Colouring random geometric graphs

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(based on joint work with Tobias Müller)

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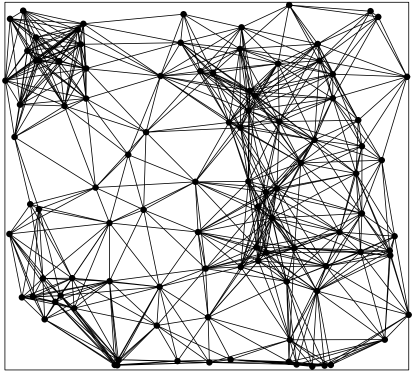
random geometric graph (RGG) in $\mathbf{R}^2$

Construct a random graph $G(n, r)$ as follows. Pick vertices $X_1, \ldots, X_n \in [0, 1]^2$ i.i.d. uniformly at random and join X_i, X_j $(i \neq j)$ by an edge if $\| X_i - X_j \| < r$. Here $n = 100$ and $r = \frac{1}{4}$.



RGG more generally

- RGG can be defined in arbitrary dimension d, with the points $X_1, \ldots, X_n$ i.i.d. according to any probability measure on $\mathbb{R}^d$ with bounded density, and where the distance between points is measured by an arbitrary norm $\| . \|$ on $\mathbb{R}^d$.
- Results hold in this general case, but we will focus initially on the case when $d = 2$, the points are i.i.d. uniform on the unit square, and we use the Euclidean norm to measure distance between points.

Erdős-Rényi random graph (ERRG)

Perhaps more familiar is the Erdős-Rényi (or binomial) random graph $G(n, p)$. It has vertex set $V = \{1, \ldots, n\}$; and for each of the $\binom{n}{2}$ candidate edges ij we flip a coin with success probability $p = p_n$ to decide whether or not to include it, independently of all other edges.

r_n in RGG and p_n in ERRG

We are interested in the behaviour of RGG as n grows large, where $r = r_n$ varies with n.

The distance r_n plays a role similar to that of the edge-probability p_n in the Erdős-Rényi model. Depending on the choice of r_n qualitatively different types of behaviour can be observed.

Always assume $r_n \to 0$ as $n \to \infty$.

connectedness in ERRG

np_n is roughly the expected degree of a vertex.

Theorem [Erdős&Rényi 1959]
Write $x_n = np_n - \ln n$. Then

$$\lim_{n\to\infty} \mathbb{P}\left[G(n, p_n) \text{ is connected }\right] = \begin{cases} 0 & \text{if } x_n \to -\infty; \\ e^{-e^{-x}} & \text{if } x_n \to x \in \mathbb{R}; \\ 1 & \text{if } x_n \to +\infty. \end{cases}$$

connectedness in RGG

$\pi n r_n^2$ is roughly the expected degree.

Theorem [Penrose 1997]
Write $x_n := \pi n r_n^2 - \ln n$. Then:

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why so similar in ERRG and RGG?

In both models the main obstruction to being connected is having an isolated vertex (a vertex with no neighbours).

The expected number of isolated vertices is $n(1-p_n)^{n-1} \approx e^{-x}$ in ERRG, and roughly $n(1-\pi r_n^2)^{n-1} \approx e^{-x}$ in RGG.

The number Z of isolated vertices has approximately the Poisson distribution $Po(\lambda)$ with mean $\lambda = e^{-x}$, and so

$$\mathbb{P}(Z=0) \approx e^{-\lambda} = e^{-e^{-x}}.$$

giant component in ERRG

Let $L(G)$ denote the largest number of vertices in a component.

Theorem [Erdős&Rényi 1960]

1. If $np_n \leq 1$ then for any $\delta > 0$

$$\mathbb{P}(L(G(n, p_n) > \delta n) \to 0$$

2. If $np_n \geq 1 + \varepsilon$ for some $\varepsilon > 0$ then there exists $\delta > 0$ such that

$$\mathbb{P}(L(G(n, p_n) > \delta n) \to 1$$

giant component in RGG

Theorem [Penrose 2003]
There is a constant $\lambda_{\text{crit}} > 0$ such that, for each $\varepsilon > 0$:

(i) If $\pi n r_n^2 \leq \lambda_{\text{crit}} - \varepsilon$ then for each $\delta > 0$

$$\mathbb{P}(L(G(n, p_n) > \delta n) \to 0$$

(ii) If $\pi n r_n^2 \geq \lambda_{\text{crit}} + \varepsilon$ then there exists $\delta > 0$ such that

$$\mathbb{P}(L(G(n, p_n) > \delta n) \to 1$$

The precise value of λ_{crit} is unknown, but experimentally $\lambda_{\text{crit}} \approx 4.5$.

graph theory: notation and terminology

$\Delta(G)$ denotes the maximum degree of a vertex in the graph G.

A *clique* (or complete subgraph) in G, is a set of pairwise adjacent vertices. The clique number $\omega(G)$ is the maximum size of a clique.

A *stable* (or independent) set is a set of pairwise adjacent vertices. The *stability number* (or independence number) $\alpha(G)$ is the maximum size of a stable set.

A k-colouring of G is a map $f : V \to \{1, \ldots, k\}$ such that $f(v) \neq f(w)$ whenever v and w are adjacent. The chromatic number $\chi(G)$ is the least k such that G has a k-colouring.

bounds on $\chi(G)$ (for any G)

$$\chi(G) \geq \omega(G)$$

$$\chi(G) \geq \frac{|V(G)|}{\alpha(G)}$$

$$\chi(G) \leq \Delta(G) + 1$$

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In general the ratios of $\chi(G)$ to these bounds can be arbitrarily large. For example, for ERRG the ratio χ/ω is big:
$\frac{\chi(G(n,\frac{1}{2}))}{\omega(G(n,\frac{1}{2}))} \to \infty$ whp for any p_n such that $\frac{1}{n} \ll p_n < 1 - \varepsilon$.

χ and ω for geometric graphs

Let G be a geometric graph in $\mathbf{R}^2$ (unit disk graph).

Then every subgraph has a vertex of degree less than $3\omega(G)$, and so $\chi(G)/\omega(G) \leq 3$.

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We can find the clique number $\omega(G)$ in polynomial time, though it is NP-hard to find the chromatic number $\chi(G)$.

χ/ω for RGG: the story in 2000

Theorem [McD RSA 2003]

(i) (sparse case) If $\frac{nr_n^2}{\ln n} \to 0$ (not too quickly) then

$$\frac{\chi(G(n, r_n))}{\omega(G(n, r_n))} \to 1 \quad \text{whp}$$

(ii) (dense case) If $\frac{nr^2}{\ln n} \to \infty$ then

$$\frac{\chi(G(n, r_n))}{\omega(G(n, r_n))} \to \frac{2\sqrt{3}}{\pi} \approx 1.103 \quad \text{whp}$$

where does this come from?

$$\omega(G(n,r)) \geq \max_x \text{ number of points } X_i \text{ in } B(x; r/2).$$

$$\chi(G(n,r)) - 1 \leq \Delta(G(n,r)) \leq \max_x \text{number of points } X_i \text{ in } B(x; r).$$

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In the dense case, the points X_i behave as if they are uniformly spread. We can approximate max above by average and obtain the value $\pi n r^2/4$ for ω.

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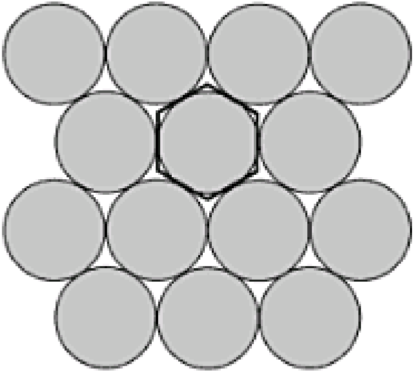

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So the number of radius $\frac{r}{2}$ disks we can pack in the unit square is about $\frac{\pi}{2\sqrt{3}}/(\frac{\pi r^2}{4}) = \frac{2}{\sqrt{3}}r^{-2}$, leading to $\alpha \sim \frac{2}{\sqrt{3}}r^{-2}$.
Since the points are uniformly spread, we find we can colour using stable sets like a triangular lattice, of size close to α. This gives a colouring using at most about $\frac{n}{\alpha}$ colours. But always $\chi \geq \frac{n}{\alpha}$, so $\chi \sim \frac{n}{\alpha} \sim \frac{\sqrt{3}}{2}nr^2$, and

$$\frac{\chi}{\omega} \sim \frac{2\sqrt{3}}{\pi} \approx 1.103.$$

in more generality

Now consider $\mathbb{R}^d$ for some fixed d, with a norm $\| . \|$. The (translational) 'packing density' δ is the greatest proportion of $\mathbb{R}^d$ that can be filled with disjoint translates of the unit ball B, where $B = \{x \in \mathbb{R}^d : \| x \| < 1\}$.

Always $0 < \delta \leq 1$. For $\mathbb{R}^2$ and the Euclidean norm, we saw that $\delta = \frac{\pi}{2\sqrt{3}} \approx 0.907$, and $1/\delta \approx 1.103$.

Assume that $\delta < 1$.

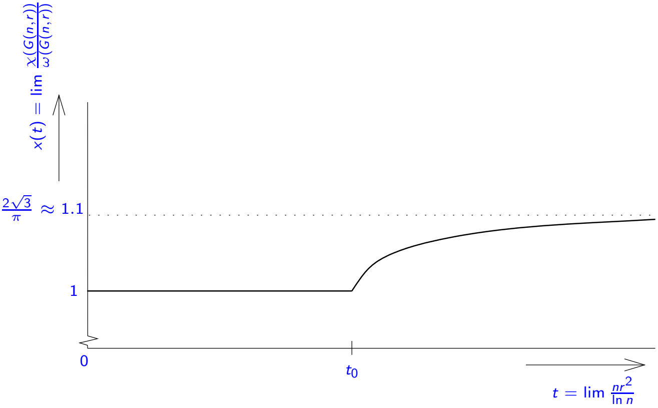
χ/ω in the 'phase change' region

Theorem [McD&Müller 2011]
There is a constant $0 < t_0 < \infty$ and a continuous function $x(t)$ for $t \in [0, \infty]$ such that, if $\frac{nr^d}{\ln n} \to t$ then

$$\frac{\chi(G(n, r_n))}{\omega(G(n, r_n))} \to x(t) \quad \text{whp.}$$

Here $x(t) = 1$ for $t \leq t_0$, $x(t)$ is strictly increasing for $t \geq t_0$ and $x(t) \to x(\infty) = 1/\delta$ as $t \to \infty$.

A picture for $\mathbb{R}^2$ and the Euclidean norm

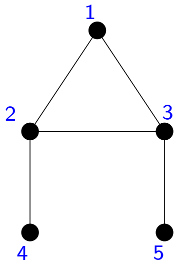


How does $x(t)$ arise?

The vertex-stable set incidence matrix

The vertex-stable set incidence matrix of G is the matrix A whose rows are indexed by the vertices of G and whose columns are indexed by the stable sets of G. If $v \in V$ and $S \subseteq V$ is a stable set then $A_{v,S} = 1$ if $v \in S$ and 0 otherwise.

vertex-stable set incidence matrix



has vertex-stable set incidence matrix:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \end{pmatrix}.$$

$A_{v,S} = 1$ if v is in the stable set S and $= 0$ otherwise.

ILP formulation of graph colouring

Let A be the vertex-stable set incidence matrix of G, with a row for each vertex and a column for each stable set.

The chromatic number $\chi(G)$ can be written as an integer linear programme (ILP) as follows:

$$\begin{aligned} \min \quad & 1^T x \\ \text{subject to} \quad & Ax \geq 1, \\ & x \geq 0, x \text{ integral.} \end{aligned}$$

fractional chromatic number

The *fractional chromatic number* $\chi_f(G)$ is the LP-relaxation of this ILP:

$$\begin{aligned} \min \quad & 1^T x \\ \text{subject to} \quad & Ax \geq 1, \\ & x \geq 0. \end{aligned}$$

Clearly $\chi_f(G) \leq \chi(G)$. Crucially, it turns out that

$$\chi(G(n,r)) \sim \chi_f(G(n,r)) \quad \text{whp.}$$

using LP duality

By LP-duality $\chi_f(G)$ also equals:

$$\begin{array}{ll} \max & 1^T y \\ \text{subject to} & A^T y \leq 1, \\ & y \geq 0. \end{array}$$

The feasible vectors y for this dual programme give each vertex v a non-negative value y_v and $\sum_{v \in S} y_v \leq 1$ for each stable set S.

For geometric graphs we shall introduce a corresponding notion of (dual) feasible **functions** defined on $\mathbb{R}^d$ (where vertices live).

collection $\mathcal{F}$ of feasible functions

Let $\mathcal{S}$ denote the collection of all 'well-spread' sets $S \subseteq \mathbb{R}^2$ with

$$x, y \in S,\ x \neq y \quad \Rightarrow \quad \|x - y\| \geq 1.$$

Let $\mathcal{F}$ denote the collection of all feasible functions $\varphi : \mathbb{R}^2 \to [0, \infty)$ that satisfy $\sum_{x \in S} \varphi(x) \leq 1$ for each set $S \in \mathcal{S}$ (and some regularity conditions).

For example $\varphi_0 := 1_{B(0,\frac{1}{2})} \in \mathcal{F}$ and $\varphi := \frac{1}{N(K)} \cdot 1_{[0,K)^2} \in \mathcal{F}$ with $N(K)$ the maximum number of points inside $[0, K)^2$ with all distances at least 1.

$M(V, \varphi)$ and χ_f: deterministic result 1

For any set $V \subseteq \mathbb{R}^d$, define $M(V, \varphi)$ to be

$$\sup_{x} \sum_{v \in V} \varphi(v - x).$$

Then we may easily see that

$$\chi_f(G(V, 1)) = \sup_{\varphi \in \mathcal{F}} M(V, \varphi).$$

$M(V, \varphi)$ and χ_f: deterministic result 2

For each $\varepsilon > 0$ there exists a positive integer m, simple $(1+\varepsilon)$-feasible, tidy functions $\varphi_1, \ldots, \varphi_m$, and a constant c such that:

$$\chi(G(V,1)) \leq (1+\varepsilon) \max_{i=1,\ldots,m} M(V, \varphi_i) + c,$$

The proof involves discretising and rounding up an optimal basic feasible solution to the (primal) LP for χ_f to obtain a solution to the ILP for χ.

generalised scan statistic

So to learn about $\chi(G_n)$ we investigate $M(V, \varphi)$ with $V = \{X_1, \ldots, X_n\}$ suitably rescaled; that is, the 'generalised scan statistic'

$$M_\varphi = \sup_x \sum_{i=1}^{n} \varphi(\frac{1}{r} X_i - x).$$

Recall that $\frac{nr^d}{\ln n} \to t$ as $n \to \infty$. We need to find the appropriate factor to multiply the maximum of an expected value in order to obtain the expected value of the maximum; as in

$$\begin{aligned} &\mathbf{E} \ \max_x \text{(number of points } X_i \text{ in } B(x,r)) \\ = &\max_x \mathbf{E} \text{ (number of points } X_i \text{ in } B(x,r)) \ \cdot \text{factor} \end{aligned}$$

the weighting value $s(\varphi, t)$

Let $H(x) = x \ln x - x + 1$ for $x \geq 1$. Then $H(1) = 0$, H is strictly increasing, and $H(x) \to \infty$ as $x \to \infty$.

Fix a non-negative bounded function φ on $\mathbb{R}^d$ with $0 < \int \varphi(x)dx < \infty$ ($\int$ means $\int_{\mathbb{R}^d}$). The weighting value $s = s(\varphi, t)$ is uniquely defined for $0 < t < \infty$ by

$$\int H(e^{s\varphi(x)})dx = \frac{1}{t}.$$

This function is decreasing, $\to \infty$ as $t \to 0$, and $\to 0$ as $t \to \infty$.

weighted integral $\xi(\varphi, t)$

Using the weighting value $s = s(\varphi, t)$ we define the weighted integral

$$\xi(\varphi, t) := \int_{\mathbb{R}^d} \varphi(x) e^{s(\varphi,t)\varphi(x)} dx.$$

Think of φ as a dual-feasible vector, and the integral as an average sum of dual variables multiplied by an appropriate factor.

Suppose $\varphi = \mathbf{1}_W$ where $\mathrm{vol}(W) = w$. Then

$$\int H(e^{s\varphi(x)})dx = \int_W H(e^s)dx = w \cdot H(e^s),$$

so $s = s(\varphi, t)$ satisfies $e^s = H^{-1}(\frac{1}{wt})$. Thus

$$\xi(\varphi, t) = \int_W e^s dx = w \cdot e^s = w \cdot H^{-1}(\frac{1}{wt}).$$

ω, χ and χ/ω

Suppose that $\frac{nr^d}{\ln n} \to t \in (0, \infty]$.

Recall that $\varphi_0 = 1_{B(0,\frac{1}{2})} \in \mathcal{F}$.

Theorem [Penrose] $\omega(G(n, r_n))/nr^d \to \xi(\varphi_0, t)$ a.s.

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Theorem [Penrose] $\omega(G(n, r_n))/nr^d \to \xi(\varphi_0, t)$ a.s.

Theorem [McD&Müller] $\chi(G(n, r))/nr^d \to \sup_{\varphi \in \mathcal{F}} \xi(\varphi, t)$ a.s.

Thus the a.s. limit of $\frac{\chi(G(n,r_n))}{\omega(G(n,r_n))}$ is

$$f_{\chi/\omega}(t) = x(t) = \frac{\sup_{\varphi \in \mathcal{F}} \xi(\varphi, t)}{\xi(\phi_0, t)}.$$

Questions

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When is the chromatic number 'local' rather than 'global'?
Let $R(G)$ be $\inf R$ such that the set W of points in some ball of radius R has $\chi(G[W]) = \chi(G)$.

If $\frac{nr^2}{\ln n} \to t > 0$ where t is sufficiently small, then $R(G(n, r_n)) \leq 2r_n$. What about larger t?