Monotonicity in Bargaining Networks
(extended abstract)

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Abstract

We study bargaining networks, discussed in a recent paper of Kleinberg and Tardos [KT08], from the perspective of cooperative game theory. In particular we examine three solution concepts, the nucleolus, the core center and the core median. All solution concepts define unique solutions, so they provide testable predictions. We define a new monotonicity property that is a natural axiom of any bargaining game solution, and we prove that all three of them satisfy this monotonicity property. This is actually in contrast to the conventional wisdom for general cooperative games that monotonicity and the core condition (which is a basic property that all three of them satisfy) are incompatible with each other. Our proofs are based on a primal-dual argument (for the nucleolus) and on the FKG inequality (for the core center and the core median). We further observe some qualitative differences between the solution concepts. In particular, there are cases where a strict version of our monotonicity property is a natural axiom, but only the core center and the core median satisfy it. On the other hand, the nucleolus is easy to compute, whereas computing the core center or the core median is #P-hard (yet it can be approximated in polynomial time).

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1 Introduction

Consider the following bargaining game that models mutually beneficial interactions between its players, and how they choose to split the joint benefit. Given a graph, suppose that there is a dollar on every edge of the graph, and a player at every node. Two players joined by an edge may split the dollar on that edge amongst themselves, if they agree on how to split it. However, each player is allowed to split the dollar with at most one neighbor. The proportion in which the players split the dollar may display asymmetries due to the position of the players in the graph. The outcome of such a game has been a central focus of *network exchange theory* [Wil99], a subfield of sociology. There has been extensive research [CE78, CEGY83] in this area in gathering experimental data on such outcomes. The goal from a theoretical perspective is to provide models that not only predict outcomes of actual experiments, but also have nice mathematical properties. In fact, game theory already provides us with several solution concepts that can be applied to such games. We intend to study natural properties of these solution concepts so as to provide a theoretical basis for comparison among them. In the process, we prove two combinatorial lemmas that seem to be of independent interest.

The recent work of Kleinberg and Tardos [KT08] brought this problem to the attention of theoretical computer science community. They consider *balanced outcomes*[1], which can be seen as an extension of Nash's bargaining theory [Nas50] to networks. Nash's bargaining theory considers a similar game where two players are allowed to split a dollar between them, but they each have an *alternative*, which is the value they get if they disagree with each other. Let a_1 and a_2 be the alternatives of the two players, and x_1 and x_2 be their corresponding share of the dollar. Nash's solution says that x_1 and x_2 are such that $x_1 + x_2 = 1$ and $x_1 - a_1 = x_2 - a_2$. (There is no agreement if $a_1 + a_2 > 1$.) Note that an outcome of the bargaining game on a network corresponds to a matching in the graph. For a node i, let x_i be his share in the outcome. A basic notion is that of *stable* outcomes: an outcome is said to be stable if for all edges (i, j) not in the matching, $x_i + x_j \geq 1$. An outcome is balanced, if in addition, each pair of matched players split the dollar according to Nash's solution, with the alternative of player i with a matching edge (i, j) being $\max_{(i,k)\in E:k\neq j}\{1 - x_k\}$. Kleinberg and Tardos [KT08] gave a polynomial time algorithm to compute the set of balanced outcomes, proving several structural properties along the way.

Another way to approach the bargaining game is via cooperative game theory, the branch of game theory that deals with the problem of *fair division*. A *co-operative game* is defined by (v, N) where N is a finite set of *players* and $v : 2^N \to \mathbf{R}$. A feasible *allocation* $x \in \mathbf{R}_+^N$ is a way to divide $v(N)$ among the players: $\sum_{i\in N} x_i = v(N)$. In fact, the bargaining game we defined is a cooperative game, with N = node set of the graph, and $\forall S \subseteq N$, $v(S)$ = the size of the maximum cardinality matching in G_S, the subgraph of G induced by S. This game, in this form, was introduced by Shapley and Shubik [SS72] and is known as the *matching game*. The *core* and the *kernel* (see Section 2 for formal definitions) are standard solution concepts for cooperative games. In fact, for the matching game, the core is exactly the set of stable outcomes, and the set of balanced outcomes is exactly the intersection of the core and the kernel.

One sees that these games have been independently discovered by multiple researchers in different areas and different contexts. This is an affirmation of the fact that these games are natural and important objects of study. Further, they display a rich combinatorial structure with connections to matching theory; for instance, it is well known that for bipartite graphs, the core is always non-empty, and is exactly the set of all optimum solutions to the dual of the standard linear program for maximum matching (the dual LP can be interpreted as a fractional minimum vertex cover).

Uniqueness: Balanced outcomes coincide with the experimental data on many of the simple graphs. However, there may be more than one balanced outcome for a given graph. In fact, for any biconnected bipartite graph, any solution that gives x to one side of the graph and $1 - x$ to the other side is a balanced outcome. For instance, consider the graph $K_{2,2}$, in which all vertices are symmetric. Yet, a balanced outcome may give very different allocations to two vertices. This asymmetry seems to be in conflict with the goal of network exchange theory

[1]The notion of balanced outcomes was proposed earlier by Rochford [Roc84] and independently by Cook and Yamagishi [CY92].

which is to identify the influence of the graph topology on the resulting allocations. Therefore, we'd like to consider solution concepts that define *unique* outcomes, and hence provide *testable* predictions. Furthermore, if a solution concept is used for a prescriptive purpose[2] rather than a descriptive purpose, then uniqueness becomes a necessity. Also, no one solution concept may be appropriate for all situations; hence we need multiple solution concepts with nice properties.

Monotonicity: Further, uniqueness lets us define other natural properties such as *monotonicity*: suppose that we introduced a new edge in the graph. Then the endpoints of this edge have more options for negotiation than before, and one would expect that they only have a higher bargaining power in the new graph. For all graphs $G = (V, E)$ and $(i, j) \notin E(G)$, let $G + (ij)$ be the graph obtained by adding the edge (i, j) to G. Let α be a (unique) solution concept with $\alpha_i(G)$ being the allocation to i. α is *monotone* if

$$\text{for all } i, j \in N, \alpha_i(G + (ij)) \geq \alpha_i(G).$$

Many other notions of monotonicity have been considered [GDSR07] for general cooperative games. $\alpha : (v, N) \to \mathbf{R}^N$ is *strongly monotone* if whenever $v, u : 2^N \to \mathbf{R}$ and $i \in N$ are such that $\forall S \subseteq N$, $u(S \cup \{i\}) - u(S) \geq v(S \cup \{i\}) - v(S)$, then $\alpha_i(u, N) \geq \alpha_i(v, N)$. α satisfies *coalitional monotonicity* if $u(T) > v(T)$ and $u(S) = v(S)$ for all $S \neq T$ implies that for all $i \in T$, $\alpha_i(u, N) \geq \alpha_i(v, N)$. When this restriction is only applied to the grand coalition, $T = N$, it is called *aggregate monotonicity* [Meg74]. Strong monotonicity implies monotonicity, whereas coalitional and aggregate monotonicity are incomparable to monotonicity.

Although monotonicity properties are very natural to expect from a solution concept, most previous results [Meg74, You85, HC98, Mas92, Hok00] conclude that monotonicity and the core condition are incompatible with each other. The Shapley value is the unique solution concept that always satisfies strong monotonicity. However, for matching games, the Shapley value may not correspond to any matching (which implies it is not in the core), and is therefore not appropriate. For instance, for a path on 3 vertices, the Shapley value assigns $2/3$ to the center and $1/6$ to the end points. We consider three solution concepts, the core center, the core median and the nucleolus, all of which are unique and are guaranteed to be in the core. However, none of these satisfy any of strong, coalitional or aggregate monotonicities in general games [Meg74, You85, HC98, Mas92, Hok00]. This is in contrast to our main result.

> **Main result**: We show that the core center, the core median and the nucleolus all satisfy monotonicity for matching games on bipartite graphs.

One can also extend the game to weighted graphs, where the weight on an edge corresponds to the dollar amount available for the end points to split. Shapley and Shubik [SS72] also studied the weighted version for the case of bipartite graphs, under the name, *assignment games*. The motivation for such games comes from two-sided markets, such as the housing market, where one side has all the buyers, each of whom is interested in buying a house, and the other side has all the sellers, each of whom wants to sell one house. The weight on an edge is the surplus generated by the exchange. Assignment games have also been thought of as the *cardinal* version of the classic Gale-Shapley stable marriage problem, with transferable utility (by Schwarz and Yenmez [SY09], for instance). Each man, instead of simply having a ranking over the women, places a value on each of the women (and so do the women). The weight of an edge is the sum of the values the end points place on each other. The outcome now involves not only a matching, but also a *transfer* of utility between matched nodes. The corresponding notion of monotonicity is that increasing the weight[3] of an edge by 1 leads to the endpoints of that edge getting a higher allocation. *For the core center and the core median, our proof also works for assignment games, whereas it remains open if the nucleolus is monotone for assignment games.*

[2] By a prescriptive purpose, we mean situations in which a central authority would propose an outcome to the players, such as the National Resident Matching Program, which matches medical students to residency positions in hospitals.

[3] W.l.o.g we may assume that weights are integral and that G is complete, by letting weights of non-edges be 0.

Computability: The core [SS72], the set of balanced outcomes [KT08] and the nucleolus [SR94] can all be computed in polynomial time. We show (via simple reductions to a problem considered by Rademacher [Rad07]) that computing the core center and the core median[4] is #P-hard; however, there are Polynomial Time Approximation Schemes to compute them. It is an open problem whether there exist fast distributed algorithms that converge to any of these concepts. Such algorithms would be particularly interesting, since they might suggest how people actually negotiate a solution.
Organization: We give the formal definitions of the solution concepts and a comparison among them in Section 2. The proofs of the monotonicity for the core center and the core median are in Section 3 and the proof for the nucleolus is in Section 4. The #P-Hardness and PTAS for core center and core median are deferred to the full version.

2 Preliminaries

2.1 The core center

Let the set of all feasible allocations for a given game be denoted by $\mathcal{A} := \{x \in \mathbf{R}_+^N : \sum_{i \in N} x_i = v(N)\}$. From now on, $\forall S \subseteq N$, we write $x(S)$ for $\sum_{i \in S} x_i$. The *core* of a game is the set of allocations such that no coalition has an incentive to secede: $\mathcal{C} := \{x \in \mathcal{A} : \forall S \subseteq N, x(S) \geq v(S)\}$. Suppose that $\mathcal{C} \neq \emptyset$, then the *core center* [GDSR07], γ, is the center of gravity of the core:

$$\gamma := \frac{\int_{x \in \mathcal{C}} x dx}{\int_{x \in \mathcal{C}} 1 dx} = \mathrm{E}_{x \in \mathcal{C}}[x],$$

where the expectation is taken over the uniform probability distribution.

Consider unweighted bipartite graphs; for these graphs it is well known that the core is a polytope[5]. Also, adding an edge to the graph is equivalent to adding an extra constraint of the form $x_i + x_j \geq 1$. In order to prove monotonicity, we need to argue that intersecting the core with a hyperplane of this form moves the center in such a way that both x_i and x_j co-ordinates increase. Such a property is not true for all polytopes in general, as can be seen from the example in Figure 1.

By eliminating all the variables on one side of the bipartite graph, it can be shown that the core is isomorphic to an order polytope in a lower dimensional space, with boundary conditions, $0 \leq x_i \leq 1$. Adding an edge is now equivalent to adding a constraint of the form $x_i \geq x_j$. We then show that for all such polytopes, adding an inequality of the form $x_i \geq x_j$ only increases the i co-ordinate of the center.

In fact, we define a range of solution concepts, for all $k \in \mathbb{N}$, called the k-core center. It is the center of gravity of a discrete grid in the core, where k denotes the granularity of the grid. We show monotonicity for k-core center for all k and conclude, by going to the limit, that the core center itself is monotone. Our proof uses the FKG inequality, and is inspired by Shepp's proof of the "XYZ" theorem [She82].

For all $k \in \mathbb{N}$, let $\Lambda_k^n \subset \mathbb{R}^n$ be the lattice spanned by the scaled standard orthogonal basis $\{\frac{1}{k}e_1, \frac{1}{k}e_2, \ldots, \frac{1}{k}e_n\}$. The *$k$-core center*

$$\gamma^k := \frac{\sum_{x \in \mathcal{C} \cap \Lambda_k^N} x}{|\mathcal{C} \cap \Lambda_k|} = \mathrm{E}_{x \in \mathcal{C} \cap \Lambda_k^N}[x].$$

Also, for ease of notation, let $\hat{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ and $\gamma^\infty := \gamma$. When $k = 1$, the k-core center is simply the average of the vertices of the core polytope.

The core center is actually strictly monotone in the following sense: unless the new edge added is between two vertices in the same biconnected component, both of their allocations strictly increase. The monotonicity

[4] [Sch09] asked about the computational complexity of the core median.
[5] It follows from the fact, mentioned earlier, that the core is the set of dual optimum solutions to the fractional vertex cover LP.

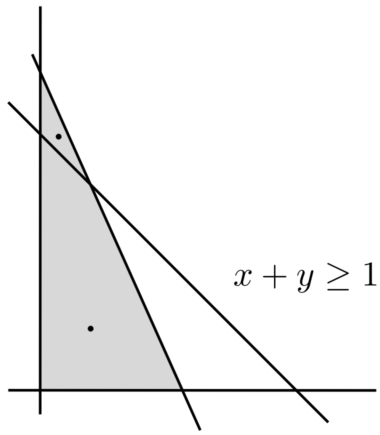

Figure 1: Consider the shaded triangle, and suppose we add the constraint $x + y \geq 1$. Note that as a result the center moves in such a way that the x co-ordinate decreases.

for the core center holds also for assignment games. In this case the core is not an order polytope anymore, but essentially the same techniques can be used for this larger class of polytopes. Formally,

Theorem 1. *For all assignment games, and for all* $k \in \hat{\mathbb{N}}, \gamma^k$ *is monotone.*

2.2 The Core Median

The core median was defined by Schwarz and Yenmez [SY09] for assignment games. It is inspired by the notion of a *median stable matching* for the Gale-Shapley stable marriage game. For a given player i in a Gale-Shapley game, one can order all the stable matchings using i's rank of the player matched to i. Now consider the stable matching that is the median[6] in this ordering and let i be matched to $\mu(i)$ in this matching. [TS98] showed that μ actually defines a matching, that is, $\mu(\mu(i)) = i$, and that it is stable. That is, the same stable matching gives the median matches to all the players simultaneously. This matching is called *the* median stable matching.

For an assignment game, similarly, for all $i \in N$, let μ_i be such that

$$\Pr_{x \in \mathcal{C}}[x_i \leq \mu_i] = \frac{1}{2} = \Pr_{x \in \mathcal{C}}[x_i \geq \mu_i].$$

The core median is the vector $\mu = (\mu_1, \mu_2, \ldots, \mu_N)$. Schwarz and Yenmez [SY09] showed that μ is always in the core for assignment games. This may not be true for general co-operative games.

The proof of monotonicity for the core median is very similar to our proof for the core center. We consider the discrete grids of the core, and define a core median for the discrete grid. The monotonicity of this is shown using similar techniques, via the FKG inequality. The monotonicity of the core median follows from taking the limit.

For all $k \in \mathbb{N}$, let μ^k be such that[7] for all $i \in N$,

$$|\{x \in \mathcal{C} \cap \Lambda_k^N : x_i < \mu_i^k\}| = |\{x \in \mathcal{C} \cap \Lambda_k^N : x_i > \mu_i^k\}|.$$

[6] If there are an odd number of stable matchings, then the median is well defined. Otherwise, if there are $2M$ stable matchings, we define the median for one side of the bipartite graph as the M^{th} stable matching and for the other side of the bipartite graph as the $M + 1^{\text{st}}$ stable matching.

[7] Again, assume that $|\{x \in \mathcal{C} \cap \Lambda_k^N\}|$ is odd. The case that it is even can be handled as in the case of the Gale-Shapley stable marriage game.

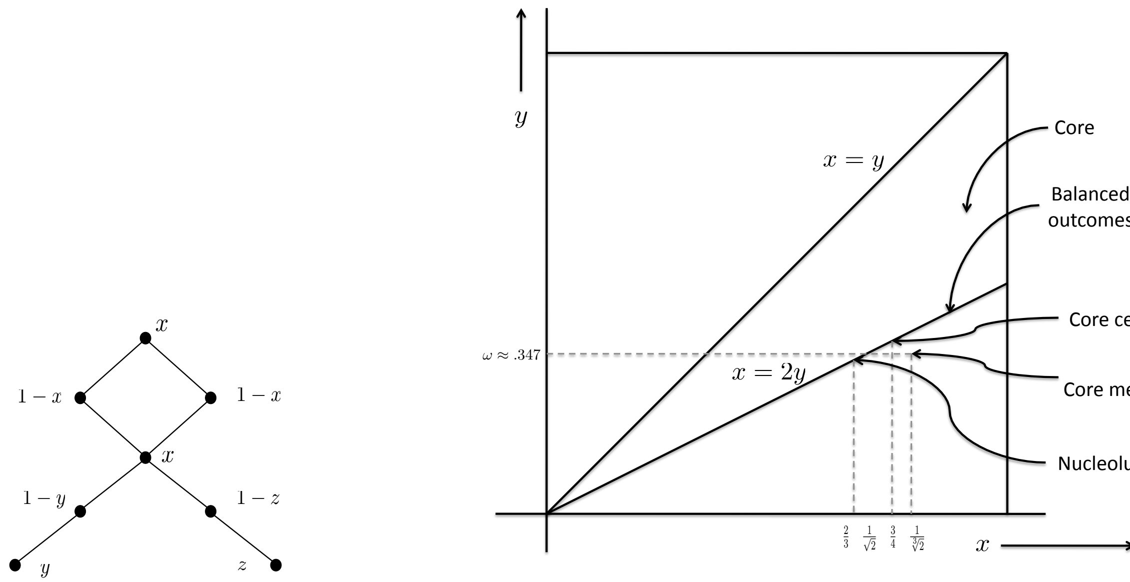

Figure 2: A graph that illustrates the differences between the solution concepts. To the right is the projection of the allocations on the x-y plane.

μ^k is called the k*-core median*.

Theorem 2. *For all assignment games, and for all* $k \in \hat{\mathbb{N}}$, μ^k *is monotone.*

2.3 The Nucleolus

As opposed to the core center, which is the center of gravity of the core, the nucleolus is the *lexicographic center* of the core. That is, the nucleolus is that point in the core, that maximizes the minimum *slack* w.r.t all of the hyperplanes defining the core. And subject to the minimum being maximized, maximizes the second to minimum slack, and so on.

For all players $S \subseteq N$, let the *satisfaction* be $s(S, x) := x(S) - v(S)$. Let $\theta(x)$ be the vector of satisfactions for all $S \subseteq N$, sorted in the increasing order. The *nucleolus* [Sch69] $\nu := \arg$ lex-$\max\{\theta(x) : x \in \mathcal{A}\}$, where lex-$\max$ is the maximum in the lexicographic ordering of vectors. It can be shown [Sch69] that the nucleolus is unique, and is a balanced outcome.

Theorem 3. *For all matching games in bipartite graphs,* ν *is monotone.*

We prove the monotonicity for the nucleolus via a reduction to a combinatorial problem on lattices (A lattice is a partially ordered set with a global minimum and a global maximum). We construct a lattice from the given graph (very much like a construction in [KT08]) and design a primal-dual algorithm on the lattice that gives the nucleolus. We then argue that by adding an edge to the graph, the run of the primal-dual algorithm changes in such a way that it gives a higher weight to the node adjacent to the new edge.

2.4 Comparison between the different solution concepts

Consider the graph in Figure 2 This graph illustrates the difference between various solution concepts. All of them lie in the unit hypercube, $0 \leq x, y, z \leq 1$. The core is the region satisfying $x \geq y, x \geq z$; the set of balanced outcomes is such that $y = z = x/2$; the nucleolus is $x = 2/3, y = z = 1/3$; the core center is $x = 3/4, y = z = 3/8$; the core median is $x = 1/\sqrt[3]{2}, y = z = \omega$, ω is a root of the equation $y^3 - 3y + 1 = 0$. Also, in the absence of the second "tail", the nucleolus remains the same, the core is the set $x \geq y$ and the set of balanced outcomes is $y = x/2$. However, the core-center changes to $x = 2/3, y = 1/3$ and the core median changes to $x = 1/\sqrt{2}, y = 1 - x$. On the other hand, if we add more tails then the x value increases for both core center and the core median, while it stays the same for the nucleolus. Thus, if a node has multiple options of the same value, then the nucleolus does not give the node a higher share, whereas the core center and the core median do.

2.5 Previously Known Results

In this section, we state some known results for reference. For all pairs $i, j \in N, i \neq j$, the *surplus*

$$\sigma_{ij}(x) := \max_{S \subseteq N : i \in S, j \notin S} \{s(S, x)\}.$$

The *kernel* of a game is $\mathcal{K} := \{x \in \mathcal{A} : \forall i, j \in N, i \neq j, \sigma_{ij}(x) = \sigma_{ji}(x)\}$. Schmeidler [Sch69] showed that the nucleolus is always contained in the kernel. (This also shows that the kernel is always non-empty.) Further, if the core is non-empty, then the nucleolus is also in the core.

Theorem 4 ([Sch69])**.** *$\nu \in \mathcal{K}$. If $\mathcal{C} \neq \emptyset$, then $\nu \in \mathcal{C}$.*

Shapley and Shubik [SS72] showed that for assignment games, the core is always non-empty.

Theorem 5 ([SS72])**.** *For any assignment game, $\mathcal{C} \neq \emptyset$.*

They also showed that the only coalitions "that matter" in an assignment game are those of size 1 and 2. Let $G = (N, E, w)$ be a weighted graph. Let M be any maximum weight matching in G. Let

$$\begin{aligned} \mathcal{C}' := \{x \in \mathbf{R}^N : \quad & \forall (i,j) \in M, x_i + x_j = w(ij), \\ & \forall (i,j) \in E, x_i + x_j \geq w(ij), \\ & \forall i \in N \text{ unmatched by } M, x_i = 0, \\ & \forall i \in N, x_i \geq 0.\} \end{aligned}$$

Lemma 6 ([SS72])**.** *$\mathcal{C}' = \mathcal{C}$.*

Given this equivalent definition of the core, it is easy to see that the core is exactly the set of *stable* outcomes as defined in Kleinberg and Tardos [KT08]. Further, it can be shown that the set of *balanced* outcomes in Kleinberg and Tardos [KT08] is $\mathcal{K} \cap \mathcal{C}$.

3 Core Center and Core Median

In this section we focus on the proof of monotonicity of the core center, and mention the changes needed to prove the monotonicity of the core median. (The proofs are very similar; the complete proof for the core median is deferred to the full version of the paper.)

Consider an assignment game defined by a weighted bipartite graph $G = (U, V, E, w)$. Let M be a maximum weight matching in G. By adding dummy vertices and edges, if needed, we may assume that M is a perfect matching. For $i \in U$, let $M(i)$ denote the vertex in V that i is matched to by M. Let $n = |U| = |V| = |M|$. Recall that our goal is to examine the effect of raising the weight of an edge $e \in E$ by 1; the modified assignment game is denoted by $G + e$. Notice that we may choose M in such a way that M is a maximum weight matching in $G + e$ as well, because the weights are assumed to be non-negative integers.

Consider the core $\mathcal{C}(G)$ of the assignment game G. Every $y \in \mathcal{C}(G)$ is an assignment of non-negative values to the nodes of G that satisfies $y_i + y_j \geq w(i,j)$ for every $\{i,j\} \in E$, and $y_i + y_{M(i)} = w(i, M(i))$ for every $i \in U$. Define a vector x by setting, for every $i \in U$, $x_i = y_i$ and $x_{M(i)} = w(i, M(i)) - y_{M(i)}$. Let $\mathcal{P}(G)$ denote the projection of $\mathcal{C}(G)$ on the coordinates of U. Then

$$\begin{aligned}\mathcal{P}(G) := \{x \in \mathbb{R}^n : \quad & \forall \{M(i), j\} \in E,\ x_j - x_i \geq w(j, M(i)) - w(i, M(i)), \\ & \forall i \in U, 0 \leq x_i \leq w(i, M(i))\}.\end{aligned}$$

Now assume w.l.o.g that $e = \{M(1), 2\}$. Let

$$\mathcal{H}_{12} = \{x \in \mathbb{R}^n : x_2 - x_1 \geq w(2, M(1)) + 1 - w(1, M(1))\}.$$

Note that $\mathcal{P}(G+e) = \mathcal{P}(G) \cap \mathcal{H}_{12}$. The core center of G (respectively, $G+e$) is easily derived from the expectation $\mathrm{E}[x]$ under the uniform probability measure on $\mathcal{P}(G)$ (respectively, $\mathcal{P}(G+e)$). In order to show monotonicity of the core center, we need to show that $\mathrm{E}[x_1]$ does not increase when the constraint $\mathcal{H}_{12}$ is added, and similarly that $\mathrm{E}[x_2]$ does not decrease. This essentially will follow by showing that $\mathcal{H}_{12}$ is negatively correlated with x_1 and positively correlated with x_2. In fact, we use the following correlation inequality to prove the above in a slightly more general setting, as explained below.

Lemma 7 (FKG Inequality). *Let $\mathcal{L} = (X, \preceq)$ be a finite distributive order lattice. Let κ be a log-super modular positive function on X. Let f, g be two real valued functions on X. If both f, g are monotonically non-decreasing in $\mathcal{L}$ then*

$$\frac{\sum_{x\in X} f(x)g(x)\kappa(x)}{\sum_{x\in X}\kappa(x)} \geq \frac{\sum_{x\in X} f(x)\kappa(x)}{\sum_{x\in X}\kappa(x)} \cdot \frac{\sum_{x\in X} g(x)\kappa(x)}{\sum_{x\in X}\kappa(x)}$$

(i.e., f, g are positively correlated). If f is monotonically non-increasing and g is monotonically non-decreasing then

$$\frac{\sum_{x\in X} f(x)g(x)\kappa(x)}{\sum_{x\in X}\kappa(x)} \leq \frac{\sum_{x\in X} f(x)\kappa(x)}{\sum_{x\in X}\kappa(x)} \cdot \frac{\sum_{x\in X} g(x)\kappa(x)}{\sum_{x\in X}\kappa(x)}$$

(i.e., f, g are negatively correlated).

As the FKG Inequality requires a finite lattice, we will need to discretize $\mathcal{P}(G)$ and then pass to the limit via the Riemann integral.

3.1 Monotonicity of Order Polytopes

Let

$$\mathcal{P} = \left\{ x \in \mathbb{R}^n : \bigwedge_{i=1}^{n} \{a_i \leq x_i \leq b_i\} \wedge \bigwedge_{t=1}^{m} \{x_{j_t} - x_{i_t} \geq c_t\} \right\}$$

be the (non-empty) feasible set of a non-homogeneous set of order constraints, where for all $i \in [n]$, $a_i, b_i \in \mathbb{Z}$, and for all $t \in [m]$, $c_t \in \mathbb{Z}$. Fix $s \in \mathbb{Z}$. Let $\mathcal{H} = \{x \in \mathbb{R}^n :\ x_2 - x_1 \geq s\}$. Let $\mathcal{P}_k = \mathcal{P} \cap \Lambda_k^n$ and $\mathcal{H}_k = \mathcal{H} \cap \Lambda_k^n$. (Recall that Λ_k^n is the lattice in $\mathbb{R}^n$ spanned by $\frac{1}{k}e_1, \frac{1}{k}e_2, \ldots, \frac{1}{k}e_n$.)

Lemma 8. *For every $k \in \mathbb{N}$, if $\mathcal{P}_k \cap \mathcal{H}_k \neq \emptyset$ then $E[x_1 :\ x \in \mathcal{P}_k \cap \mathcal{H}_k] \leq E[x_1 :\ x \in \mathcal{P}_k]$ and $E[x_2 :\ x \in \mathcal{P}_k \cap \mathcal{H}_k] \geq E[x_2 :\ x \in \mathcal{P}_k]$, where expectations are taken with respect to the uniform probability measure over the corresponding (finite) set.*

Proof. Let $Q = \{x \in \Lambda_k^n :\ \bigwedge_{i=1}^n \{a_i \leq x_i \leq b_i\}\}$. Define a binary relation $\preceq$ on Q by setting $x \preceq y$ iff $x_1 \geq y_1$ and for every $j = 2, 3, \ldots, n$, $x_j - x_1 \leq y_j - y_1$. By a lemma of Shepp [She82], $(Q, \preceq)$ is a distributive order lattice. The join and the meet are given by $(x \vee y)_j = \min\{x_1, y_1\} + \max\{x_j - x_1, y_j - y_1\}$ and $(x \wedge y)_j = \max\{x_1, y_1\} + \min\{x_j - x_1, y_j - y_1\}$.

Define $f : Q \to [0,1]$ and $g, \kappa : Q \to \{0,1\}$ as follows. For $x \in Q$, $f(x) = x_1$, $g(x)$ is the indicator of $x_2 - x_1 \geq \gamma$, and $\kappa(x)$ is the indicator of $x \in \mathcal{P}$. We now wish to apply the FKG Inequality. First we verify that the conditions on f, g, κ hold. By definition of $\preceq$, f is monotonically non-increasing. In order to show that g is monotonically non-decreasing, we simply have to verify that if $x \preceq y$ and $g(x) = 1$, then $g(y) = 1$. This is clearly true, as $y_2 - y_1 \geq x_2 - x_1 \geq \gamma$, where the first inequality follows from the definition of $\preceq$ and the second inequality follows from $g(x) = 1$.

We now show that κ is log-super modular. In this case we simply have to verify that if $\kappa(x) = \kappa(y) = 1$ then $\kappa(x \vee y) = \kappa(x \wedge y) = 1$. So, consider an arbitrary constraint in $\mathcal{P}$. We will verify that if x and y satisfy it, then so do $x \vee y$ and $x \wedge y$. There are three types of constraints in $\mathcal{P}$, and we deal with each type separately. Firstly, consider a constraint of the form $x_j - x_i \geq c$. By our assumption, also $y_j - y_i \geq c$. Now, $(x \vee y)_j - (x \vee y)_i = \max\{x_j - x_1, y_j - y_1\} - \max\{x_i - x_1, y_i - y_1\}$. Without loss of generality, the second term is maximized at $x_i - x_1$. Therefore, $(x \vee y)_j - (x \vee y)_i \geq (x_j - x_1) - (x_i - x_1) = x_j - x_i \geq c$. Similarly for $x \wedge y$ we have that $(x \wedge y)_j - (x \wedge y)_i = \min\{x_j - x_1, y_j - y_1\} - \min\{x_i - x_1, y_i - y_1\}$. Without loss of generality, the first term is minimized at $x_j - x_1$, so $(x \wedge y)_j - (x \wedge y)_i \geq (x_j - x_1) - (x_i - x_1) = x_j - x_i \geq c$. Secondly, consider a constraint of the form $x_i \geq a_i$. So also $y_i \geq a_i$ Now, $(x \vee y)_i = \min\{x_1, y_1\} + \max\{x_i - x_1, y_i - y_1\}$. Without loss of generality, the first term is minimized at x_1, so $(x \vee y)_i \geq x_1 + x_i - x_1 = x_i \geq a_i$. Similarly, $(x \wedge y)_i = \max\{x_1, y_1\} + \min\{x_i - x_1, y_i - y_1\}$. Without loss of generality, the second term is minimized at $x_i - x_1$, so $(x \wedge y)_i \geq x_i \geq a_i$. Thirdly, consider a constraint of the form $x_i \leq b_i$. Using $(x \vee y)_i = \min\{x_1, y_1\} + \max\{x_i - x_1, y_i - y_1\}$, assume without loss of generality that the second term is maximized at $x_i - x_1$, so $(x \vee y)_i \leq x_i \leq b_i$. Using $(x \wedge y)_i = \max\{x_1, y_1\} + \min\{x_i - x_1, y_i - y_1\}$, assume without loss of generality that the first term is maximized at x_1, so $(x \wedge y)_i \leq x_i \leq b_i$. This completes the argument that κ is log-super modular.

Thus, we can apply the FKG Inequality to conclude that

$$\frac{\sum_{x \in Q} f(x) g(x) \kappa(x)}{\sum_{x \in Q} \kappa(x)} \leq \frac{\sum_{x \in Q} f(x) \kappa(x)}{\sum_{x \in Q} \kappa(x)} \cdot \frac{\sum_{x \in Q} g(x) \kappa(x)}{\sum_{x \in Q} \kappa(x)},$$

or

$$\mathrm{E}[f(x) g(x) :\ x \in \mathcal{P}_k] \leq \mathrm{E}[f(x) :\ x \in \mathcal{P}_k] \cdot \mathrm{E}[g(x) :\ x \in \mathcal{P}_k].$$

As $\mathrm{E}[g(x) :\ x \in \mathcal{P}_k] = \Pr[g(x) = 1 :\ x \in \mathcal{P}_k]$, dividing by $\mathrm{E}[g(x) :\ x \in \mathcal{P}_k]$ gives

$$\mathrm{E}[f(x) :\ x \in \mathcal{P}_k \wedge g(x) = 1] \leq \mathrm{E}[f(x) :\ x \in \mathcal{P}_k].$$

The second claim follows by symmetry. Set $x' = -x$, reverse the inequalities by setting, for all $i \in [n]$, $-b_i \leq x'_i \leq -a_i$, for all $t \in [m]$, $x'_{i_t} - x'_{j_t} \geq -c_t$, and $x'_1 - x'_2 \geq -s$, and apply the above argument. ■

Corollary 9. *if $\mathcal{P} \cap \mathcal{H} \neq \emptyset$ then $E[x_1 :\ x \in \mathcal{P} \cap \mathcal{H}] \leq E[x_1 :\ x \in \mathcal{P}]$ and $E[x_2 :\ x \in \mathcal{P} \cap \mathcal{H}] \geq E[x_2 :\ x \in \mathcal{P}]$, where expectations are taken with respect to the uniform probability distribution μ over the corresponding (finite measure Borel) set.*

Proof. Consider the polytope $\mathcal{P}$ (and similarly the polytope $\mathcal{P} \cap \mathcal{H}$). This polytope has dimension $n' \leq n$ and its vertices have integer coordinates. Let L denote the n'-dimensional affine hull of $\mathcal{P}$. For a positive integer k, let Λ_k^n denote the lattice in $\mathbb{R}^n$ spanned by the scaled standard basis $\frac{1}{k}e_1, \frac{1}{k}e_2, \ldots, \frac{1}{k}e_n$. As $\Lambda_k^n \cap L$ includes the vertices of $\mathcal{P}$, the affine hull of $\Lambda_k^n \cap L$ has dimension n', so it is a lattice in L. Consider the function $f : L \to \mathbb{R}$, defined by $f(x) = x_1$ for $x \in \mathcal{P}$ and $f(x) = 0$ otherwise. As $\mathcal{P}$ has a measure 0 boundary, this function is Riemann integrable, and therefore

$$\frac{\sum_{x \in \Lambda_k^n \cap \mathcal{P}} f(x)}{|\Lambda_k^n \cap \mathcal{P}|} \to \int f(x) d\mu = E[x_1 :\ x \in \mathcal{P}],$$

as $k \to \infty$. A similar argument applies to $E[x_2 :\ x \in \mathcal{P}]$, and to the same expectations over $\mathcal{P} \cap \mathcal{H}$. Applying Lemma 8 completes the proof. ■

3.2 Monotonicity of the Core Center and Core Median

Proof of Theorem 1. Notice that $\mathcal{P}(G)$ and $\mathcal{H}_{12}$ are special cases of $\mathcal{P}$ and $\mathcal{H}$ as needed by Lemma 8 and Corollary 9. Hence, applying the Lemma gives us that for all $k \in \hat{\mathbb{N}}$, $\gamma_2^k(G+e) \geq \gamma_2^k(G)$, and $\gamma_1^k(G+e) \leq \gamma_1^k(G)$. Since $\gamma_{M(1)}^k = w(1, M(1)) - \gamma_1^k$ for both G and $G+e$, we have that $\gamma_{M(1)}^k(G+e) \geq \gamma_{M(1)}^k(G)$. Thus γ^k is monotone. ■

The proof of Theorem 2 follows closely the above proof for the core center. Monotonicity is implied by showing that the distribution of x_1 (respectively, x_2) in $\mathcal{P} \cap \mathcal{H}$ is dominated by (respectively, dominates) the distribution of x_1 (respectively, x_2) in $\mathcal{P}$. This simply requires replacing the function f in Lemma 8 by the indicator functions for $x_1 \geq a$ for every possible threshold a, then repeating the rest of the argument. The complete proof will appear in the full version of the paper.

4 The Nucleolus

In order to prove the monotonicity of the nucleolus we look at the algorithm that computes the nucleolus in assignment and matching games on a graph G using iterative linear programming.

Algorithm Lex-Center

- $F \leftarrow M$; $A \leftarrow E \setminus M$; $X \leftarrow \mathcal{C}$.
- While $A \neq \emptyset$, do
  - $\epsilon \leftarrow \max_{x \in X} \min_{(i,j) \in A} \{x_i + x_j - w(ij)\}$.
  - $Y \leftarrow \arg\max_{x \in X} \min_{(i,j) \in A} \{x_i + x_j - w(ij)\}$.
  - $F' \leftarrow \{(i,j) \in A : \forall x \in Y, x_i + x_j - w(ij) = \epsilon\}$.
  - $F \leftarrow F \cup F'$.
  - $A \leftarrow F \setminus F'$.
  - $X \leftarrow Y$.
- Output X.

Theorem 10. *[SR94] The output of Algorithm Lex-Center, X is $\{\nu\}$.*

For the case in which $w(ij) = 1$ for all $e = (i,j)$ where $G = (U, V, E)$ is unweighted bipartite graph we simplify the algorithm as follows. Construct a graph H with U as its vertices: $(i,j) \in E(H)$ if and only if $(M(i), j) \in E(G)$ where M is a maximum matching. The constraint $x_{M(i)} + x_j \geq 1$ is equivalent to $x_j \geq x_i$. Hence if there is a cycle in H, then for any vector in the core x_i is constant on the cycle. Further, in the first iteration of Algorithm Lex-Center, ϵ would be zero and F' would correspond to the set of all edges in any strongly connected component of H. X would remain equal to $\mathcal{C}$.

Contract the strongly connected components of H to get a DAG. Add two additional vertices to this DAG, s and t, and add edges from s to all the vertices and from all the vertices to t. The result DAG is denote by D. We set $x_s = 0$ and $x_t = 1$. It is easy to see that the algorithm above is reduced to the following algorithm for computing the projected Lex-Center on U using the DAG D:

Algorithm-new Lex-Center

- $F \leftarrow \emptyset$; $A \leftarrow E(D)$; $X \leftarrow \mathcal{C}_{|U}$.
- While $A \neq \emptyset$, do

- $\epsilon \leftarrow \max_{x \in X} \min_{(i,j) \in A} \{x_j - x_i\}$.
- $Y \leftarrow \arg\max_{x \in X} \min_{(i,j) \in A} \{x_j - x_i\}$.
- $F' \leftarrow \{(i,j) \in A : \forall x \in Y, x_j - x_i = \epsilon\}$.
- $F \leftarrow F \cup F'$.
- $A \leftarrow F \setminus F'$.
- $X \leftarrow Y$.

- Output X.

We note that the DAG D actually defines an order lattice L (the partial order relations are the transitive closure of the DAG) on vertices V with global minimum (source) s and a global maximum (sink) t. We first show the following lemma.

Lemma 11. *Suppose L is a lattice with source s and sink t, and w gives weights on the edges of a DAG D representing L. Then the system of inequalities:*

$$\forall (i,j) \in E(D), x_i - x_j \geq w(e),$$
$$x_s = 0, x_t = 1,$$

is feasible if and only if all s-t paths in D have weight at most 1.

Proof. The proof is a simple application of Farkas' Lemma. ∎

The above lemma motivates us to define the following algorithm.

Algorithm-primal-dual

- $F \leftarrow \emptyset$; $w(e) \leftarrow 0$ for all $e \in E(D)$.
- While $F \neq E(D)$, do
  - increase $w(e)$ for $e \in E(D) - F$ until there is a path from s to t of length 1.
  - F be the set of edges on a length 1 path from s to t.
- for any v let $d(v)$ be the distance (on any path) from s to v.
- output $d(v)$

We note that while running the primal-dual algorithm the longest path from s to v for each vertex v is monotonically increasing until vertex v participates in a length 1 path from s to t. At that step $d(v)$ is determined as the distance from s on this path. We call that time the freezing time of x.

Theorem 12. *The primal-dual algorithm computes the Lex-Center projected on U. That is $d(v) = x(v)$ for all $v \in U$.*

Proof. Consider the primal-dual Algorithm run on D and any particular iteration. Using Lemma 11 with the weights from the algorithm, we see that the weights of the new edges frozen in that iteration correspond to ϵ in one iteration of the Lex-Center Algorithm. The new edges frozen in an iteration of Primal-Dual Algorithm correspond to F' in the Lex-Center Algorithm. Thus $d(i) = x_i$. ∎

Given a DAG D assume that the edge (u, v) is not in $E(D)$ but it is a relation that follows from the lattice L. Then $D_1 = D \cup (u, v)$ is also a DAG that corresponds to the same lattice L. We first show that the output of the primal-dual algorithm is the same if we apply it to D or D_1 that correspond to L.

Lemma 13. *Given a DAG D that corresponds to a lattice L. Let (u, v) a relation in the lattice (i.e. it is in the transitive closure). Then Then the outputs of primal dual algorithm on D and on D_1 are the same. In particular, the output of the primal-dual algorithm on all D that correspond to the same lattice L are the same.*

Proof. We run the algorithm on D_1. The new edge (u, v) is in the transitive closure of D and hence there is a directed path from $u_0 = u$ to $u_k = v$ denoted by $(u_0, u_1), (u_1, u_2), \ldots, (u_{k-1}, u_k)$. Clearly, any path P that uses (u, v) has a corresponding path P' that replaces (u, v) by $(u_0, u_1), (u_1, u_2), \ldots, (u_{k-1}, u_k)$. We note that while running the algorithm all non-frozen edges have the same value. Hence as long as not all edge $(u_0, u_1), (u_1, u_2), \ldots, (u_{k-1}, u_k)$ are frozen the length of P' is always at least as long as the length of P. Therefore (u, v) will be frozen no earlier than any of the edges $(u_0, u_1), (u_1, u_2), \ldots, (u_{k-1}, u_k)$. Hence the $d(u)$ and $d(v)$ remains the same as in D and the edge (u, v) has not affected the value given for any edge or vertex. ■

In order to show the monotonicity of the nucleolus we need to show that the primal-dual algorithm is monotone.

We would like to compare the optimal assignment d on a DAG D that corresponds to a lattice L with the optimal function d' on the graph D' which is created from L by adding an edge (u, v). There are 3 possibilities. If $(u, v) \in L$ then $L' = L$. If $(v, u) \in L$ then by adding (u, v) the graph would have a cycle and lattice collapses to a new lattice L' by contracting all cycles (they must contain (u, v)) to a single vertex (which we call z). In the third case neither (u, v) nor (v, u) are in L and hence the new lattice L' has the same set V as the lattice L and the extra edge (u, v). Next we prove monotonicity of the function d.

Theorem 14. *Given a DAG D that corresponds to a lattice L, let D' be the graph $D \cup (u, v)$ after contracting cycles. D' corresponds to some lattice L'. Then, let d by the output assignment for D and d' the output assignment for D'. Then $d'(v) \geq d(v)$ and $d'(u) \leq d(u)$ if L' has not collapsed and $d'(z) \geq d(v)$ and $d'(z) \leq d(u)$ and L' collapsed (i.e. cycles have been contracted).*

Proof. The first case where $(u, v) \in L$ then $L' = L$ and by Lemma 13 above $d' = d$ and we are done. Next we consider the third case where (v, u) is not in L. In this case D' is still a directed acyclic graph on the same set of vertices V. We will show that $d'(u) \leq d(u)$. In a symmetric fashion (by replacing the directions of all edges) one would get that $d'(v) \geq d(v)$. We note that all paths in D' are also in D. Run the primal-dual algorithm on D' until the vertex u freezes. If (u, v) has not been frozen yet, then running the algorithm on D would result in the same freezing of u and hence $d(u) = d'(u)$. If (u, v) has frozen, then $d(u) \geq d'(u)$ since in D the value of $d(u)$ may continue to increase. This completes the case that (v, u) is not in L. At last we consider the case that $(v, u) \in L$. Then we have cycles which are contracted to a vertex z (hence we have a lattice L' on less points). Note that u was the maximum vertex in L that participates in any cycle and v was the minimum such vertex. Next, we note that every path in D can be transformed to a path in D' by contracting the appropriate vertices to z. The path in D' cannot be longer than the path in D. Run the primal-dual algorithm on L until the first time a contracted vertex y freezes. Clearly, z could not freeze before in L'. Hence $d'(z) \geq d(y)$. Since v is the minimum vertex in the contracted part we know that $d(v) \leq d(y)$. Hence, we conclude that $d'(z) \leq d(v)$. Similarly, by replacing the directions of all edges we can conclude that $d'(z) \leq d(u)$. ■

Proof of Theorem 3. By applying Theorems 12 and 14, we see that if one adds an edge $(M(i), j)$ to G, it corresponds to adding (i, j) to D and as a result, $d(j)$ increases and $d(i)$ decreases. Therefore ν_j and $\nu_{M(i)}$ both increase, and the nucleolus is monotone. ■

References

[CE78] K. S. Cook and R. M. Emerson. Power, equity, and commitment in exchange networks. *American Sociological Review*, 43(5):721–739, Oct 1978.

[CEGY83] K. S. Cook, R. M. Emerson, M. R. Gillmore, and T. Yamagishi. The distribution of power in exchange networks: Theory and experimental results. *American Journal of Sociology*, 89:275–305, 1983.

[CY92] K. S. Cook and T. Yamagishi. Power in exchange networks: A power-dependence formulation. *Social Networks*, 14:245–265, 1992.

[GDSR07] J. Gonzlez-Daz and E. Snchez-Rodrgue. A natural selection from the core of a TU game: the core-center. *International Journal of Game Theory*, 36(1), September 2007.

[HC98] David Housman and Lori Clark. note: Core and monotonic allocation methods. *International Journal of Game Theory*, 27(4):611–616, 1998.

[Hok00] Toru Hokari. note: The nucleolus is not aggregate-monotonic on the domain of convex games. *International Journal of Game Theory*, 29(1):133–137, 2000.

[KT08] J. Kleinberg and E. Tardos. Balanced outcomes in social exchange networks. In *ACM Symposium on the Theory of Computing*, 2008.

[Mas92] Michael Maschler. The bargaining set, kernel, and nucleolus. In R.J. Aumann and S. Hart, editors, *Handbook of Game Theory with Economic Applications*, volume 1 of *Handbook of Game Theory with Economic Applications*, chapter 18, pages 591–667. Elsevier, July 1992.

[Meg74] Nimrod Megiddo. On the nonmonotonicity of the bargaining set, the kernel and the nucleolus of game. *SIAM Journal on Applied Mathematics*, 27(2):355–358, 1974.

[Nas50] J. Nash. The bargaining problem. *Econometrica*, 18:155–162, 1950.

[Rad07] L. A. Rademacher. Approximating the centroid is hard. In *Annual Symposium on Computational Geometry*, pages 302 – 305, 2007.

[Roc84] Sharon C. Rochford. Symmetrically pairwise-bargained allocations in an assignment market. *Journal of Economic Theory*, 34(2):262–281, 1984.

[Sch69] D. Schmeidler. The nucleolus of a characteristic function game. *SIAM J. Appl. Math*, 17:1163–1170, 1969.

[Sch09] Michael Schwarz, 2009. Personal Communication.

[She82] L. A. Shepp. The XYZ conjecture and the FKG inequality. *Annals of Probability*, 1982.

[SR94] T. Solymosi and T. E. S. Raghavan. An algorithm for nding the nucleolus of assignment games. *Internat. J. Game Theory*, 23:119–143, 1994.

[SS72] L. S. Shapley and M. Shubik. The assignment game I: The core. *Internat. J. Game Theory*, 1:111–130, 1972.

[SY09] Michael Schwarz and M. Bumin Yenmez. Median stable matching. Working Paper 14689, National Bureau of Economic Research, January 2009.

[TS98] Chung-Piaw Teo and Jay Sethuraman. The Geometry of Fractional Stable Matchings and Its Applications. *Mathematics of Operations Research*, 23(4):874–891, 1998.

[Wil99] D. Willer, editor. *Network Exchange Theory*. Praeger, 1999.

[You85] H. P. Young. Monotonic solutions of cooperative games. *International Journal of Game Theory*, 14(2):65–72, June 1985.