ← {G \ {(v, u)}} ∪ {(v, v'), (v', u)};
        else Let G ← {G \ {(u, v)}} ∪ {(u, v'), (v', v)};
        Let G ← G ∪ {(u, x), (x, u)};
    end
end
if d(u) = 4 then
    if (u, u_0) ∈ G or (u_0, u) ∈ G then
        if u_0 is far to u_3 or u_1 is far to u_2 then Let v ← u_0, v' ← u_1, x ← u_2, x' ← u_3;
        else Let v ← u_0, v' ← u_3, x ← u_1, x' ← u_2;
    end
    else Let v be the sibling of u in T closest to a neighbor v' ≠ u_0 of u and and x, x'
    be the closest neighbors of u different to v' and u_0;
    if (v, u) ∈ G then Let G ← {G \ {(v, u)}} ∪ {(v, v'), (v', u)};
    else Let G ← {G \ {(u, v)}} ∪ {(u, v'), (v', v)};
    Let G ← G ∪ {(u, x), (x, x')(x', u)};
end
else
    if (u, u_0) ∈ G or (u_0, u) ∈ G then Let v ← u_0, v' ← u_1, j ← 2;
    else Let v be the sibling of u in T closest to a neighbor v' ≠ u_0 of u and and
    u_j, u_{j+1}, u_{j+2} be the three consecutive neighbors of u different to v' and u_0;
    if (v, u) ∈ G then Let G ← {G \ {(v, u)}} ∪ {(v, v'), (v', u)};
    else Let G ← {G \ {(u, v)}} ∪ {(u, v'), (v', v)};
    Let G ← G ∪ {(u, u_j), (u_j, u_{j+1}), (u_{j+1}, u_{j+2}), (u_{j+2}, u)};
end
for i ← 1 to d(u) − 1 do if d(u_i) > 1 then G = TwoAntantennae(G, T, u_i, u);
```

4. NP hardness

In this section we give the proof of the NP hardness result for two antennae.

Proof of Theorem 1.2. It is done by reduction from the well-known NP-hard problem of existence of a Hamiltonian cycle in 3-regular planar graphs. Consider a 3-regular planar graph $G = (V, E)$ and replace each vertex v_i by a vertex-graph (meta-vertex) G_{v_i} shown in Figure 18a. Furthermore, replace each edge $e = \langle v_i, v_j \rangle$ of G by an edge-graph (meta-edge) G_e shown in Figure 18b.

Each meta-vertex has three parts connected in a cycle, with each part consisting of a pair of vertices (called *connecting vertices*) connected by two paths. Each meta-edge G_e has a pair of connecting vertices at each endpoint: these vertices coincide with the connecting vertices in the corresponding parts of the meta-vertices G_{v_i} and G_{v_j}. This means that after each vertex and each edge is replaced, each connecting vertex is of degree 4.



(a) Vertex graph (The dotted ovals delimit the three parts.)

(b) Edge graph (The connecting vertices are black.)

Fig. 18: Meta-vertex and meta-edge for the NP hardness proof

Take the resulting graph G' and embed it in the plane in such a way that:

(1) the distance (in the embedding) between neighbours in G' is at most 1,
(2) the distance between non-neighbours in G' is at least x, and
(3) the smallest angle between incident edges in G' is at least α.

Let us call the resulting embedded graph G''. Note that such an embedding always exists, see [3]: We have the freedom to choose the length of the paths in the meta-graphs the way we need as we can stretch the configurations apart to fit everything in without violating the embedding requirements. The only constraining places are the midpoints of the meta-edges and the three places in each meta-vertex where the parts are connected to each other. These can be embedded as shown in the right part of Figure 19. Note that the need to embed these parts without violating embedding requirements gives rise to the equations defining x and α (see Figure 19). This completes details of the main construction. □

The proof of the Theorem is based on the following claim:

Claim 4.1. *There is a Hamiltonian cycle in G if and only if there exists an assignment of two antennae with sum of angles less than α and range less than x to the vertices of G'' such that the resulting connectivity graph is strongly connected.*

Proof. First we show that if G has a Hamiltonian cycle then there exists the assignment of such antennae that makes the resulting connectivity graph of G'' strongly connected. Figure 20 shows antenna assignments in the meta-edges corresponding to edges used and

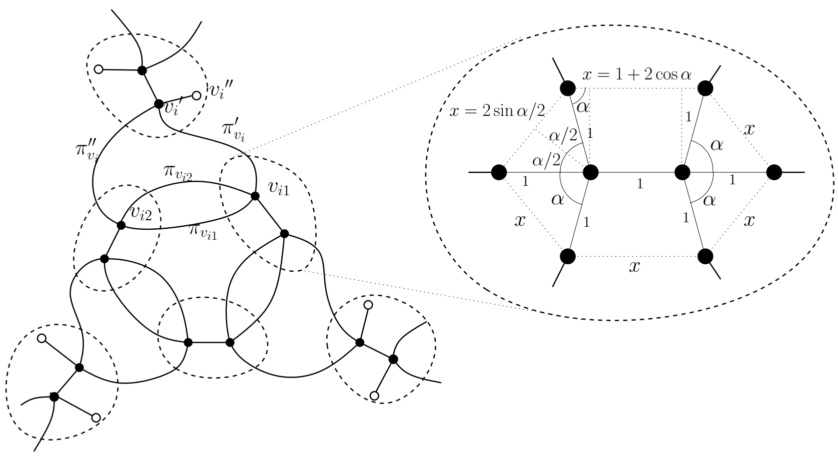


Fig. 19: Connecting meta-edges with meta-vertices. The dashed ovals show the places where embedding is constrained.

not used by the Hamiltonian cycle, respectively. Figure 21 shows the antenna assignments in a meta-vertex. Since each vertex of G has one incoming, one outgoing and one unused incident edge, and each edge is either used in one direction, or not used at all, this provides the full description of antenna assignments in G''.

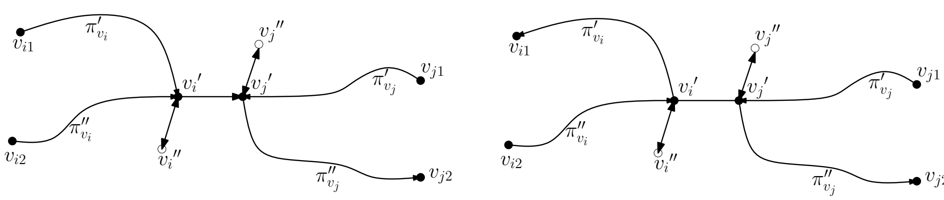


Fig. 20: Left: antenna assignments in a meta-edges corresponding to an edge used in the Hamiltonian cycle from v_i to v_j. Right: antenna assignments in a meta-edge corresponding to an unused edge.

Observe that the connecting pair of vertices at the meta-vertex uses two antennae towards the meta-edge it is connected to if and only if this meta-edge is outgoing; otherwise only one antenna is used towards the meta-edge and another is used towards the next part of the meta-vertex. It is easy to verify that the resulting connectivity graph is strongly connected:

(1) if the edge $e = \langle v_i, v_j \rangle$ is not used in the Hamiltonian path in the direction from v_i to v_j, then the near half of the meta-edge G_e (i.e. v'_j, v''_j, π'_{v_j} and π''_{v_j}) together with the connecting part of the meta-vertex G_{v_j} form a strongly connected subgraph,
(2) in each meta-vertex the part corresponding to the outgoing edge is reachable from

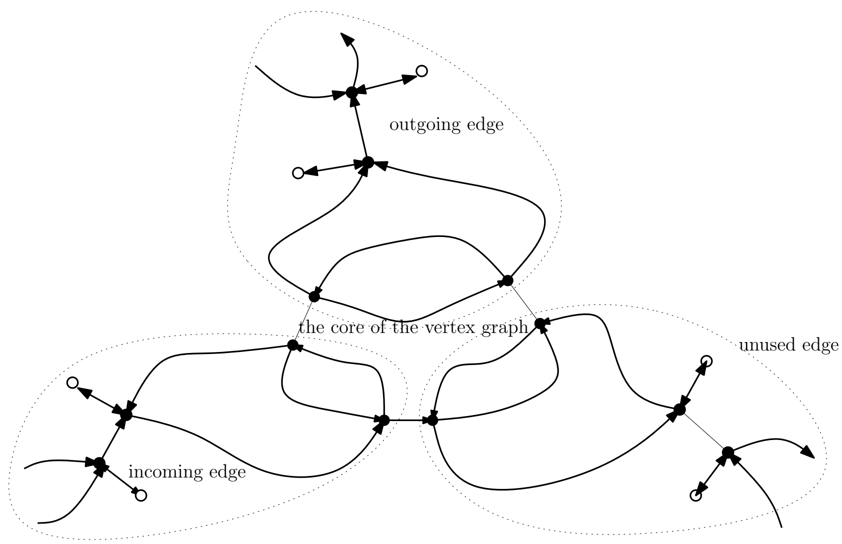


Fig. 21: Antenna assignments at the meta-vertex and incident meta-vertices.

the part corresponding to the unused edge, which is in turn reachable from the part corresponding to the incoming edge, and

(3) all vertices of a meta-edge corresponding to an outgoing edge $\langle v_i, v_j \rangle$ are reachable from either v_{i1} or v_{i2}; furthermore the destination vertices v_{j1} and v_{j2} are reachable from all these vertices.

Combining these observations with the fact that the Hamiltonian cycle spans all vertices yields that the resulting graph is strongly connected.

Next we show that if it is possible to orient the antennae in G'' such that the resulting graph is strongly connected then there exists a Hamiltonian cycle in G. Recall that G'' is constructed in such a manner that no antenna of range less than x and angle less than α can reach two neighbouring vertices, and that no antenna can reach a vertex that is not a neighbor in G''.

Assume an orientation of antennae such that the resulting graph is strongly connected. First, consider a pair of connecting vertices v_{i1} and v_{i2}. Since both path $\pi_{v_{i1}}$ and $\pi_{v_{i2}}$ are connected only to them, v_{i1} and v_{i2} must together use at least two antennae towards these two paths.

Let us call a meta-edge corresponding to edge $\langle v_i, v_j \rangle$ *directed* if in the connectivity graph there is an edge $\left\langle v'_i, v'_j \right\rangle$. Without loss of generality assume the direction is from v'_i to v'_j, i.e. v'_i used an antenna to reach v'_j. Since v''_i is reachable only from v'_i (and hence v'_i used its second antenna on v''_i), this means that there is no antenna pointing from v'_i towards the paths π'_{v_i} and π''_{v_i}. Therefore, the only way for the vertices of these two paths to be reachable is to have both connecting vertices (which for simplicity we call v_{i1} and v_{i2}, respectively) use an antenna towards these paths. Since they already used two antennae to ensure reachability of π_{v_i1} and π_{v_i2} are reachable, they have no antenna left to connect to another part of the meta-vertex.

Consider now the other half of the meta-edge. Observe that since v'_j must use one antenna on v''_j, it can use at most one antenna towards the paths π'_{v_j} and π''_{v_j}. Hence, either v_{j1} or v_{j2} must use an antenna towards one of these paths. Since these vertices must use two more antennae to ensure that the paths π_{v_j1} and π_{v_j2} are reachable, only one antenna is left for connecting to other parts of the meta vertex. Note that this argument holds both for receiving ends of directed meta-edges, as well as for non-directed meta-edges.

However, this means that in a meta-vertex there can be at most one outgoing directed meta-edge – otherwise there is no way to make the meta-vertex connected. Since each meta-vertex must have at least one outgoing directed meta-edge (otherwise the rest of the graph would be unreachable) and at least one incoming directed meta-edge (otherwise it would not be reachable from the rest), from the fact that the whole graph is strongly connected it follows that each meta-vertex must have exactly one undirected meta-edge, one directed incoming meta-edge and one directed outgoing meta-edge. Obviously, these correspond to unused/incoming/outgoing edges in the original graph G, with the directed edges forming the Hamiltonian cycle. □

5. Conclusion

We have provided an algorithm which, when given as input a set of n points (representing sensors) in the plane and an integer $1 \leq k \leq 5$, produces a strongly connected spanning graph so that each sensor uses at most k directional antennae of angle 0 and range at most $2 \cdot \sin\left(\frac{\pi}{k+1}\right)$ times the optimal. We also show that the problem of approximating the optimal range is NP-hard for 2 antennae, some approximation factors and sum of antennae angles. There are several interesting open problems including 1) looking at tradeoffs when the angle of the antennae is $\varphi > 0$, 2) deriving better lower bounds, 3) investigating more realistic antenna propagations, and 4) studying network connectivity in more dynamic settings of the nodes and/or antennae.

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