Algorithmics of Directional Antennae

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Introduction

Given a set of sensors with omnidirectional antennae forming a connected network.

How can omnidirectional antennae be replaced with directional antennae in such a way that the connectivity is maintained while the angle and the range are the smallest possible?

Outline

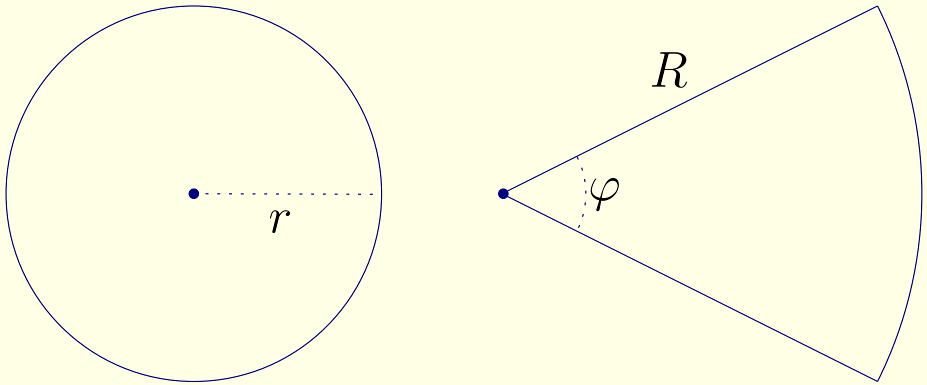
- Motivation
- Orientation Problem
  - In the Line.
  - In the Plane.
    - * Complexity.
    - * Optimal Range Orientation.
    - * Approximation Range Orientation.
  - In the Space.
- Variations of the Antenna Orientation Problem.

Motivation

Motivation

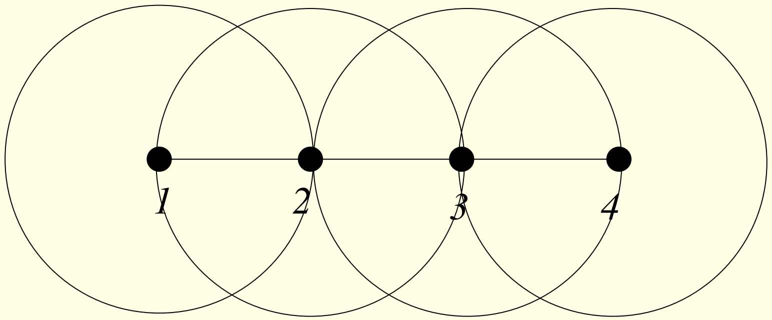
The energy necessary to transmit a message is proportional to the coverage area.

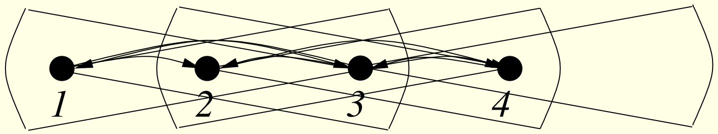
- An omnidirectional antenna with range r consumes energy proportional to πr^2.
- A directional antenna with angle φ and range R consumes energy proportional to $\varphi R^2/2$.



Motivation

With the same amount of energy, a directional antenna with angle α can reach further.





Capacity of Wireless Networks

Consider a set of sensors that transmit W bits per second with antenna sender of angle α and a receiver of angle β.

Assume that sensors are place in such a way that the interference is minimum.

Further, assume that traffic patterns and transmission ranges are optimally chosen.

Then the network capacity (amount of traffic that the network can handle) is at most $\sqrt{\frac{2\pi}{\alpha\beta}}W\sqrt{n}$ per second.

Capacity of Wireless Networks

		Receiver	
		Omnidirectional	Directional (β)
Sender	Omnidirectional	$\sqrt{\frac{1}{2\pi}}W\sqrt{n}$ [1]	-
	Directional (α)	$\sqrt{\frac{1}{\alpha}}W\sqrt{n}$ [2]	$\sqrt{\frac{2\pi}{\alpha\beta}}W\sqrt{n}$ [2]

1. *Gupta and Kumar*. The capacity of wireless networks. 2000.

2. *Yi,Pei and Kalyanaraman*. On the capacity improvement of ad hoc wireless networks using directional antennas. 2003.

Security Enhance with Directional Antennae

The use of directional antennae enhance the network security since the radiation is more restrict.

Hu and Evans[1] designed several authentication protocols based on directional antennae.

Lu et al[2] employed the average probability of detection to estimate the overall security benefit level of directional transmission over the omnidirectional one.

In [3] examined the possibility of key agreement using variable directional antennae.

[1] *Hu and Evans*. Using directional antennas to prevent wormhole attacks. 2004

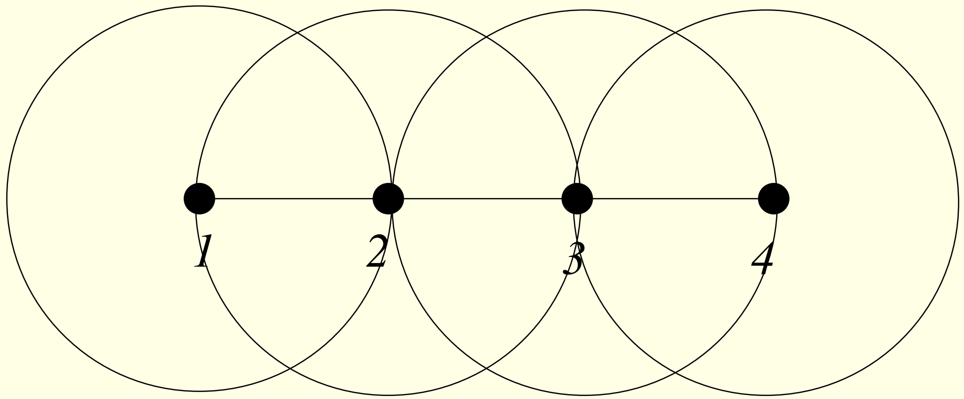
[2] *Lu, Wicker, Lio, and Towsley*. Security Estimation Model with Directional Antennas. 2008

[3] *Imai, Kobara, and Morozov*. On the possibility of key agreement using variable directional antenna. 2006

Antenna Orientation Problem in the Line

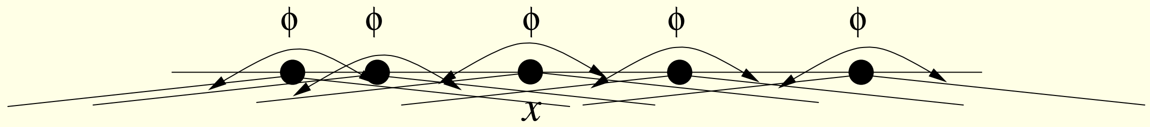
Antenna Orientation Problem in the Line

Given a set of sensors in the line equipped with one directional antennae each of angle at most $\varphi \geq 0$. Compute the minimum range r required to form a strongly connected network by appropriately rotating the antennae.

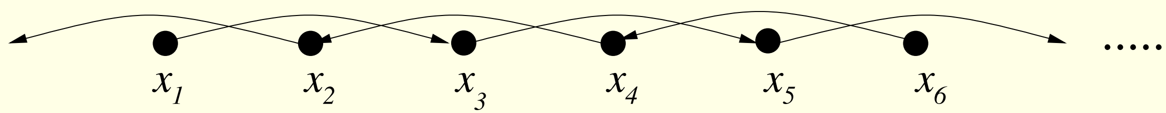


Antenna Orientation Problem in the Line

Given $\varphi \geq \pi$. The strong orientation can be done trivially with the same range required when omnidirectional antennae are used.



Given $\varphi < \pi$. The strong orientation can be done with range bounded by two times the range required when omnidirectional antennae are used.

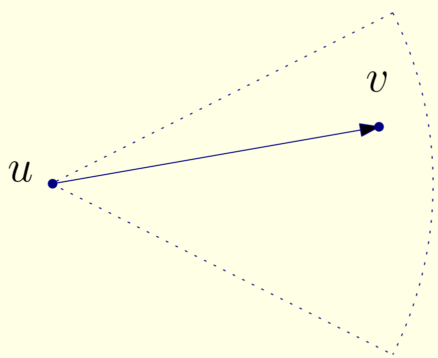


Antenna Orientation Problem in the Plane

Antenna Orientation Problem

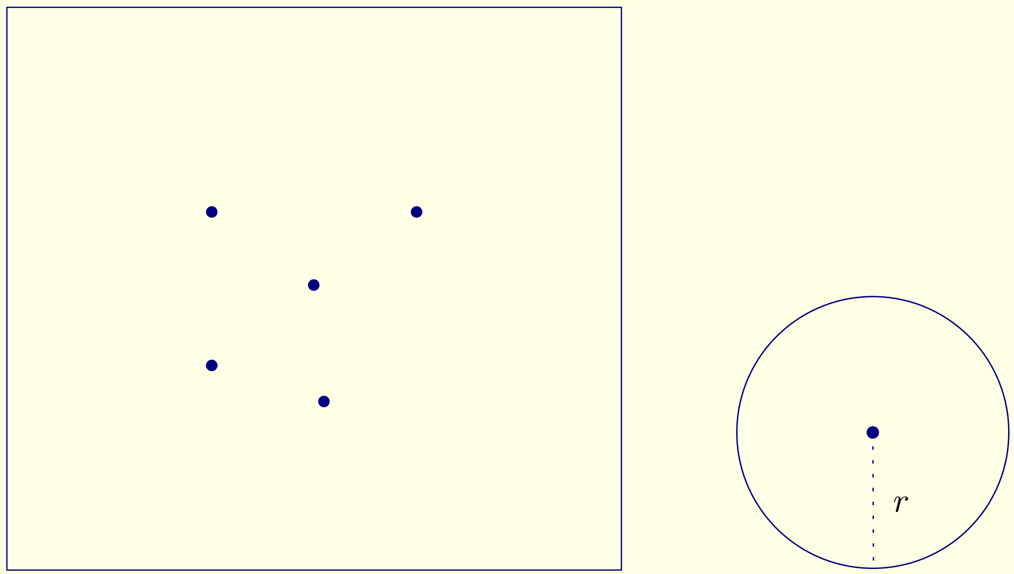
Given a set of sensors in the plane equipped with one directional antennae each of angle at most φ.

Compute the minimum range such that by appropriately rotating the antennae, a directed, strongly connected network on S is formed.



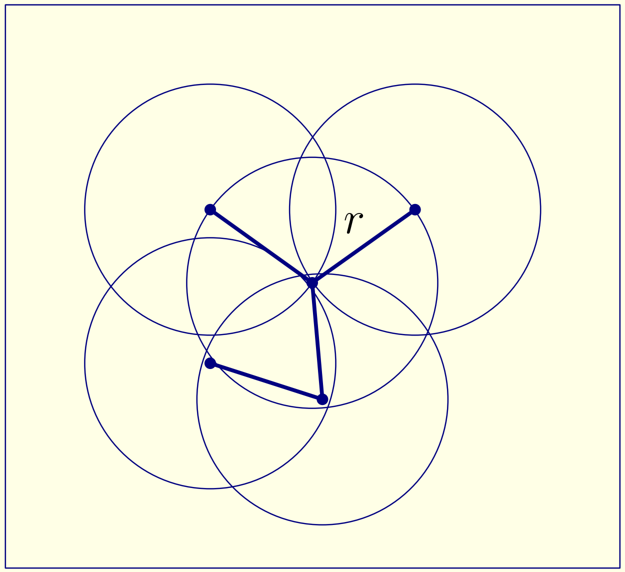
Antenna Orientation Problem

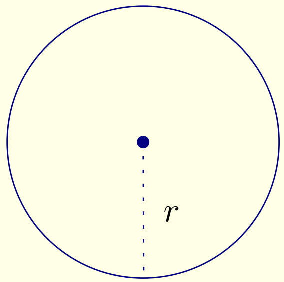
Given n sensors in the plane with omnidirectional antennae, the optimal range can be compute in polynomial time.



Antenna Orientation Problem

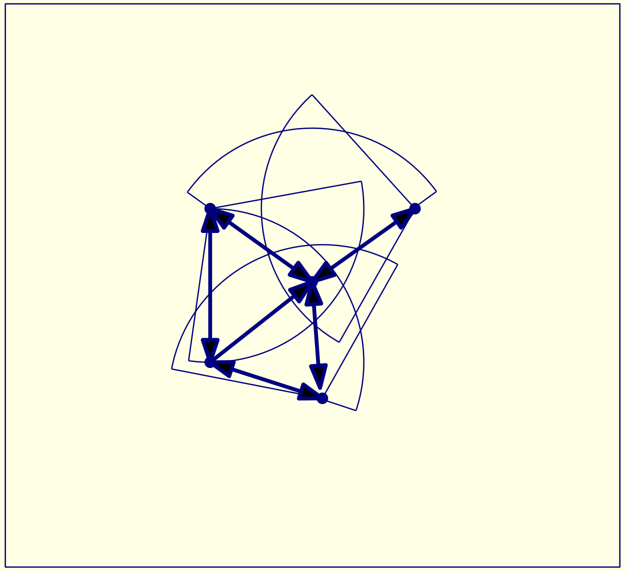
They create an omnidirectional network. Actually, the longest edge of the MST is the optimal range.

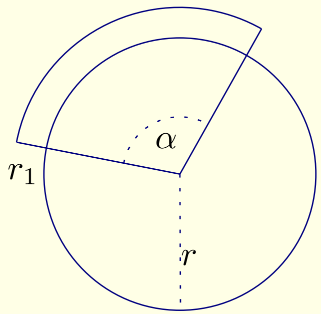




Antenna Orientation Problem

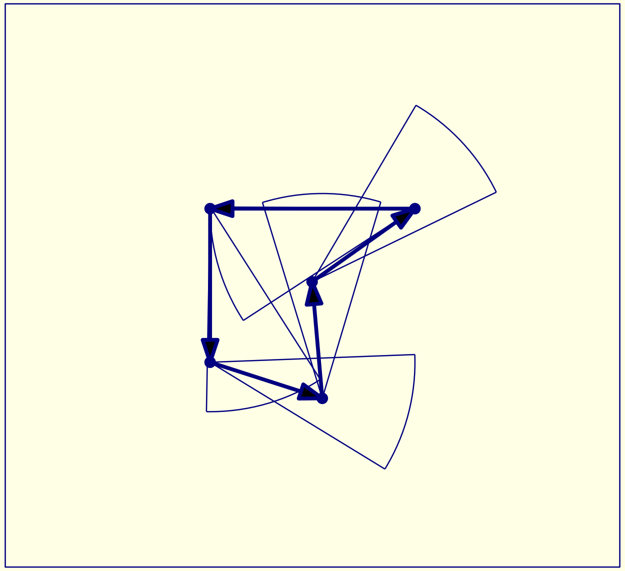
Given a directional antenna with angle α. What is the minimum radius r_1 to create a strongly connected network?

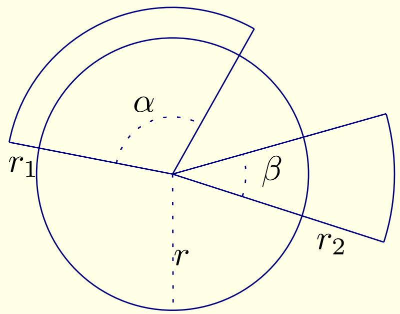




Antenna Orientation Problem

Given a directional antenna with angle β. What is the minimum radius r_2 to create a strongly connected network?





Antenna Orientation Problem

When the angle is small, the problem is equivalent to the bottleneck traveling salesman problem or Hamiltonian cycles that minimizes the longest edge. A 2-approximation is given by Parker and Rardin[4].

For which angles the two problems are equivalent?

By reduction to the problem of finding Hamiltonian circuit in bipartite planar graphs of degree three[5], it can be proved that the problem is NP-Complete when the angle is less than $\pi/2$ and a approximation range less than $\sqrt{2}$ times the optimal range.

[4] *Parker and Rardin.* Guaranteed performance heuristics for the bottleneck traveling salesman problem. 1984

[5] *tai, Papadimitriou, and Szwarcfiter.* Hamilton Paths in Grid Graphs. 1982

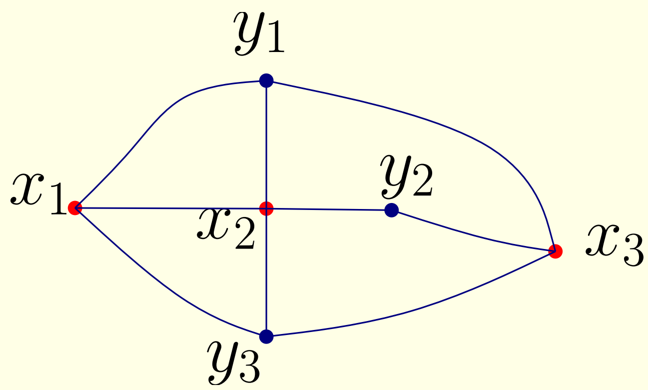
Computational Complexity

Theorem 1 (Caragiannis et al[6].) Decide whether there exists an orientation of one antenna at each sensor with angle less that $2\pi/3$ and optimal range is NP-Complete. The problem remains NP-complete even for the approximation range less than $\sqrt{3}$ times the optimal range.

[6] *Caragiannis, Kaklamanis,Kranakis, Krizanc and Wiese.* Communication in Wireless Networks with Directional Antennae. 2008

Proof (1/6)

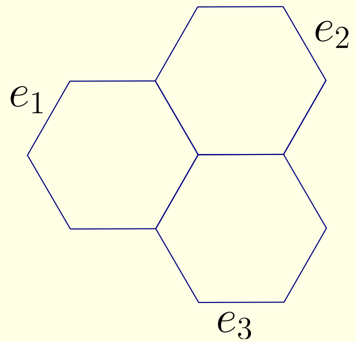
By reduction to the problem of finding Hamiltonian circuit in bipartite planar graphs of maximum degree three [7]. Take a valid instance of a bipartite planar graph $G = (V_0 \cup V_1, E)$.

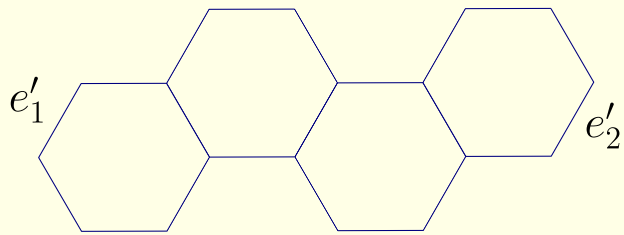


[7] *tai, Papadimitriou, and Szwarcfiter*. Hamilton Paths in Grid Graphs. 1982

Proof (2/6)

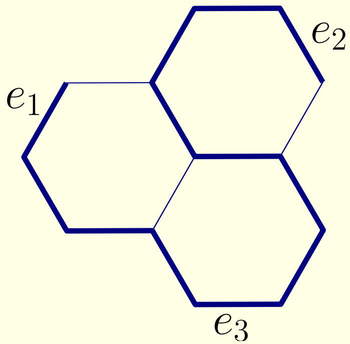
Replace every vertex by three hexagons and every edge by a necklace (path of hexagons).

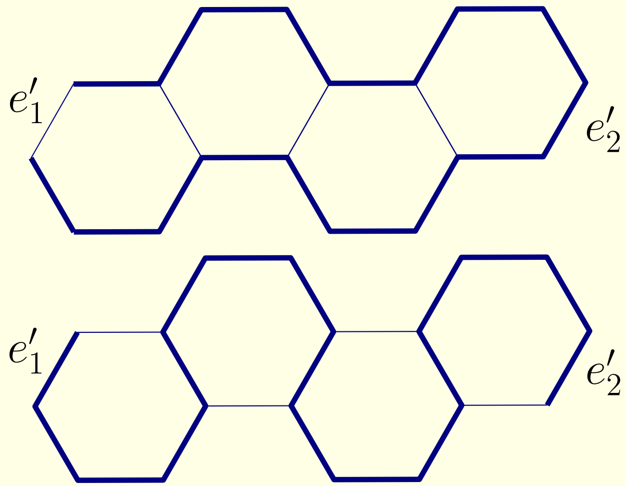




Proof (3/6)

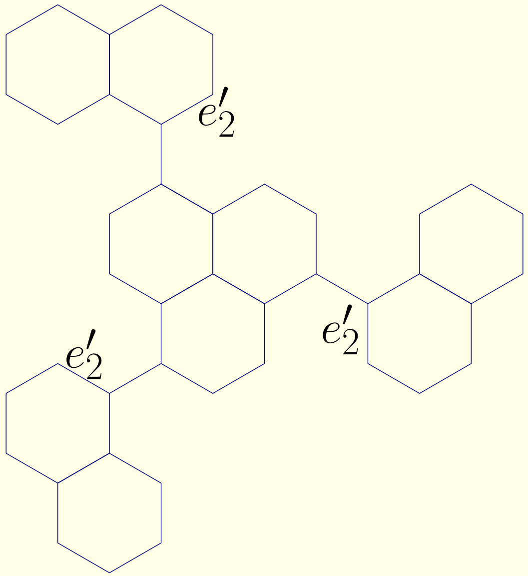
Hexagons and necklaces have the following Hamiltonian paths.





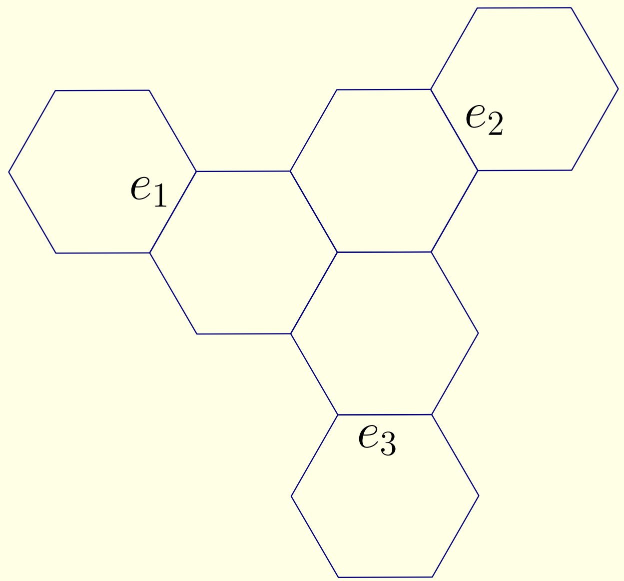
Proof (4/6)

Edges are connected to a red vertex using a special gadget.



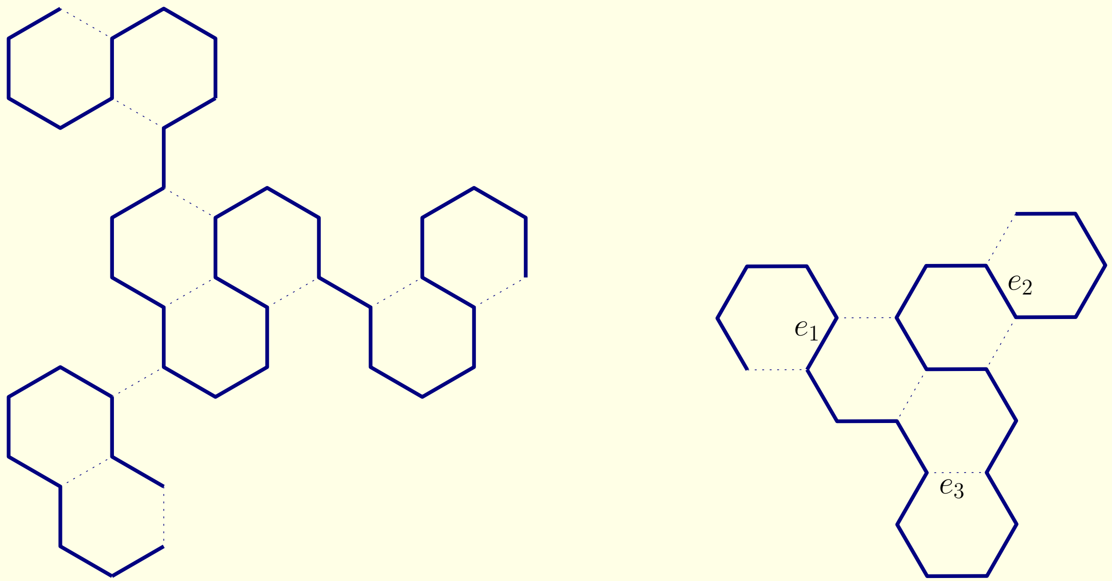
Proof (5/6)

Edges are connected to a black vertex as follows.



Proof (6/6)

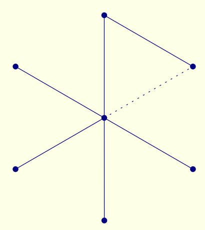
An orientation of one antenna implies a Hamiltonian cycle.



Optimal Range Orientation

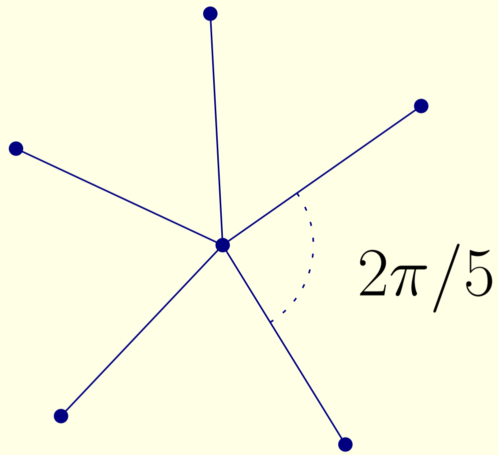
On the opposite side, what is the minimum angle necessary to create a strongly connected network if the range of the directional antennae is the same as the omnidirectional antenna?

Consider an MST T on the set of points. If the maximum degree of T is 6, by a simple argument we can find an MST with the same weigth and maximum degree 5.



Optimal Range Orientation

Therefore, by the pigeon hole principle, there exist two vertices that form an angle with their parent of at least $2\pi/5$.

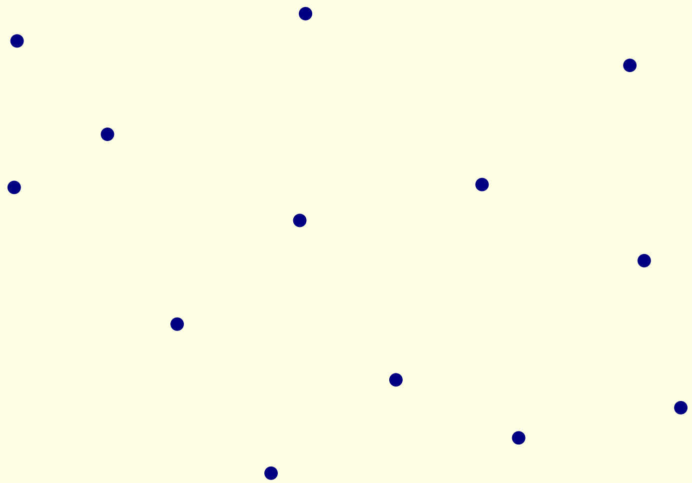


Optimal Range Orientation

Theorem 2. There exists an orientation of the directional antennae with optimal range when the angles of the antennae are at least $8\pi/5$.

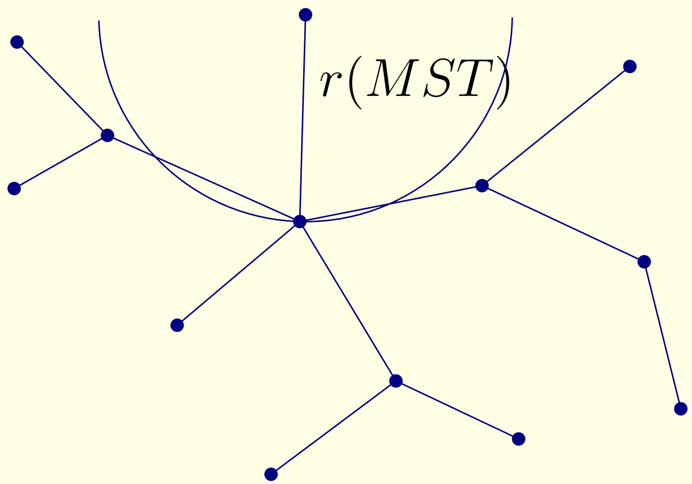
Antenna Orientation With Approximation Range

Theorem 3. Given an angle φ with $\pi \leq \varphi < 8\pi/5$ and a set of points in the plane, there exists a polynomial algorithm that computes a strong orientation with radius bounded by $2\sin(\varphi/2)$ times the optimal range.



Proof (1/10)

Consider a Minimum Spanning Tree on the Set of Points.



Proof (2/10)

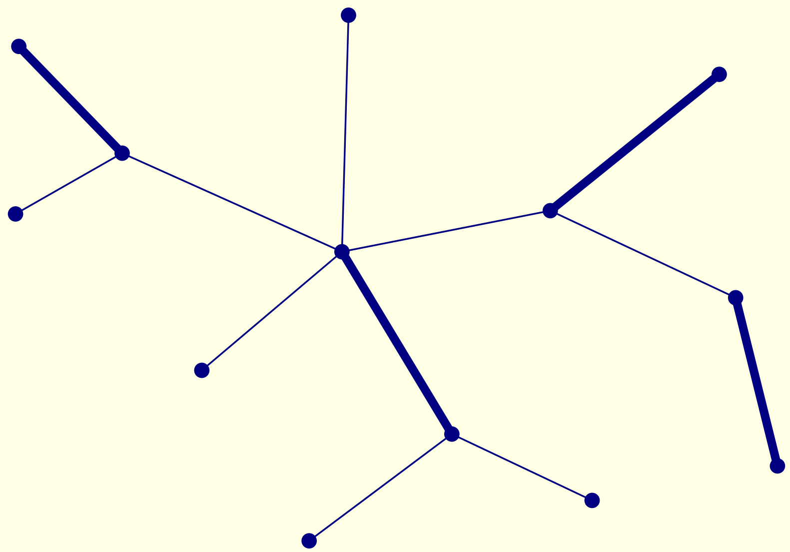
Let $r^*(\varphi)$ be the optimal range when the sector of angle is φ.

Let $r(MST)$ be the longest edge of the MST on the set of points.

Observe that for $\varphi \geq 0$, $r^*(\varphi) \geq r(MST)$.

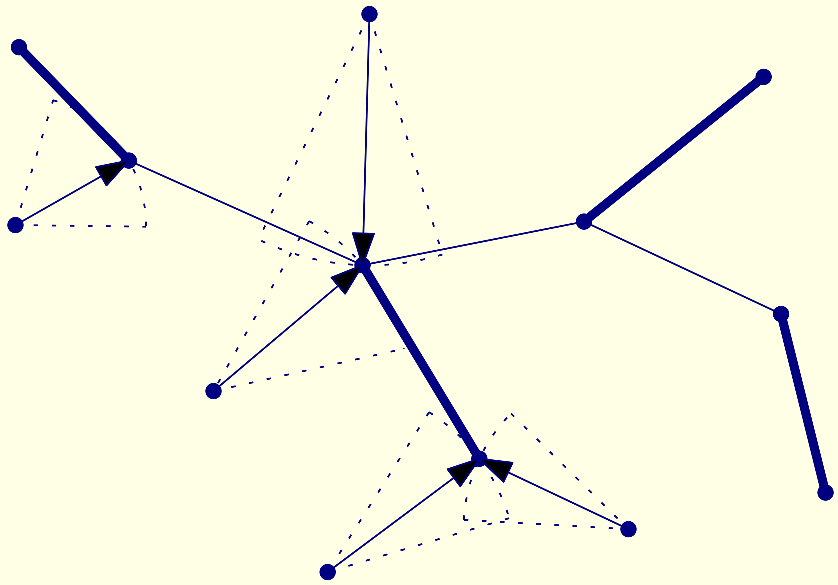
Proof (3/10)

Find a maximal matching such that each internal vertex is in the matching. This can be done by traversing T in BFS order.



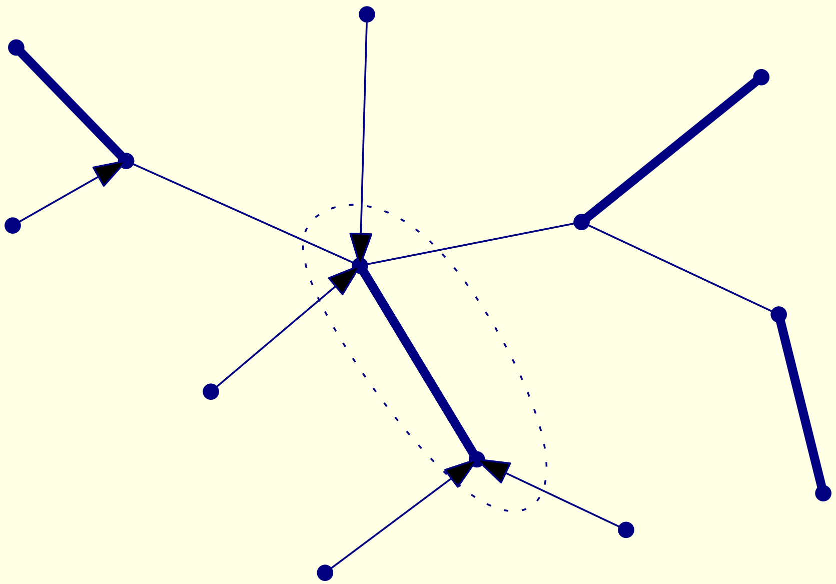
Proof (5/10)

Orient unmatched leaves to they immediate neighbors.



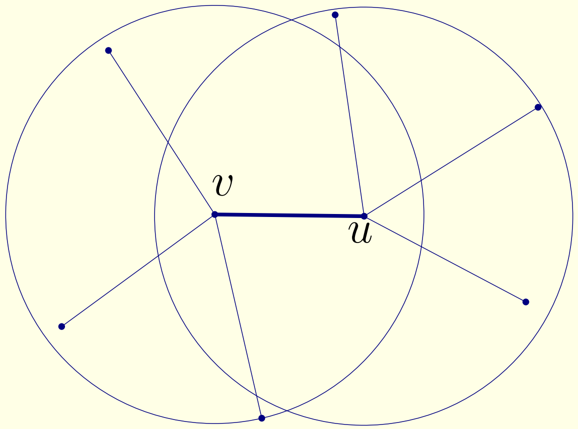
Proof (6/10)

Consider a pair of matched vertices



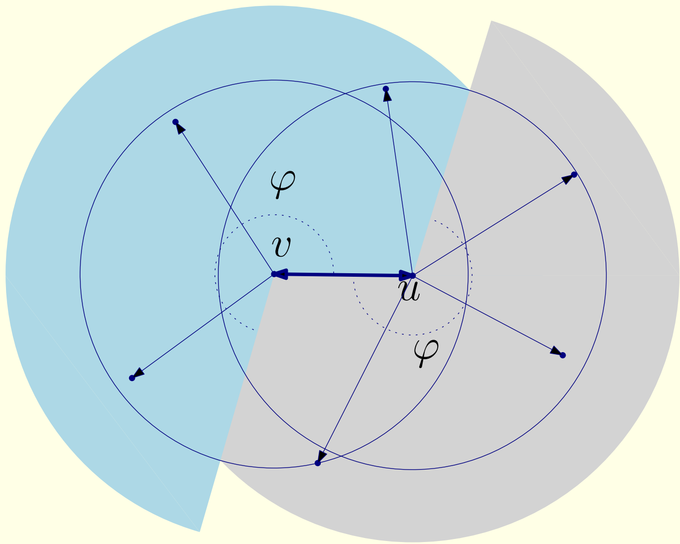
Proof (7/10)

Let $\{u, v\}$ be an edge in the matching. Consider the smallest disks of same radius centered at u and v that contain all the neighbors of u and v in the MST respectively.



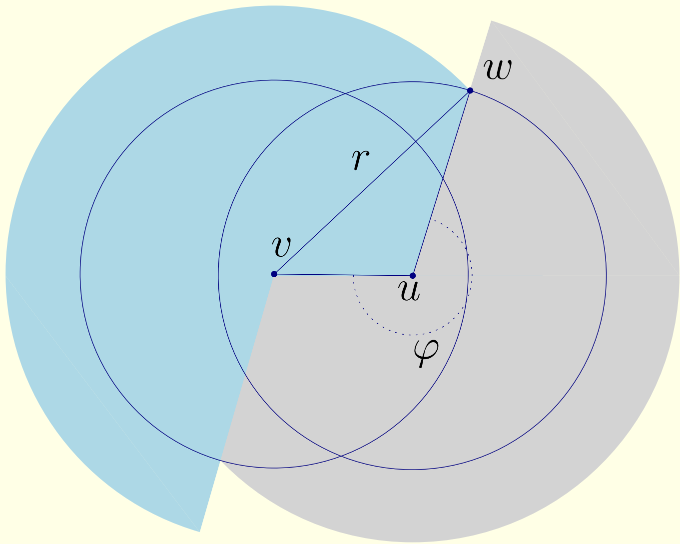
Proof (8/10)

Orient the directional antennae at u and v with angle φ in such a way that both disks are covered.



Proof (9/10)

To calculate the smallest radius necessary to cover both disk. Consider the triangle uvw.



Proof (10/10)

From the Law of Cosine we can determine r.

$$
\begin{aligned}
r &= \sqrt{|uv|^2 + |uw|^2 - 2|uv||uw|\cos(2\pi - \varphi)} \\
&\leq \sqrt{2 - 2\cos(2\pi - \varphi)} \\
&\leq 2\sin(\tfrac{2\pi - \varphi}{2}) \\
&\leq 2\sin(\pi - \varphi/2) \\
&\leq 2\sin(\varphi/2)
\end{aligned}
$$

Antenna Orientation Problem in the Space

Sensors in the Space

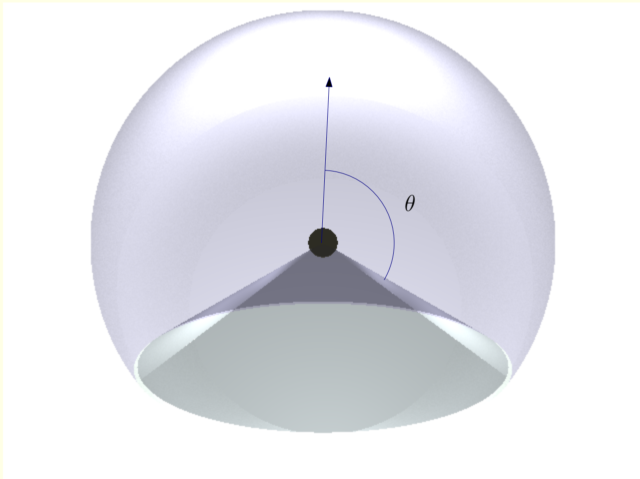
Due to the fact that sensors may lay in distinct altitudes, the previous algorithms do not work correctly in the space.

We model an antenna in 3D space with solid angle Ω as a spherical sector of radius one.

An omnidirectional antenna has solid angle 4π.

Sensors in the Space

The apex angle of a spherical sector is the maximum planar angle between any two generatrices of the spherical sector. Thus, $\Omega = 2\pi(1 - \cos\theta)$



Complexity of the Antenna Orientation Problem in the Space

Theorem 4. Decide whether there exists a strong orientation when each sensor has one directional antenna with solid angle less than π and optimal range is NP-Complete.

Proof

Consider a set S of n points in the plane.

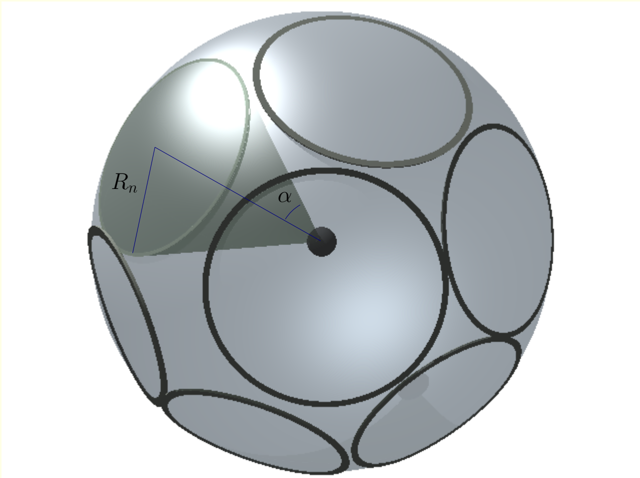
From Archimedes relation, any plane that cuts the coverage area of any 3D directional antennae through the apex with angle Ω has plane angle that satisfies $\cos(\theta) \leq 1 - \frac{\Omega}{2\pi}$.

Therefore $\theta < 2\pi/3$ if and only if $\Omega < \pi$.

A strong orientation of the directional antennae with angle less than $2\pi/3$ in 2D implies a strong orientation of directional antennae with angle less than π in 3D. The opposite is also true.

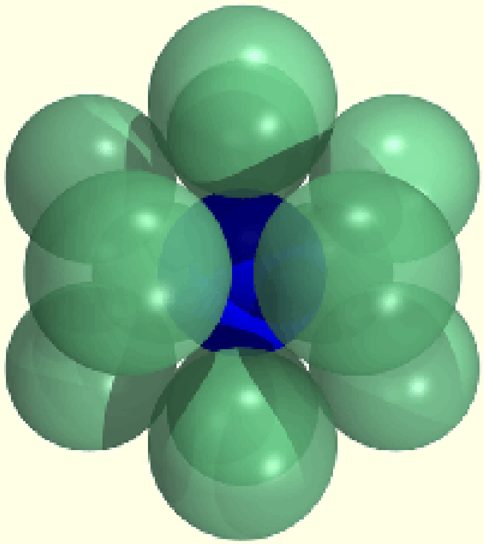
Tammes' Radius

The Tammes radius is the maximum radius of n equal non-overlapping circles on the surface of the sphere. We denote it by R_n.



Kissing Number and Tammes' Radius

The Kissing number is the number of balls of equal radius that can touch an equivalent ball without any intersection,



Kissing Number and Tammes' Radius

In particular, the Tammes' Radius is equivalent to the kissing number when all the balls have the same radius.

The maximum degree of an MST is equal to the kissing number. In 3D it is 12.

Optimal Range Orientation in the Space

Theorem 5. There exists an orientation of the directional antennae in 3D with optimal range when the solid angles of the antennae are at least $18\pi/5$.

Proof (1/4)

Let T be an MST on the points.

Let B_p be the sphere centered at p of minimum radius that covers all the neighbors of p in T.

For each neighbor u of p in T, let u' be the intersection point of B_p with the ray emanating from p toward u

Proof (2/4)

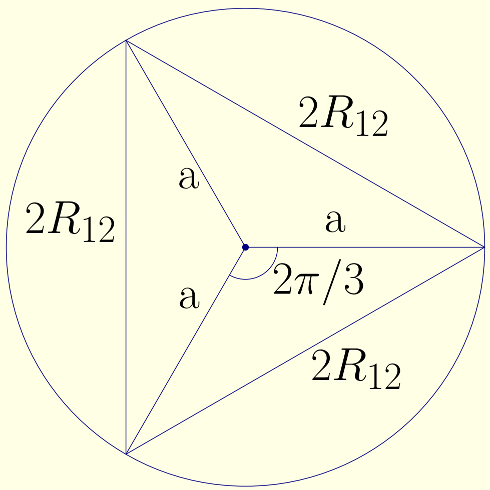
Thus, we have an unit sphere with at most 12 points.

Compute the Delauney Triangulation on the points of the sphere.

Orient the antenna in opposite direction of the center of largest triangle.

Proof (3/4)

Observe that every edge of the DT has length at least twice the Tammes' Radius $R_{12} = \sin \frac{63^o 26}{2}$. Thus, every triangle is greater than the equilateral triangle of side $2R_{12}$.



Proof (4/4)

$$
\begin{aligned}
a &= R_{12}/\sqrt{3} \\
\alpha &\leq arcsin(a) \\
\Omega &\geq 4\pi - 2\pi(1 - \cos(\alpha)) \\
&= 2\pi(1 + \cos(\alpha)) \\
&= 2\pi\left(1 + \cos\left(\arcsin\left(\frac{2R_{12}}{\sqrt{3}}\right)\right)\right) \\
&\geq \frac{18\pi}{5},
\end{aligned}
$$

Antenna Orientation With Approximation Range

Theorem 6. Given a solid angle φ with $2\pi \leq \varphi < 18\pi/5$ and a set of points in the space, there exists a polynomial algorithm that computes a strong orientation with radius bounded by $\frac{\sqrt{\Omega(4\pi-\Omega)}}{\pi}$ times the optimal range.

Proof (1/3)

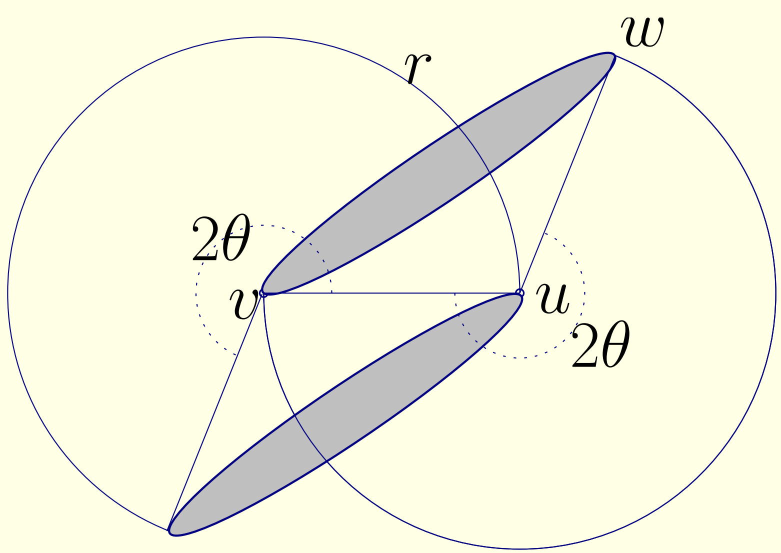
Let T be the MST on the set of points.

Consider a maximal matching such that each internal vertex is matched.

Orient unmatched leaves to their immediate neighbors.

Proof (2/3)

For every two matched vertices, orient the directional antennae in such a way that the uncover vertices of u are covered by v.



Proof (3/3)

Consider the uncovered cone. From the law of cosine we can determine r.

Let θ be the apex angle of Ω.

$$
\begin{aligned}
r &= \sqrt{|uv|^2 + |uw|^2 - 2|uv||uw|\cos(2\theta)} \\
&\leq \sqrt{2 - 2\cos(2\theta)} \\
&= 2\sin(\theta) \\
&= 2\sqrt{1 - \cos^2(\theta)} \\
&= 2\sqrt{1 - (1 - \frac{\Omega}{2\pi})^2} \\
&= \frac{\sqrt{\Omega(4\pi - \Omega)}}{\pi}
\end{aligned}
$$

Summary of the Antenna Orientation Problem

2D		3D	
Angle	Range	Solid Angle	Range
$\varphi < \frac{2\pi}{3}$	NP-C	$\Omega < \pi$	NP-C
$\frac{2\pi}{3} \leq \varphi < \pi$	Open	$\pi \leq \Omega < 2\pi$	Open
$\pi \leq \varphi < \frac{8\pi}{5}$	$2\sin(\varphi/2)$	$2\pi \leq \varphi < \frac{18\pi}{5}$	$\frac{\sqrt{\Omega(4\pi-\Omega)}}{\pi}$
$\varphi \geq \frac{8\pi}{5}$	1	$\Omega \geq \frac{18\pi}{5}$	1

Variations of the Antenna Orientation Problem

Variations of the Antenna Orientation Problem

Assume that each sensor has $k > 1$ directional antennae such that the sum is at most φ. What is the minimum range necessary to create a strongly connected network by appropiatly rotating the antennae.

Given a connected network formed by a set of sensors with omnidirectional antennae and an angle $\varphi \geq 0$. Compute the minimum number of arcs in the network in such a way that the resulting network is strongly connected and the stretch factor does not depend on the size of the network.

Thanks