Algorithmics of Directional Antennae: Strong Connectivity with Multiple Antennae

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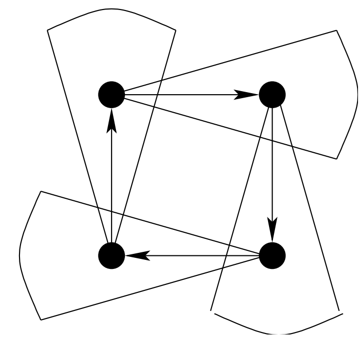
Setting

- Set of sensors represented as a set of points S in the 2D plane.



- Each sensor has k directional antennae.
- All antennae have the same transmission range r.
- Each antenna has a transmission angle, forming a coverage cone up to distance r.
  - there is a directed edge from u to v iff v lies in the cone of some antenna of u

Antenna Spread

The transmission angle(spread) of antennae is limited to φ, where φ is

- either the sum of angles for antennae in the same node, or
- the maximum transmission angle of the antennae.

The sum of angles case corresponds to energy consumption per node, the presented results will be for that case.

The Problem

Given a set of points S, number of antennae k per node, a transmission range r and an angle limit φ, set the transmission direction and angle for each antenna in such a way that the resulting directed graph is strongly connected.

Typically, we fix k and φ and try to minimize r for a given point set S.

The Transmission Range

Let $r_{(k,\varphi)-OPT}(S)$ denote the optimal (shortest) range allowing solution.

Let $r_{MST}(S)$ be the shortest range r such that UDG(S, r) is connected.

- obviously, $r_{MST} \leq r_{(k,\varphi)-OPT}$

As establishing $r_{(k,\varphi)-OPT}$ might be NP-hard, we will compare the radius r produced by a solution to r_{MST}.

- for simplicity, we re-scale S to get $r_{MST} = 1$

- later, we will discuss comparing to $r_{(k,\varphi)-OPT}$

Overview

- Introduction
- Upper Bounds
- Lower Bounds/NP-Hardness
- Conclusions/Open Problems

Upper Bounds - Results

#	Antennae Spread	Antennae Range	Paper
1	$0 \leq \varphi < \pi$	2	[PR84]
1	$\pi \leq \varphi < 8\pi/5$	$2\sin(\pi - \varphi/2)$	[CKK+08]
1	$8\pi/5 \leq \varphi$	1	[CKK+08]
2	$0 \leq \varphi < 2\pi/3$	$\sqrt{3}$	[DKK+10]
2	$2\pi/3 \leq \varphi < \pi$	$2\sin(\pi/2 - \varphi/4)$	[BHK+09]
2	$\pi \leq \varphi < 6\pi/5$	$2\sin(2\pi/9)$	[BHK+09]
2	$6\pi/5 \leq \varphi$	1	[BHK+09]
3	$0 \leq \varphi < 4\pi/5$	$\sqrt{2}$	[DKK+10]
3	$4\pi/5 \leq \varphi$	1	[BHK+09]
4	$0 \leq \varphi < 2\pi/5$	$2\sin(\pi/5)$	[DKK+10]
4	$2\pi/5 \leq \varphi$	1	[BHK+09]
≥ 5	$0 \leq \varphi$	1	[BHK+09]

Basic Observations

- The angle between two incident edges of an MST of a point set is at least $\pi/3$.
- For every point set there exists an MST of maximal degree 5.
- All angles incident to a vertex of degree 5 of the MST are between $\pi/3$ and $2\pi/3$ (included).

Corollary 1. *With $k \geq 5$ antennae, each of spread 0, there exists a solution with range 1.*

- Assign an antenna for each incident edge of the MST.

Upper Bound Techniques

All results are based on locally modifying the MST, using various techniques when k is smaller than the degree of the node in the MST to locally ensure strong connectivity:

- use antenna spread to cover several neighbours by one antenna, or
- use neighbour's antennae to locally ensure strong connectivity

Upper Bounds

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Antenna Range 1

Theorem 1. *For any* $1 \leq k \leq 5$*, there exists a solution with range* 1 *and antenna spread* $\frac{2(5-k)\pi}{5}$*.*

- exclude k largest incident angles
- this leaves k segments of total spread $\frac{2(5-k)\pi}{5}$

Upper Bounds

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2 antennae, spread π, range $2\sin(2\pi/9)$

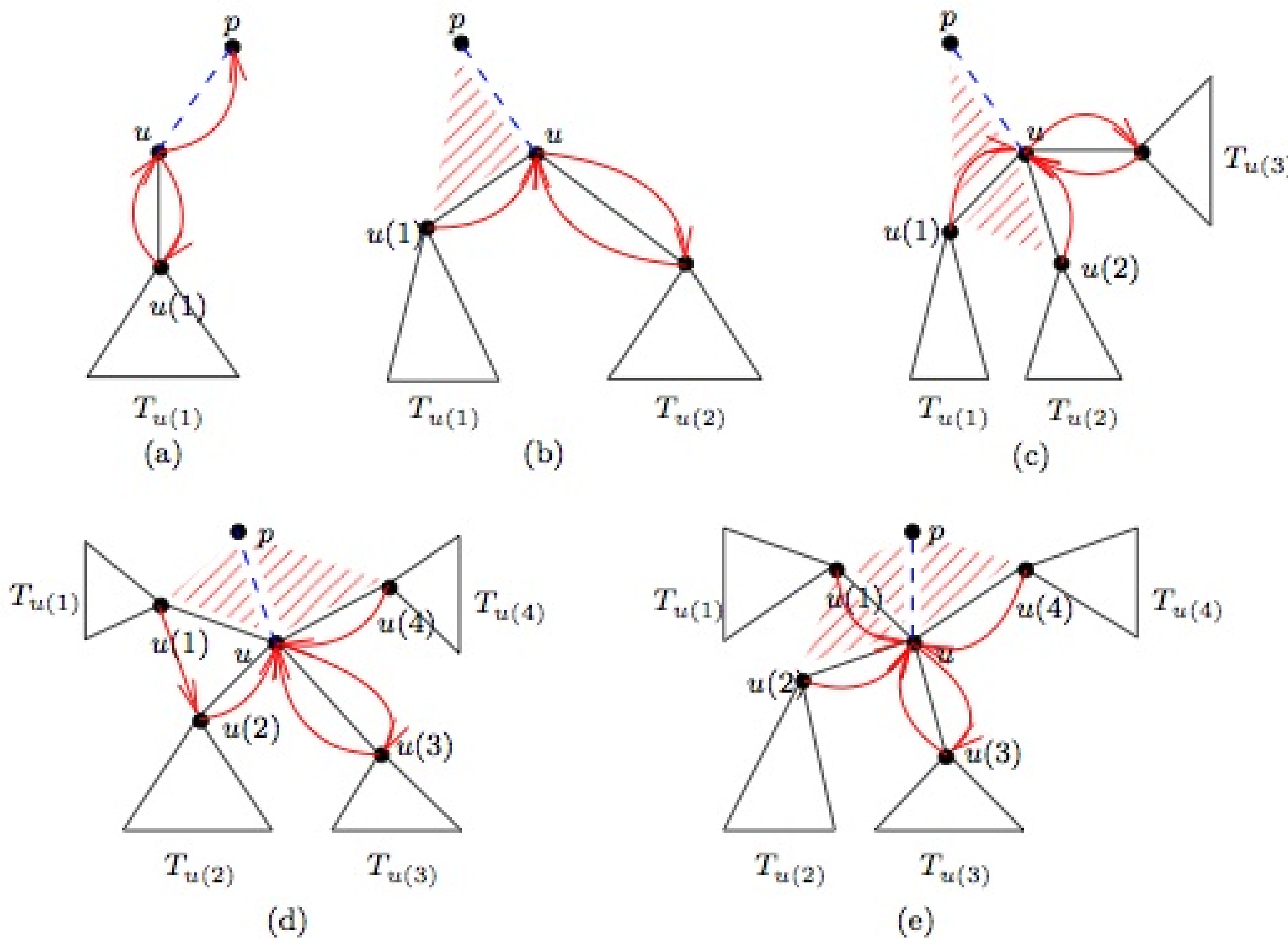
Definition 2. *A vertex p is a nearby target vertex to a vertex $v \in T$ if $d(v,p) \leq 2\sin(2\pi/9)$ and p is either a parent or a sibling of v in T.*

Definition 3. *A subtree T_v of T is nice iff for any nearby target vertex p the antennae at vertices of T_v can be set up so that the resulting graph (over vertices of T_v) is strongly connected and p is covered by an antenna from v.*

Theorem 2. *There is a way to set up 2 antennae per vertex, with antenna spread of π and range $2\sin(2\pi/9)$ in such a way that the resulting graph is strongly connected.*

Proof: By proving that T_v is nice for all v, by induction on the depth of T_v.

Induction step - case analysis on the number of children of u



Upper Bounds

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≥ 5	$0 \leq \varphi$	$2\sin(\pi/6) = 1$	[BHK+09]

Antenna spread 0

Theorem 3. *For any $1 \leq k \leq 5$, there exists a solution with range $2\sin(\frac{\pi}{k+1})$ and antenna spread 0.*

- by induction on the depth of T
- not connecting child solutions to the parent vertex, but removing all leaves, applying the induction hypothesis, then returning the leaves and showing how to connect them

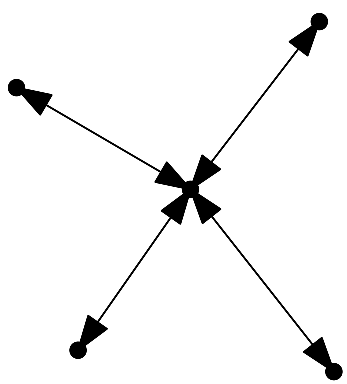
Note that since the spread is 0, a solution can be represented as a directed graph $\overrightarrow{G}$ with maximum out-degree k and edge lengths at most $2\sin(\frac{\pi}{k+1})$.

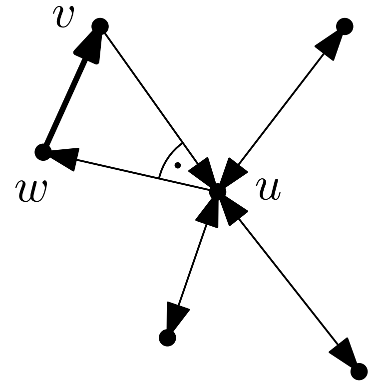
4 antennae, spread 0, range $2\sin(\pi/5)$

Induction hypothesis: Let T be an MST of a point set of radius at most x. Then, there exists a solution $\overrightarrow{G}$ for T such that:

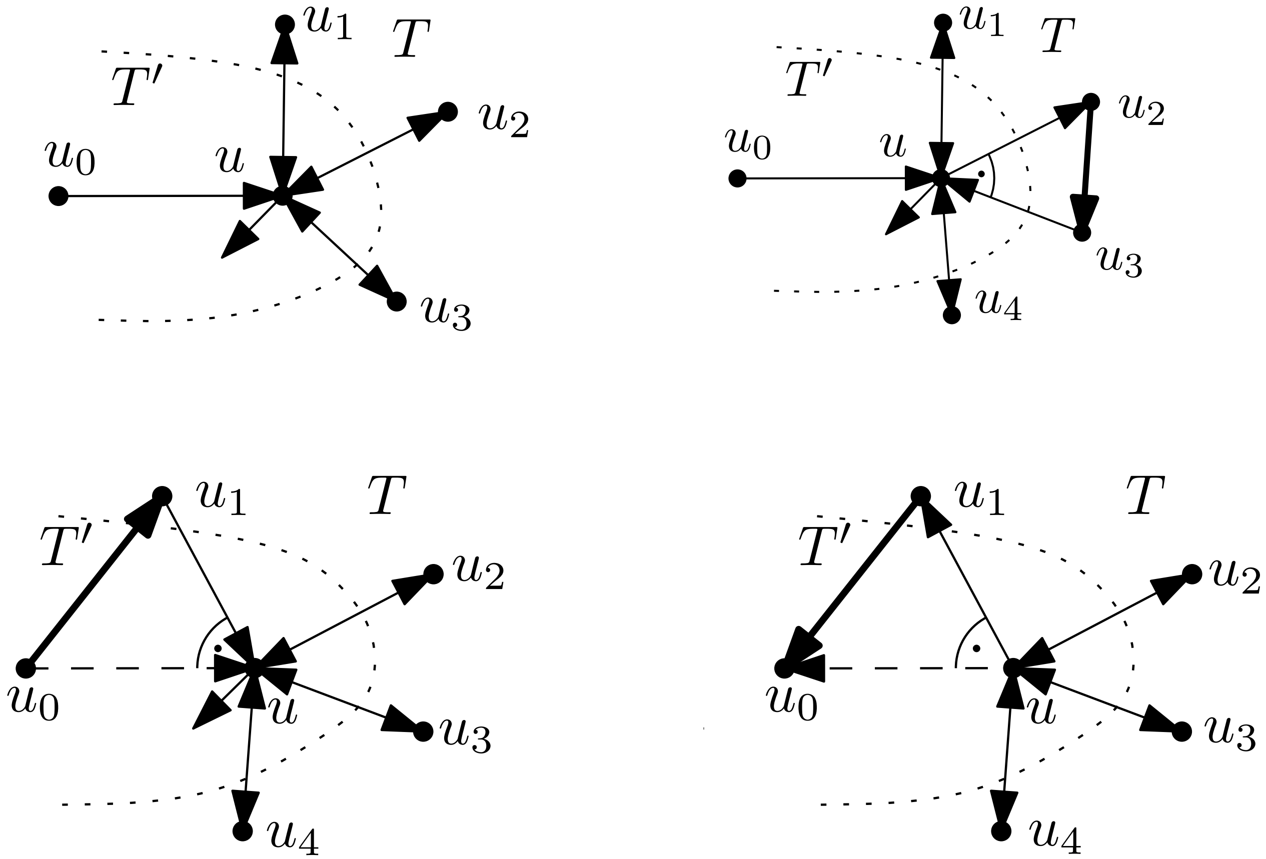
- the out-degree of u in $\overrightarrow{G}$ is one for each leaf u of T
- every edge of T incident to a leaf is contained in $\overrightarrow{G}$

Base step:

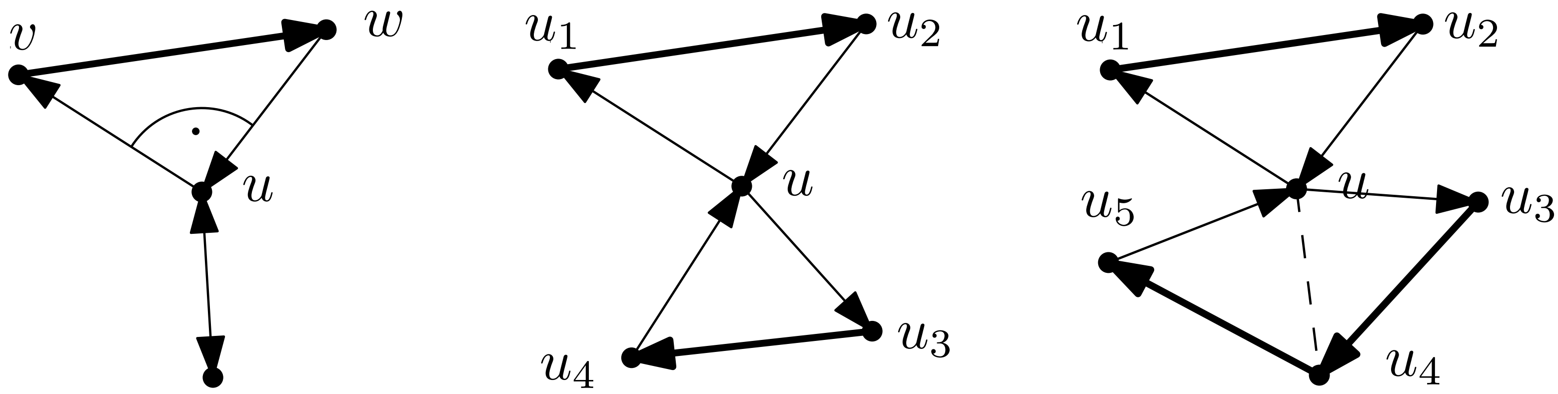




4 antennae, spread 0 - Inductive Step



2 antennae, spread 0 - Base Step



Problem: Can't ensure the inductive hypothesis ((u, u_4) not used in the solution).

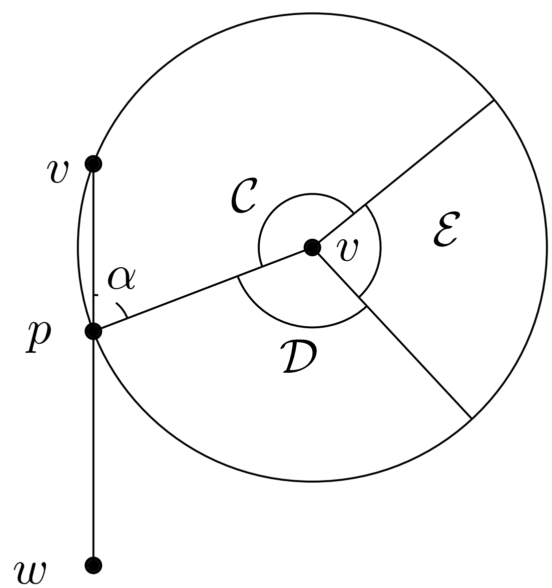
2 antennae, spread 0 - Induction Hypothesis

Let T be an MST of a point set of radius at most x. Then, there exists a solution $\overrightarrow{G}$ for T such that:

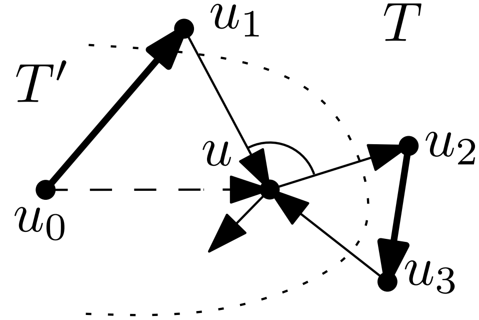
- The out-degree of u in $\overrightarrow{G}$ is one for each leaf u of T.
- Every edge of T incident to a leaf is contained in $\overrightarrow{G}$, or
- a leaf is connected to its two consecutive siblings and the edges of T incident to these siblings are also contained in G.

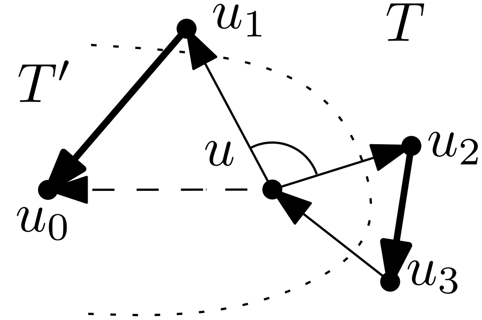
2 antennae, spread 0 - Technical Lemma

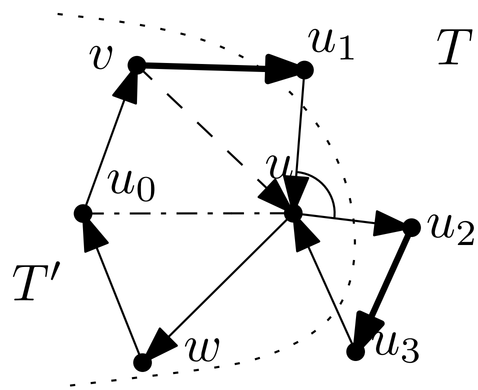
Lemma 4. *Let u, v and w be three consecutive children of p in T such that $\angle(upw) \leq \pi$. Then in any case that requires use of 2 antennae at v to solve T_v there exists a child of v that is close to either u or w.*

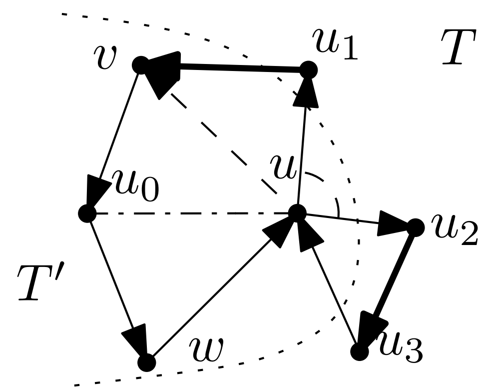


2 antennae, spread 0 - Inductive Step

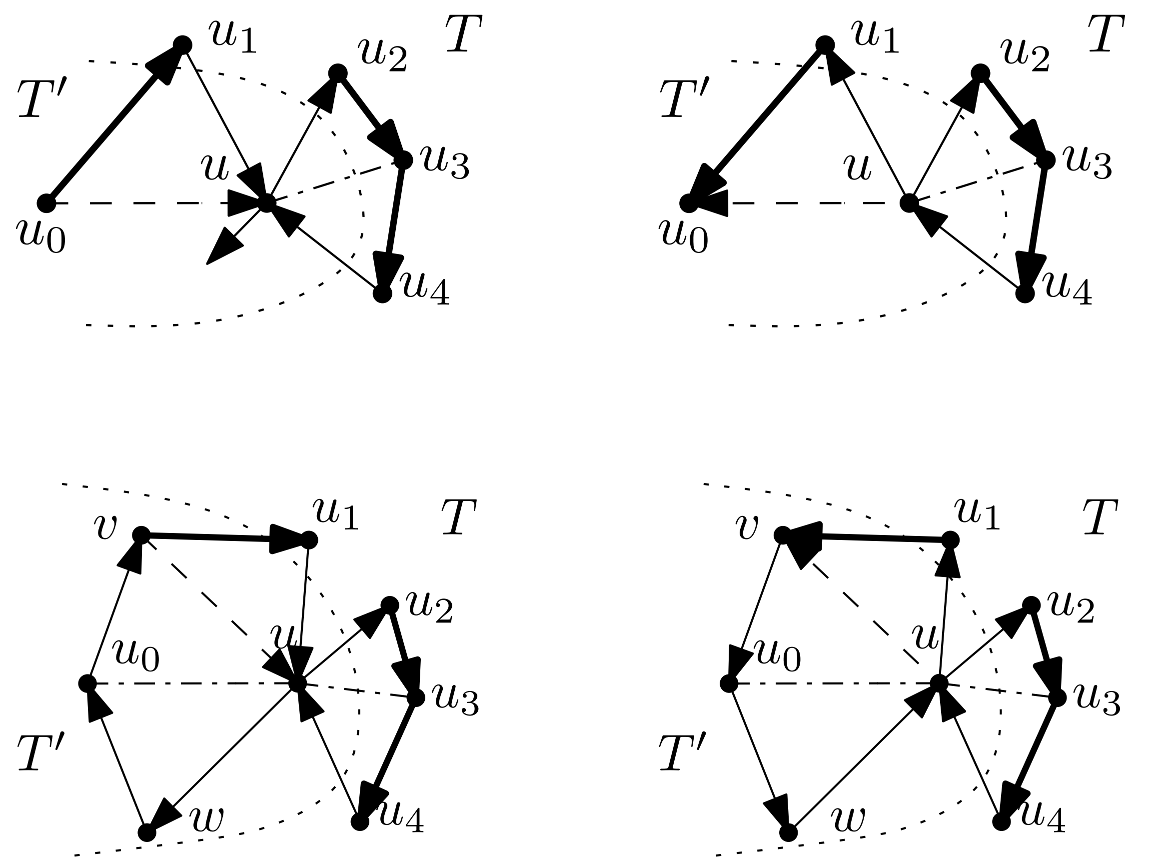








2 antennae, spread 0 - Inductive Step



Upper Bounds - How Good?

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Lower Bounds - Small Angles

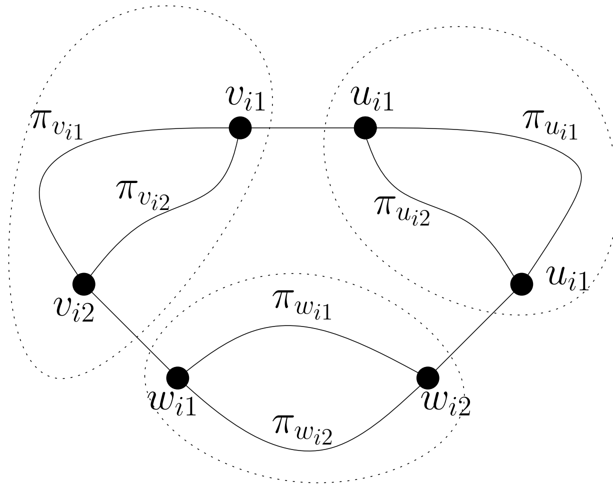
- Consider regular $k+1$-star. With angle less then $\frac{2\pi}{k+1}$, the central vertex can't reach all leaves using k antennae, hence a leaf must connect to another leaf, using range at least $2\sin(\frac{\pi}{k+1})$.
- Hence our results for spread 0 are optimal. . .
- . . . with respect to r_{MST}.
- But what about $r_{(k,\varphi)-OPT}$? In regular $k+1$-star also $r_{(k,\varphi)-OPT}$ is large!

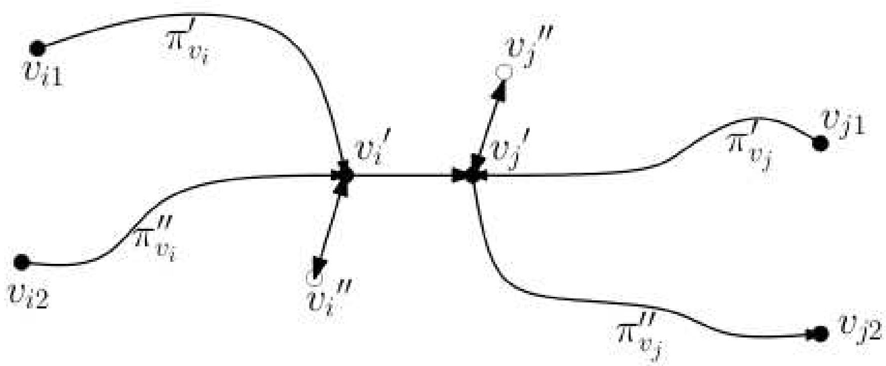
HP-Hardness for 2-antennae with limited spread and radius

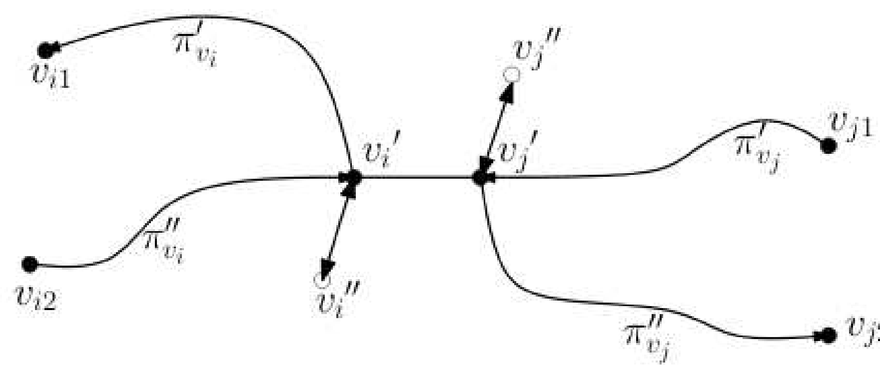
Theorem 4. *For $k = 2$ antennae, if the angular sum of the antennae is less then φ then it is NP-hard to approximate the optimal radius to within a factor of x, where x and φ are the solutions of equations $x = 2\sin(\varphi) = 1 + 2\cos(2\varphi)$.*

- $x \approx 1.30, \varphi \approx 0.45\pi$.
- The proof is by reduction from the problem of finding Hamiltonian cycles in degree three planar graphs.

HP-Hardness for $k = 2$: Key Gadgets





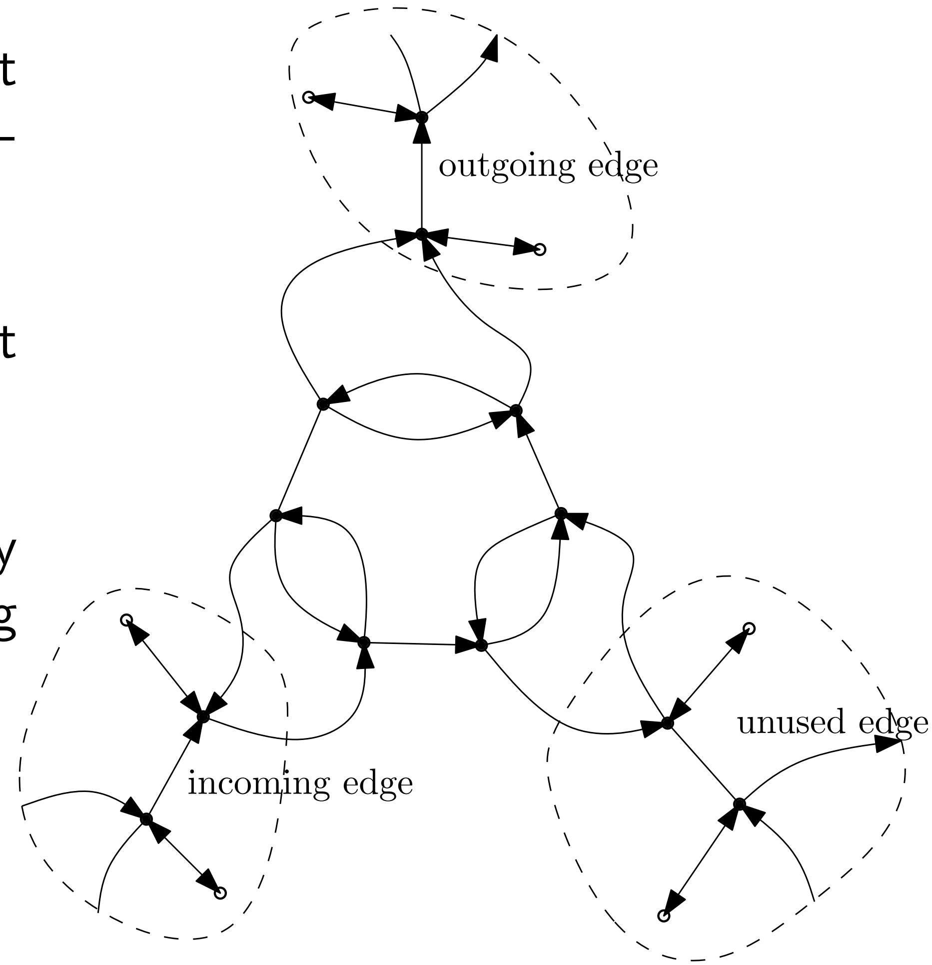


HP-Hardness for $k = 2$: Key Observations

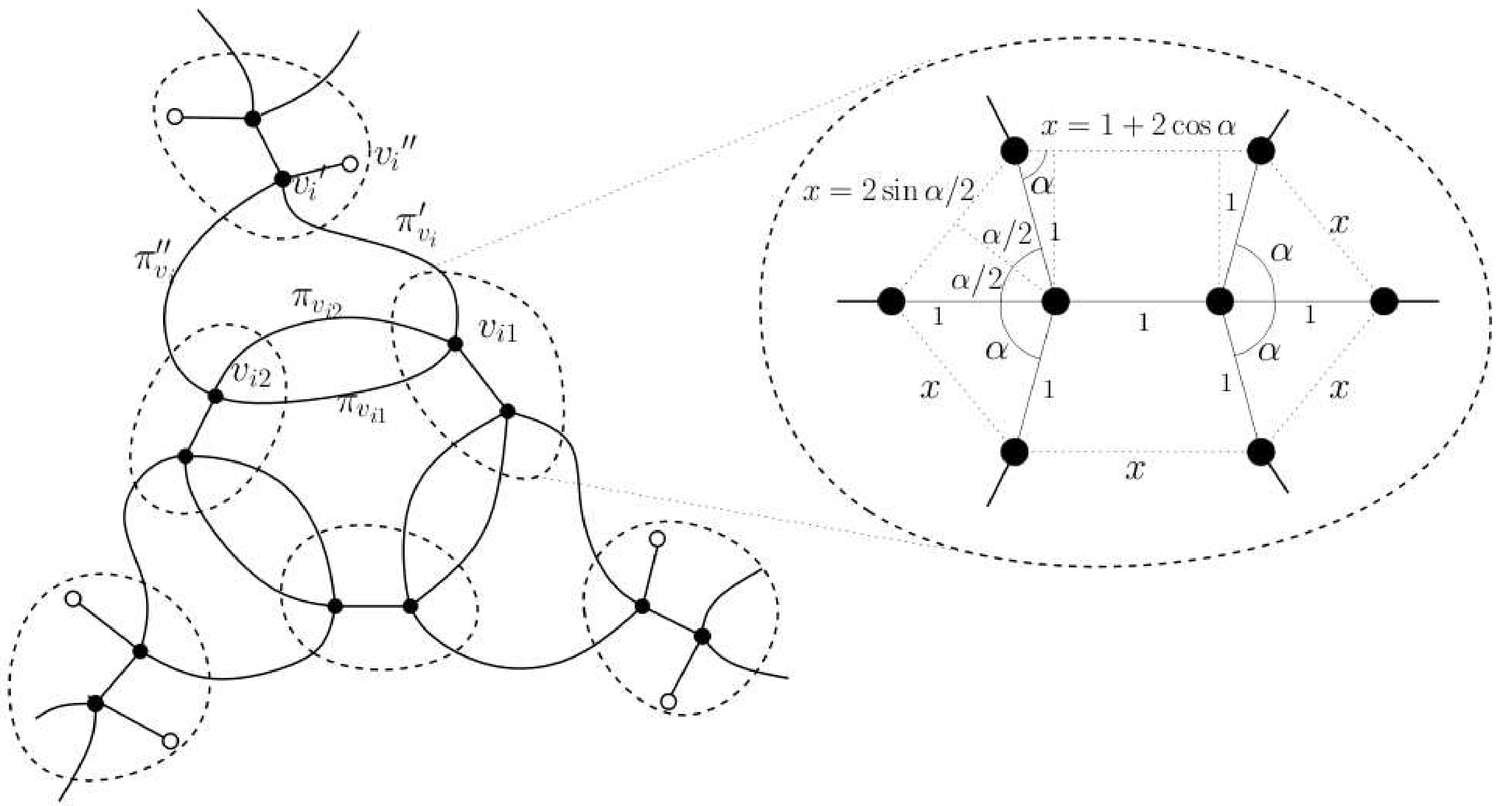
- each meta-vertex must have at least incoming and one outgoing meta-edge

- each meta-vertex can have at most one outgoing meta-edge

- hence each meta-vertex has exactly one outgoing and one incoming meta-edge



HP-Hardness for $k = 2$: x and φ



Lower Bounds: What about $k = 3$ and $k = 4$?

- Let G be a connected graph. A vertex v is an c-separator iff $G \setminus \{v\}$ has at least c connected components.

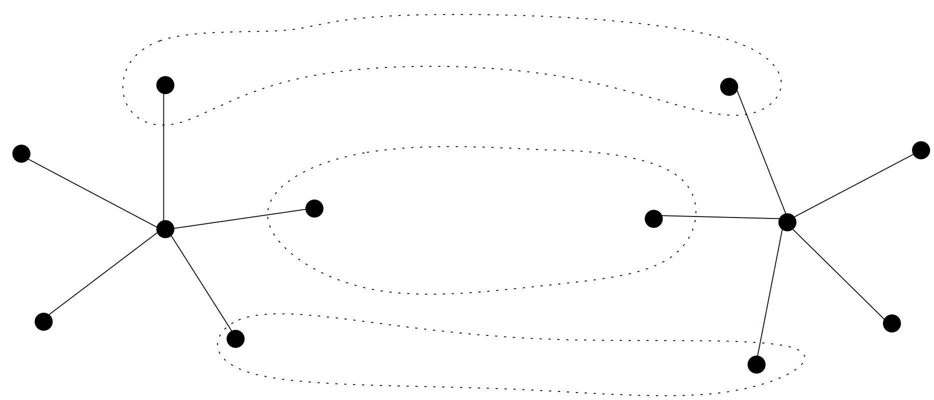
- Define $r_k(S)$ to be the smallest radius r such that UDG(S, r) does not contain a $k+1$-separator vertex.

- Obviously, $r_k(S) \leq r_{(k,0)-OPT}(S)$.

- Our hypothesis is that $r_4(S) = r_{(4,0)-OPT}(S)$ and the solution can be computed polynomially.

Lower Bounds: $k = 3$

It is not true that $r_3(S) = r_{(3,0)-OPT}(S)$.



Our hypothesis: $r_{(3,0)-OPT}$ is the smallest radius r that ensures that UDG(S, r) does not contain such a pair of separator vertices.

Conclusions/Open Problems

- there are still gaps between the lower and upper bounds
- especially for non-zero φ
- the x and φ in the NP-hardness results might possibly be improved
- consider different model variants
  - directional receivers
  - temporal aspects (antennae steering, ...)
- and different problems... (Laco)

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