Local Algorithms: Past, Present, Future

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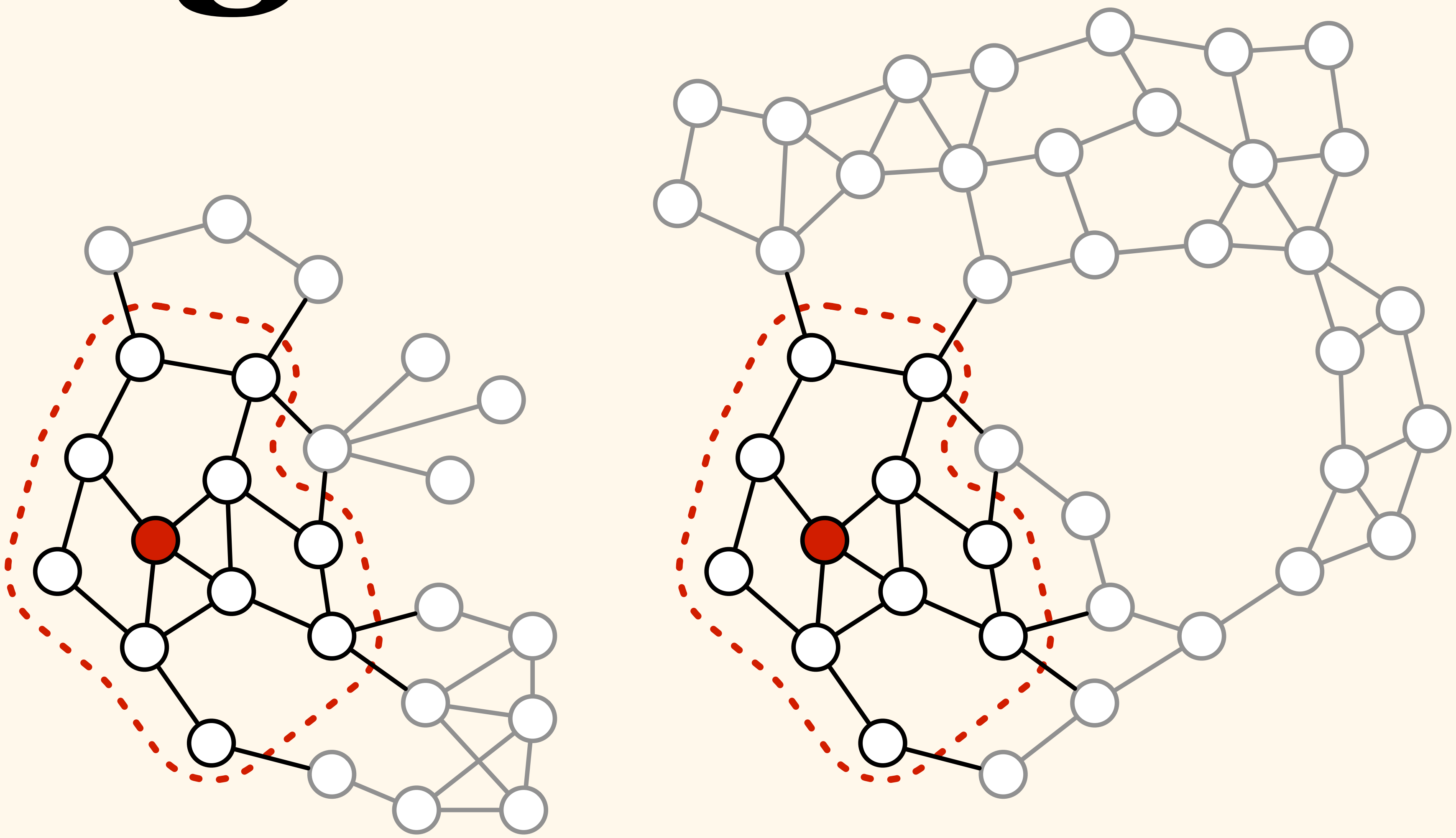
About This Tutorial

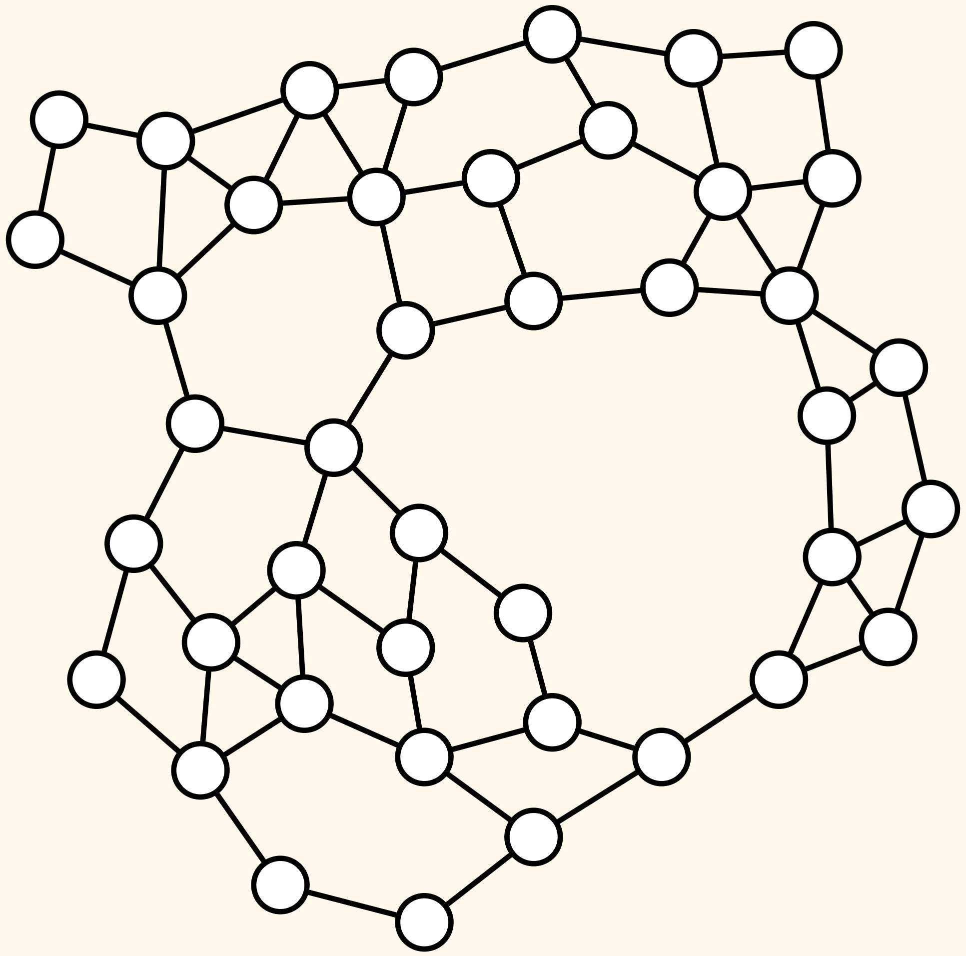
- Two parts:
  - Part A, Tuesday 11:00–12:30
  - Part B, Wednesday 11:00–12:30
- www.iki.fi/suo/tut
  - Slides, additional material, further reading

Background



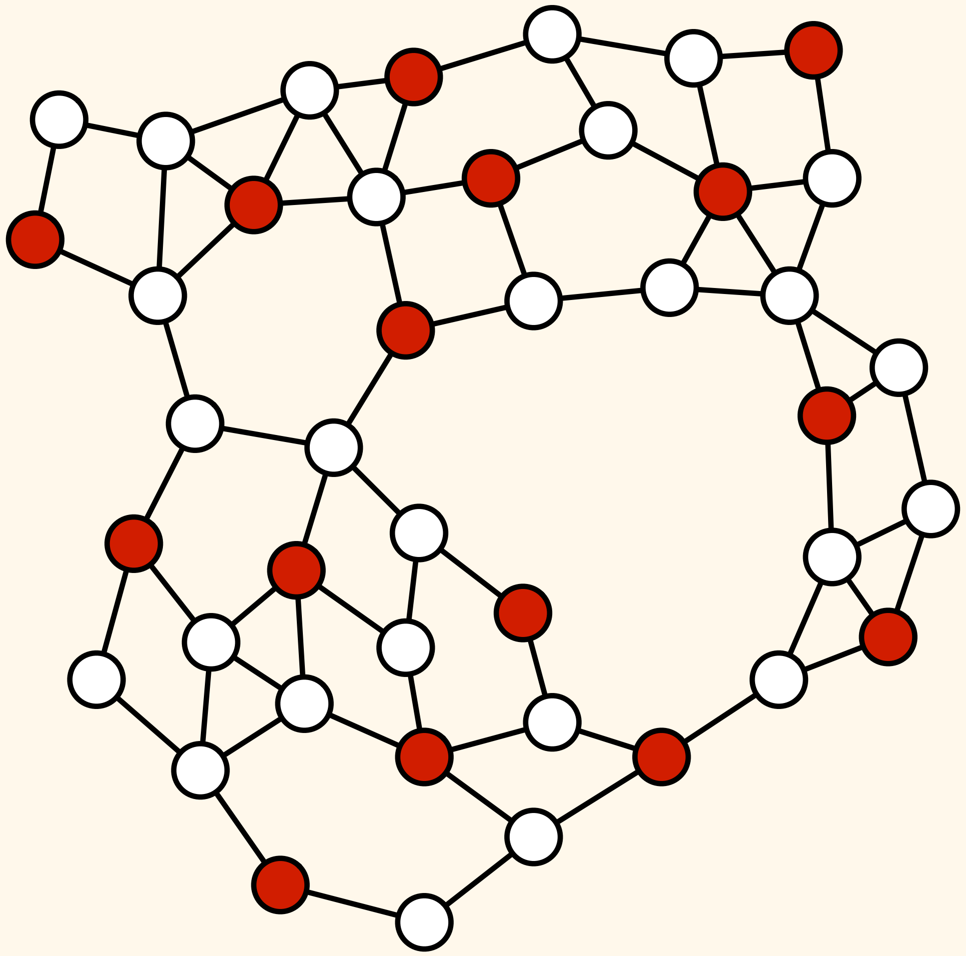
Setting

- Graphs



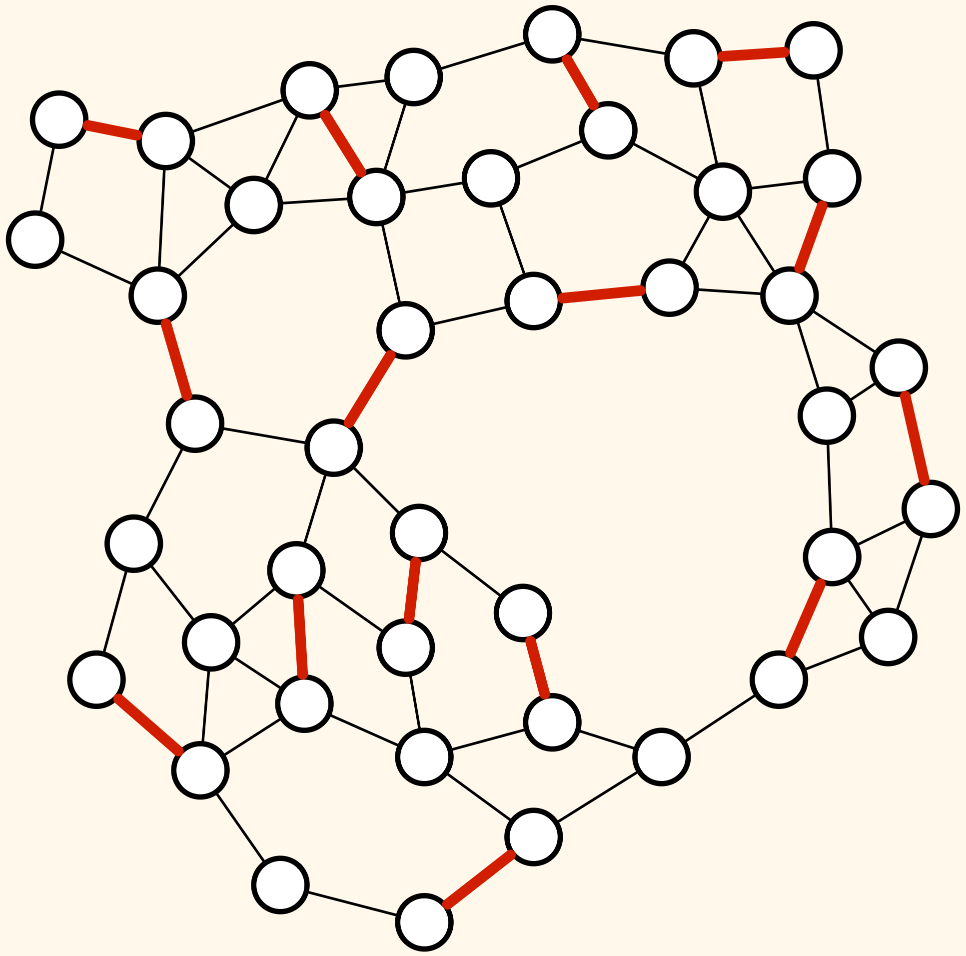
Setting

- Graphs
- Algorithms for graph problems
    - *Independent sets*



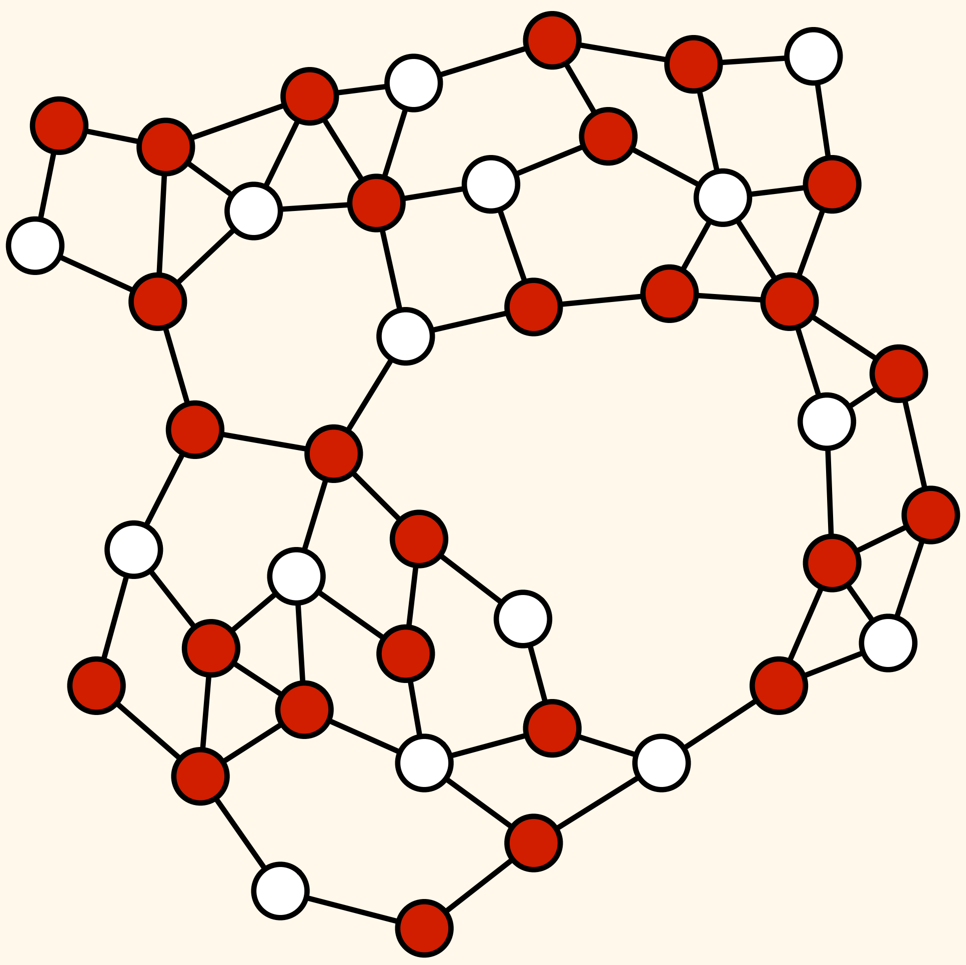
Setting

- Graphs
- Algorithms for graph problems
    - Independent sets, *matchings*



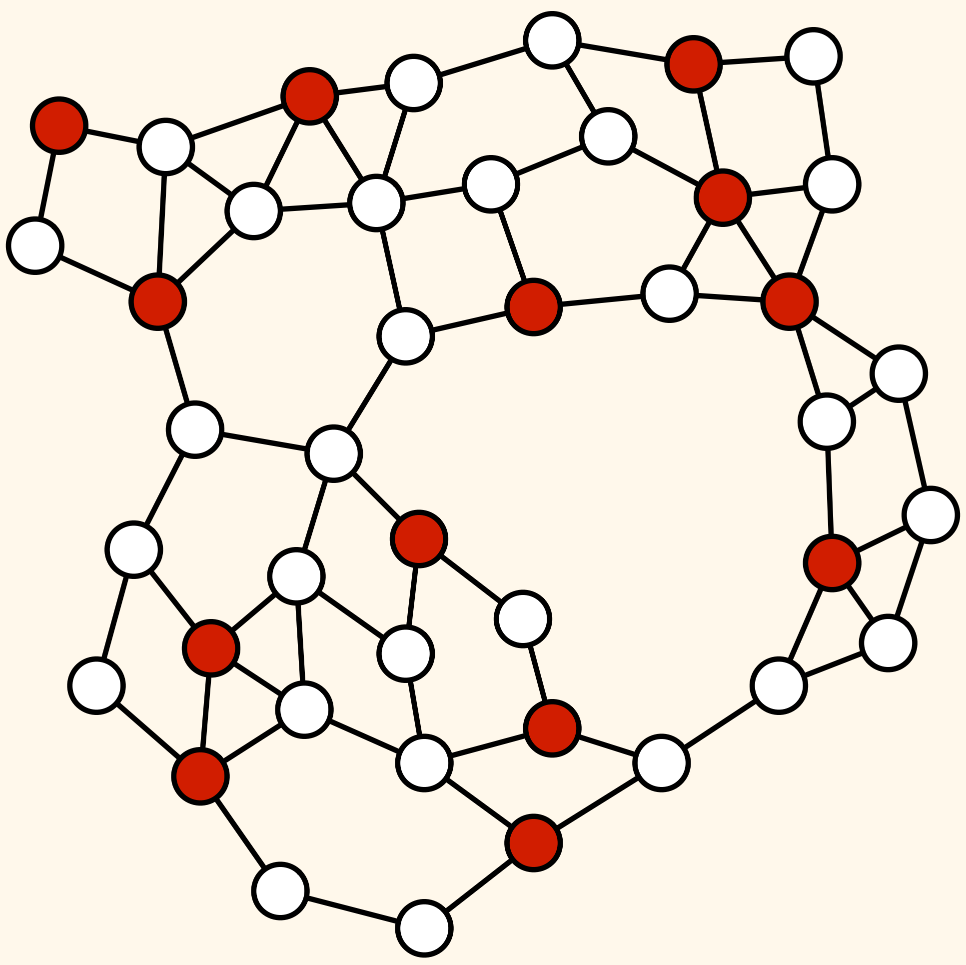
Setting

- Graphs
- Algorithms for graph problems
  - Independent sets, matchings, *vertex covers*



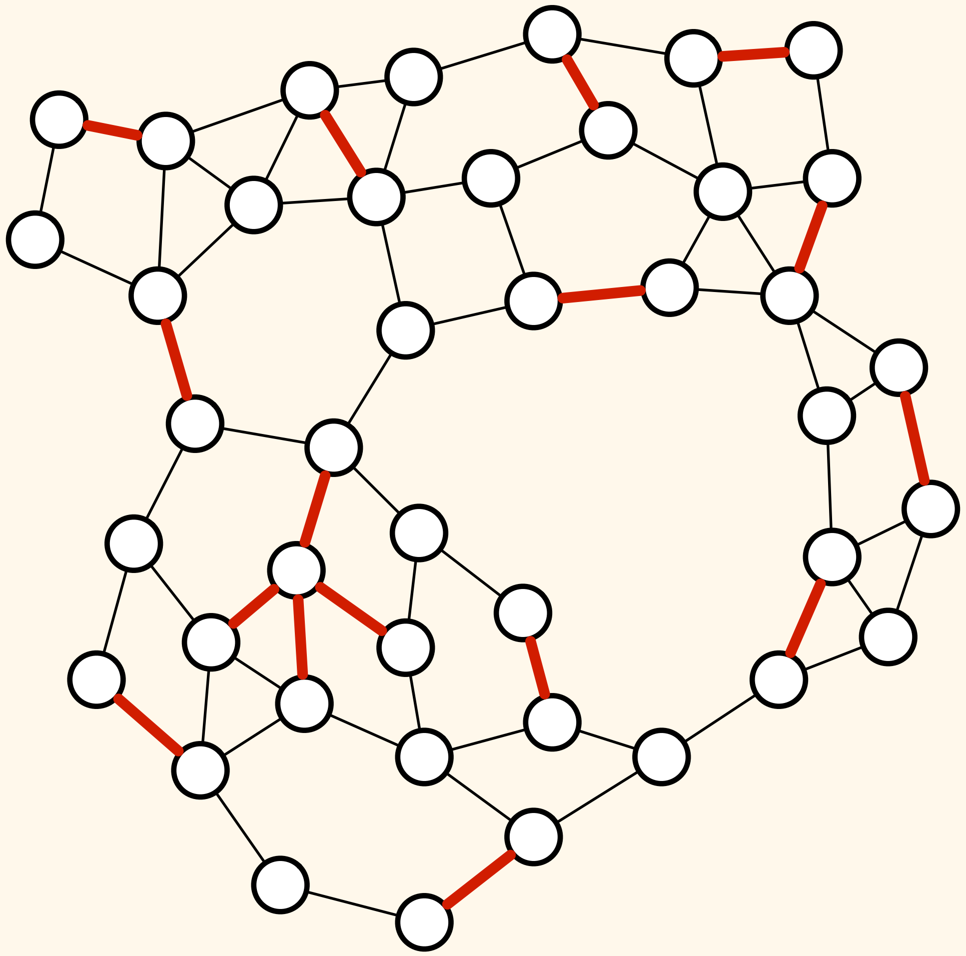
Setting

- Graphs
- Algorithms for graph problems
  - Independent sets, matchings, vertex covers, *dominating sets*



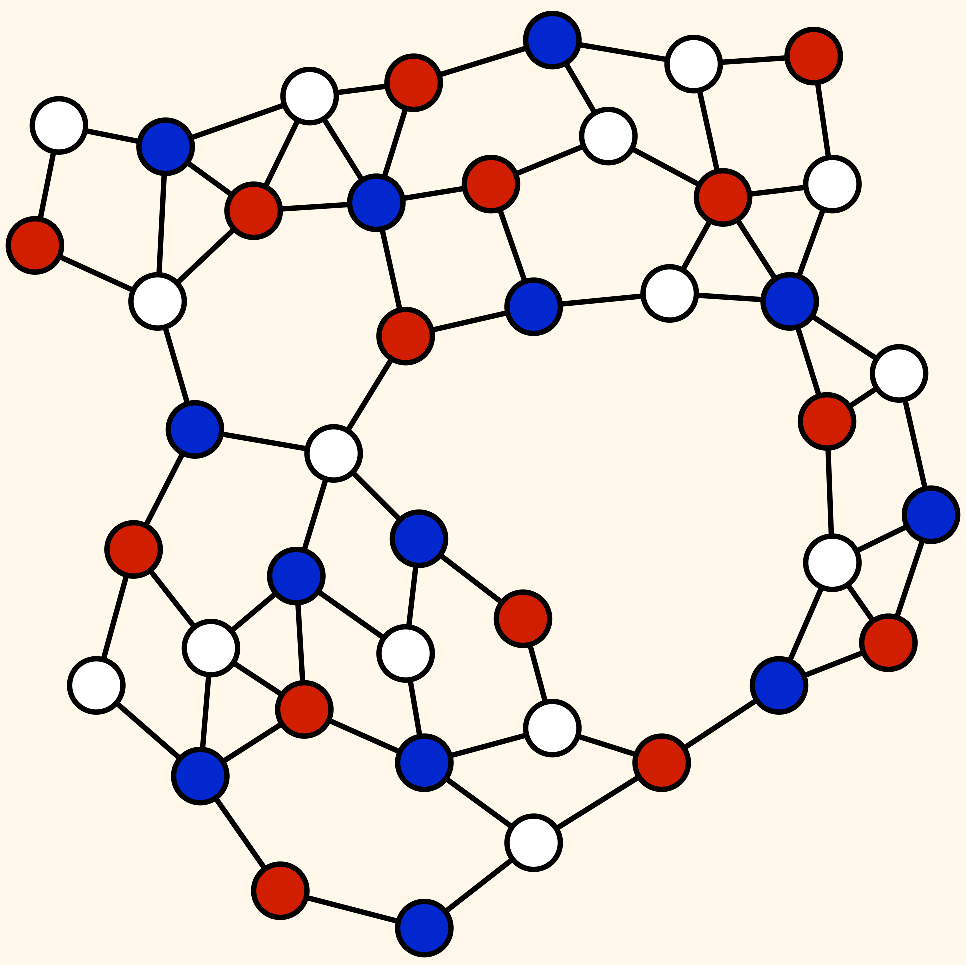
Setting

- Graphs
- Algorithms for graph problems
  - Independent sets, matchings, vertex covers, dominating sets, *edge dominating sets*



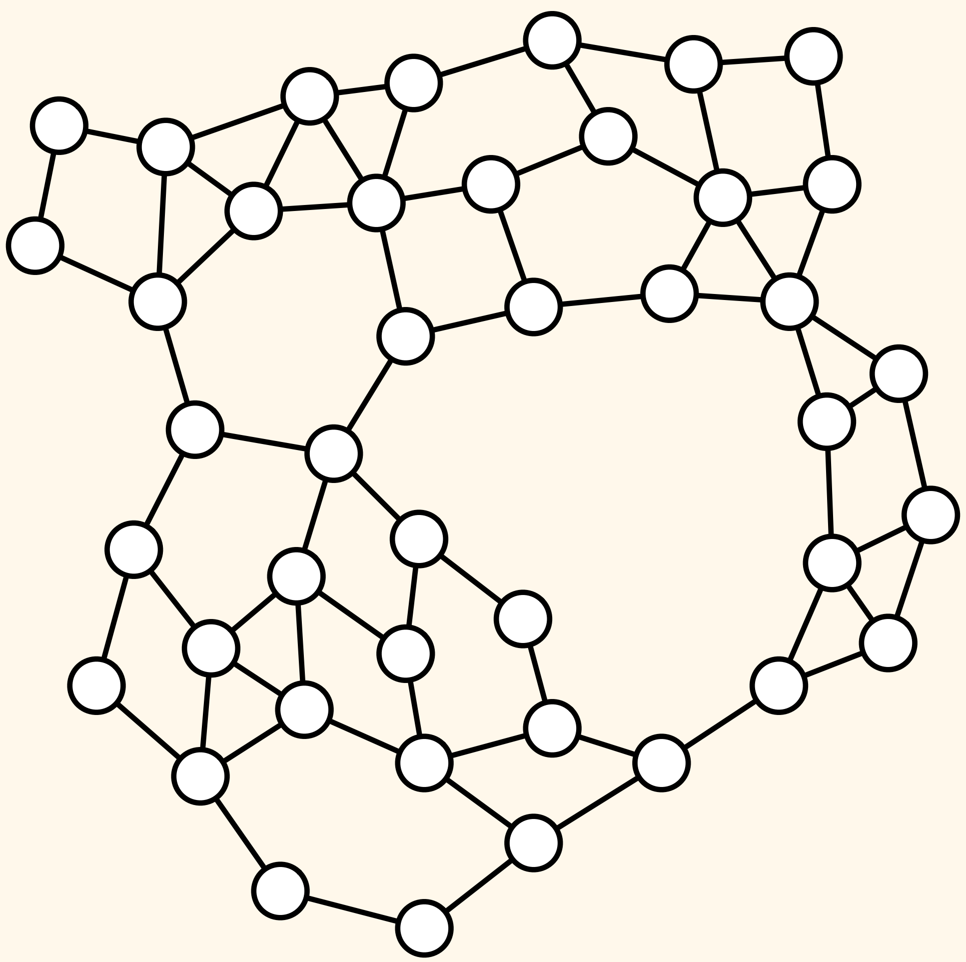
Setting

- Graphs
- Algorithms for graph problems
  - Independent sets, matchings, vertex covers, dominating sets, edge dominating sets, *graph colourings*



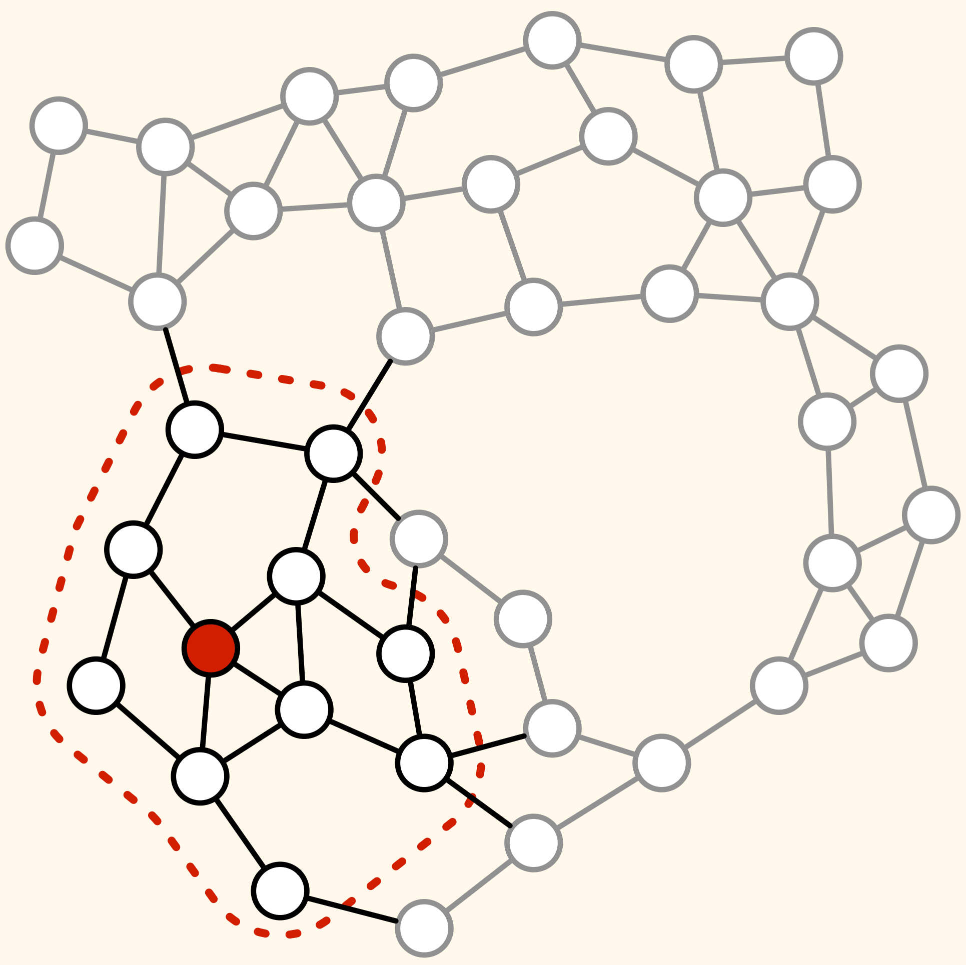
Setting

- Graphs
- Algorithms for graph problems
  - Independent sets, matchings, vertex covers, dominating sets, edge dominating sets, graph colourings, …



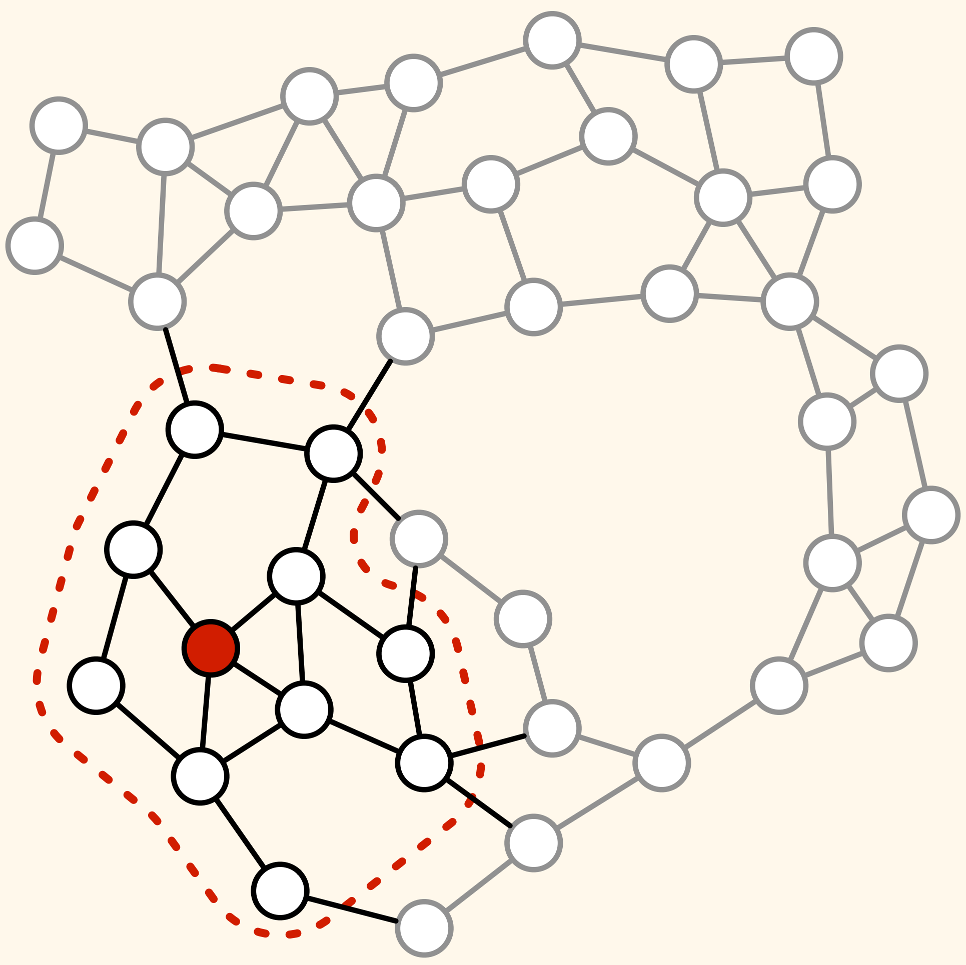
Local Algorithms

- *Local neighbourhood*: nodes at distance r
  - Here $r = O(1)$, independent of number of nodes
  - Shortest-path distance, number of edges



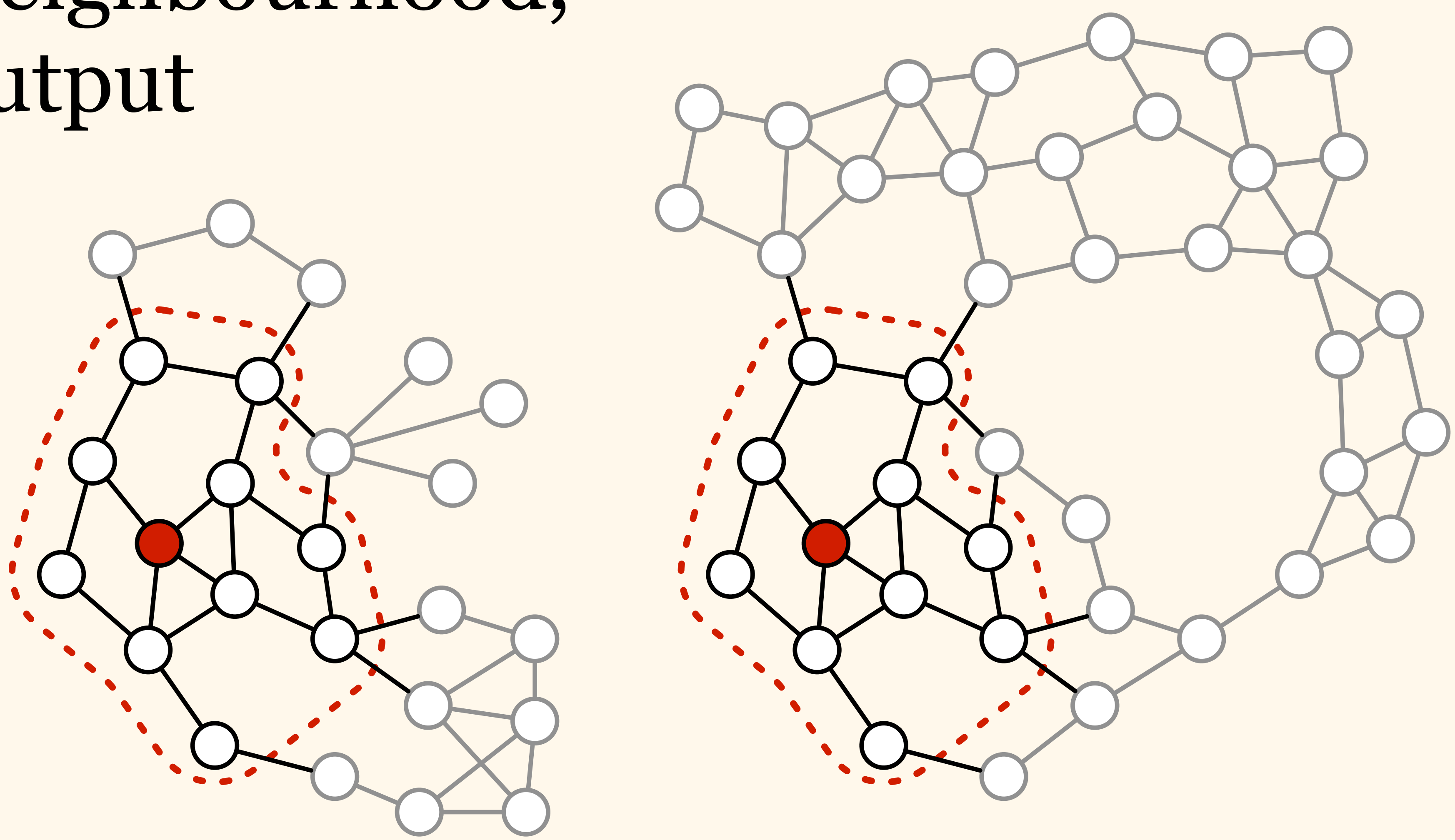
Local Algorithms

- *Local algorithm*:
  each node operates based on its local neighbourhood only
  - Output is a function of local neighbourhood



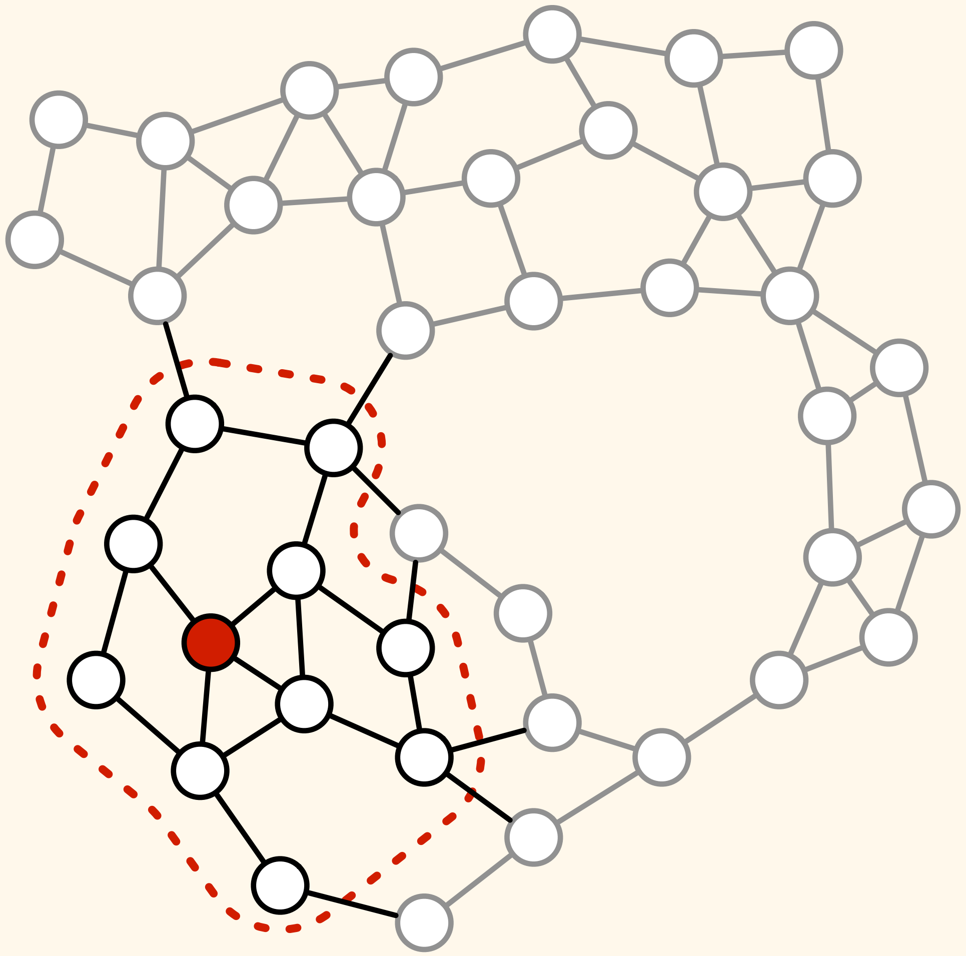
Local Algorithms

- Same neighbourhood, same output



Local Algorithms

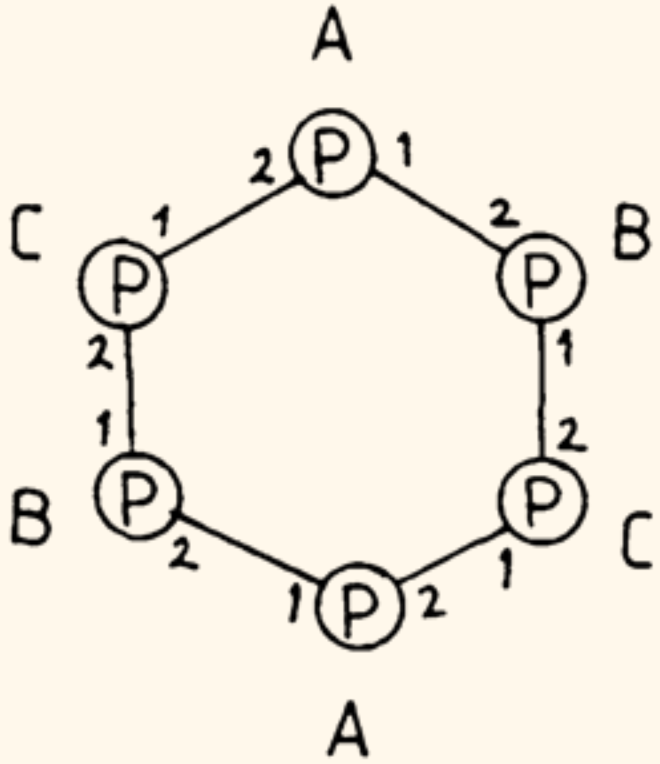
- Equivalently:
    - *Constant-time distributed algorithm*
    - Time = number of synchronous communication rounds



Advantages

- Fast and scalable distributed algorithm
  - By definition…
- Fault-tolerant and robust
  - Changes in input (or network structure): only *local changes in output*
  - We can quickly *recover from any failures*
- But do these exist?

Past

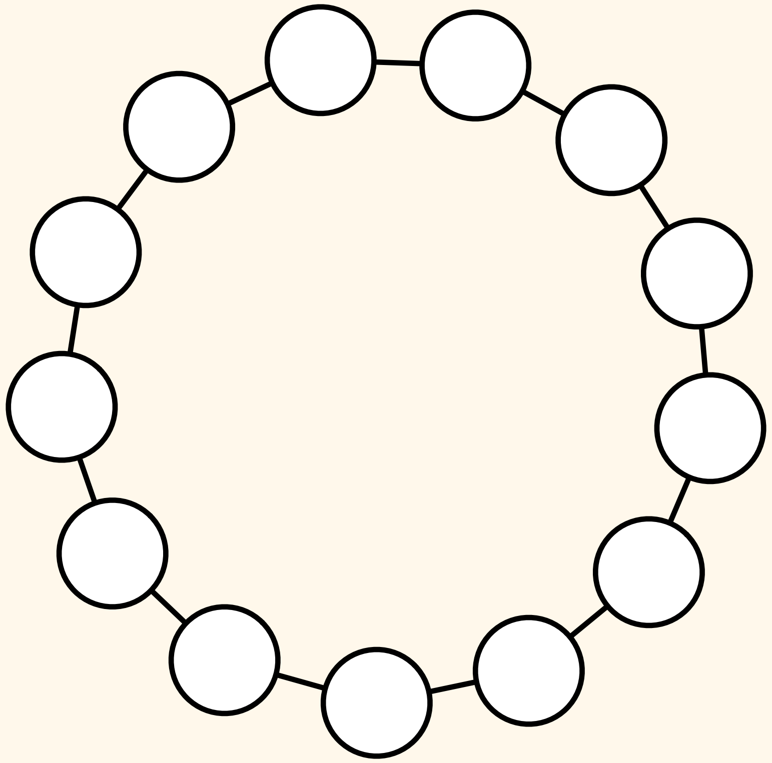


Bad News

- Long history of very strong negative results
  - *Linial* (1992)
  - *Naor & Stockmeyer* (1995)
  - *Czygrinow, Hańćkowiak & Wawrzyniak* (2008)
  - *Lenzen & Wattenhofer* (2008)
  - using, e.g., results that date back to *Ramsey* (1930)

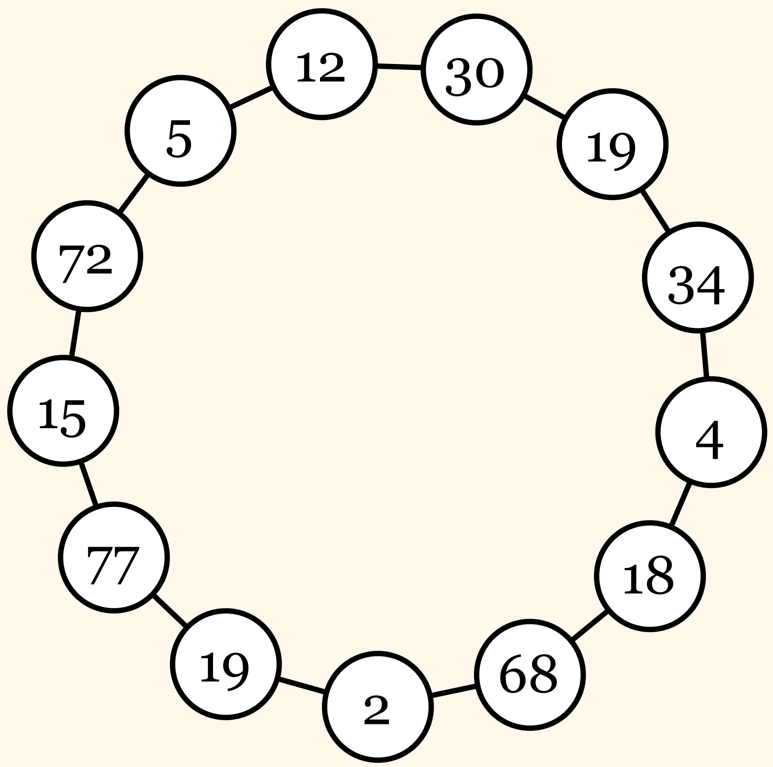
Bad News

- Even if your graph is a *cycle*...



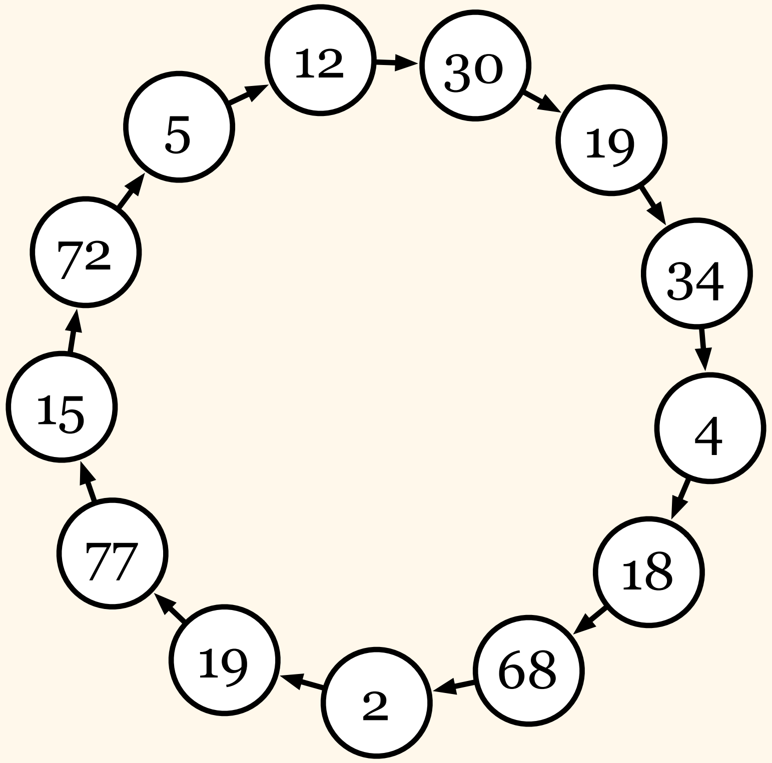
Bad News

- Even if your graph is a cycle...
- And even if you have *unique node identifiers*...



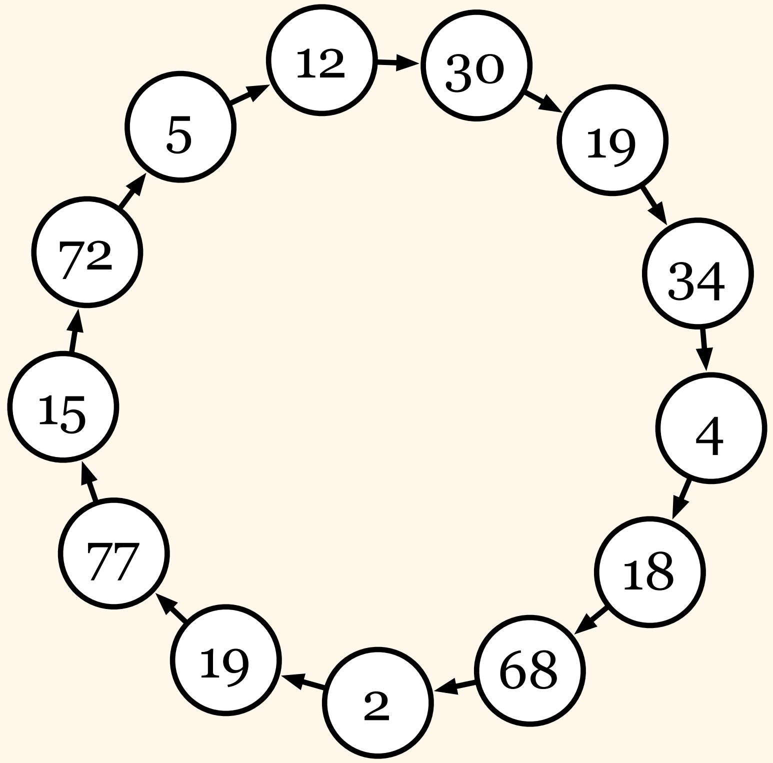
Bad News

- Even if your graph is a cycle...
- And even if you have unique node identifiers...
- And *orientation*...



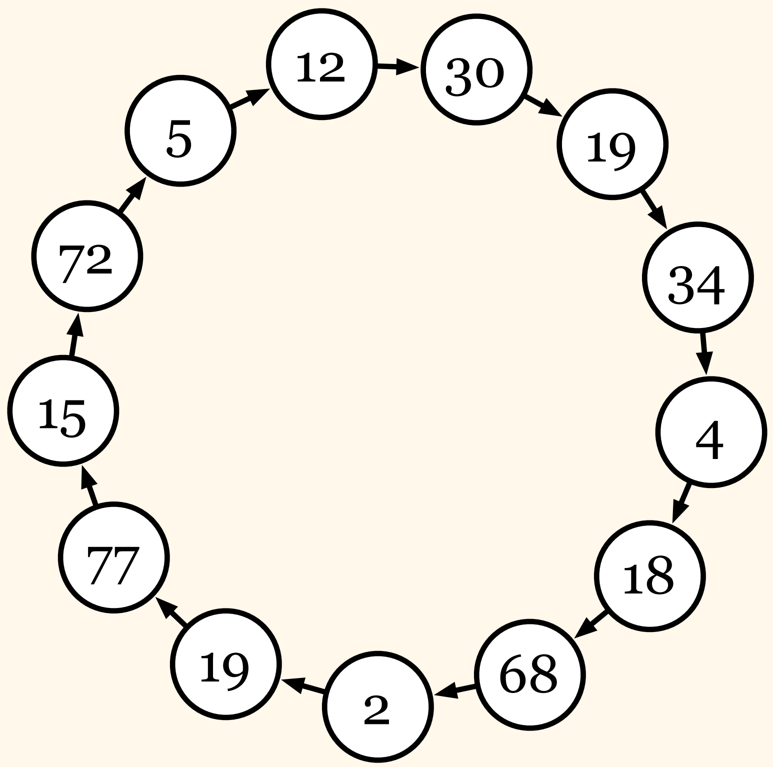
Bad News

- Even if your graph is a cycle...
- And even if you have unique node identifiers...
- And orientation...
- Then no matter which local algorithm you use, there is a *"bad input"*



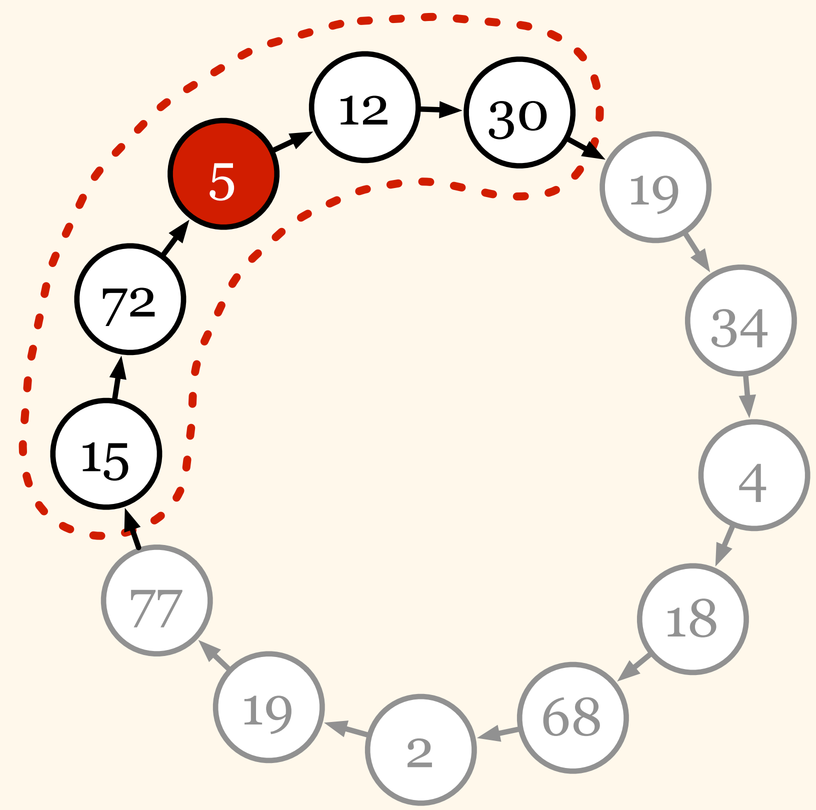
Bad News

- "Bad input":
  - *Almost all nodes will produce the same output*
  - Graph colouring not possible
  - You can find only trivial independent sets, matchings, vertex covers, dominating sets, …



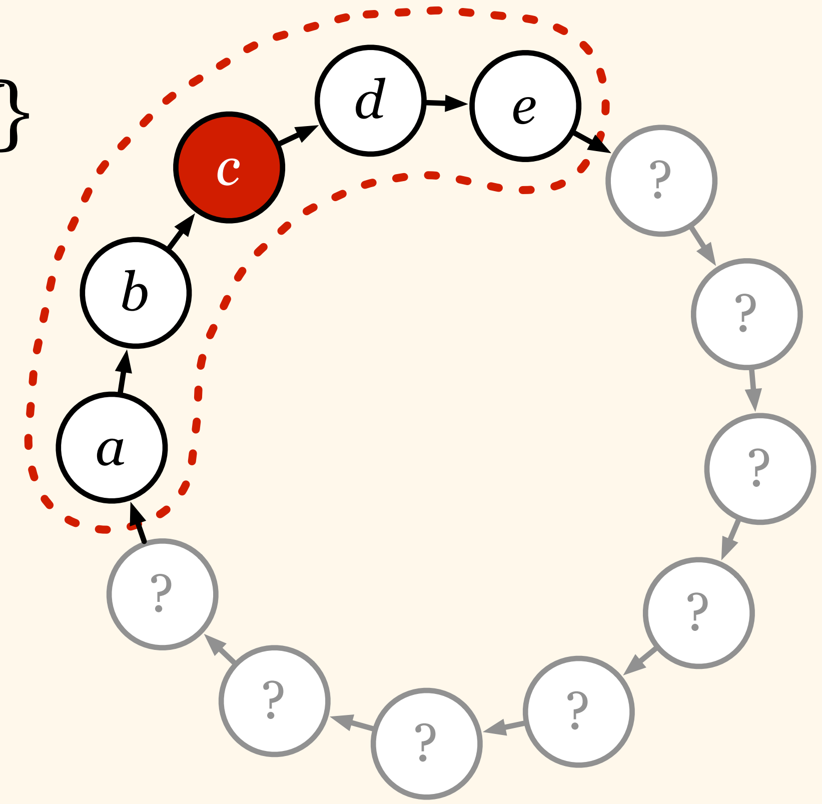
Bad News

- Example: A is a local algorithm with $r = 2$, outputs from $\{1, 2, \ldots, k\}$
    - Focus on oriented cycles
    - A maps 5-tuples of identifiers to local outputs
    - $A(15, 72, 5, 12, 30) = \ldots$



Bad News

- Example: A is a local algorithm with $r = 2$, outputs from $\{1, 2, \ldots, k\}$
  - Set of identifiers: $I = \{1, 2, \ldots, N\}$
  - Let $X = \{a, b, c, d, e\} \subseteq I$, $a < b < c < d < e$
  - Define the *colour $C(X)$* of X: $C(X) = A(a, b, c, d, e)$



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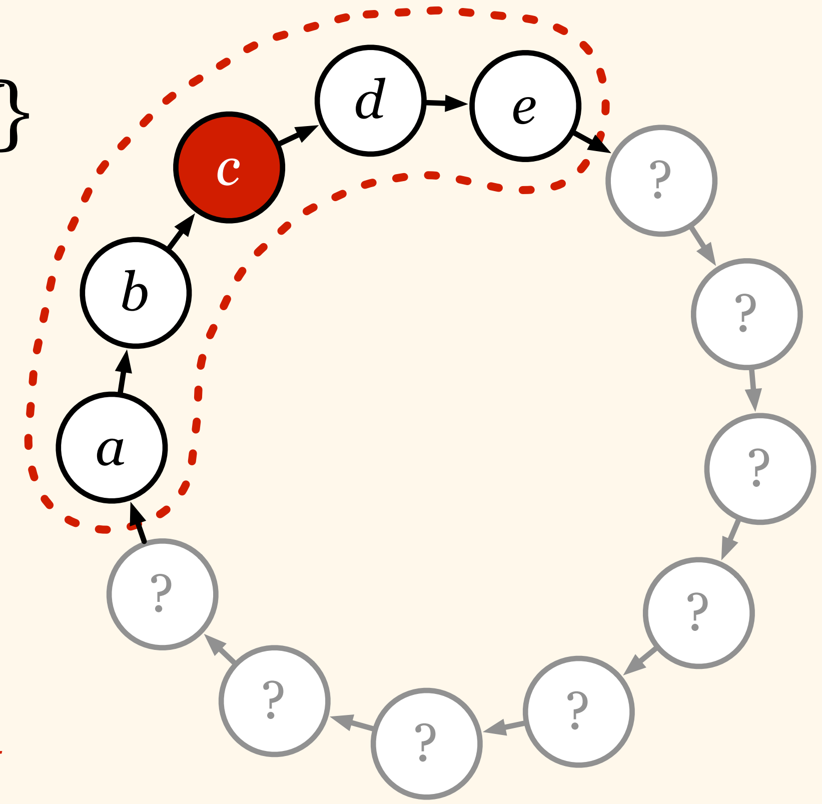
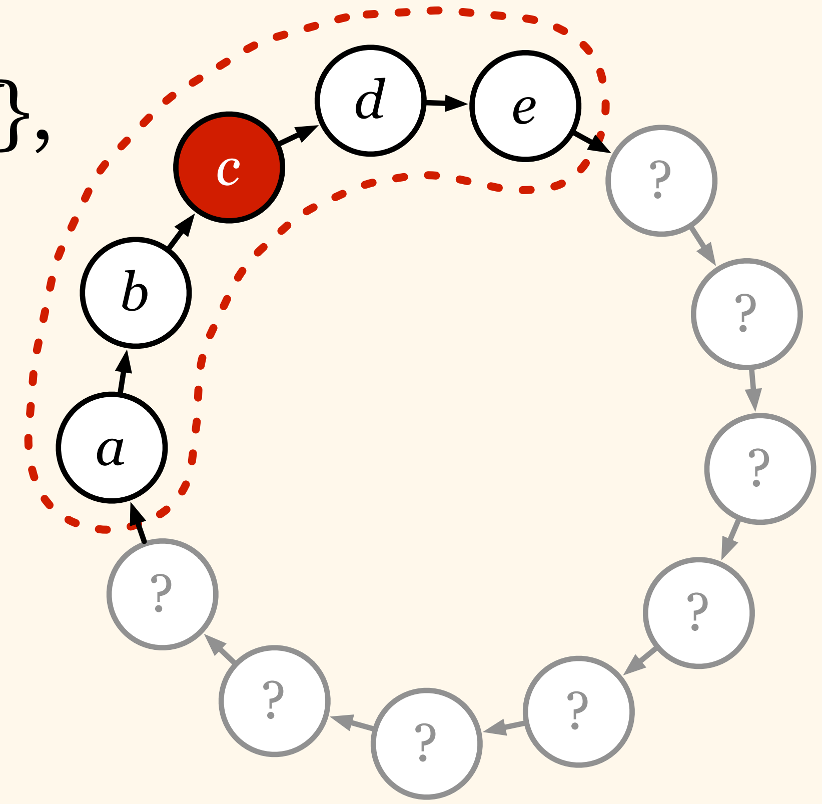
  - Define the colour $C(X)$ of X: $C(X) = A(a, b, c, d, e)$
  - We will colour *all 5-subsets of I*



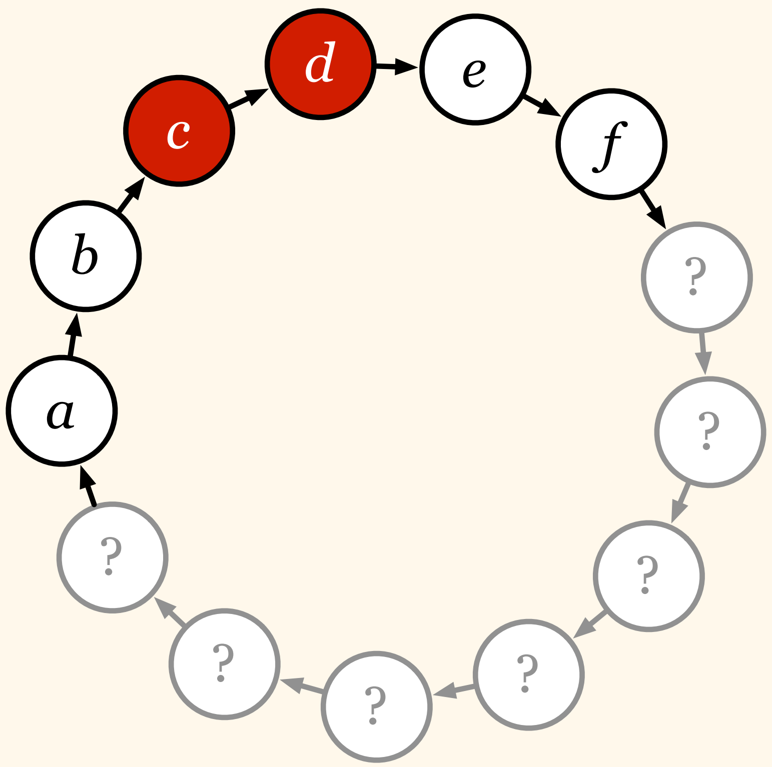
Bad News

- Example: A is a local algorithm with $r = 2$, outputs from $\{1, 2, ..., k\}$
  - Set of identifiers: $I = \{1, 2, ..., N\}$, colouring $C(X)$ of 5-subsets
  - *Ramsey*: if N is large enough, there exists a large *monochromatic* subset $M \subseteq I$
    - All 5-subsets $X \subseteq M$ have the same colour $C(X)$



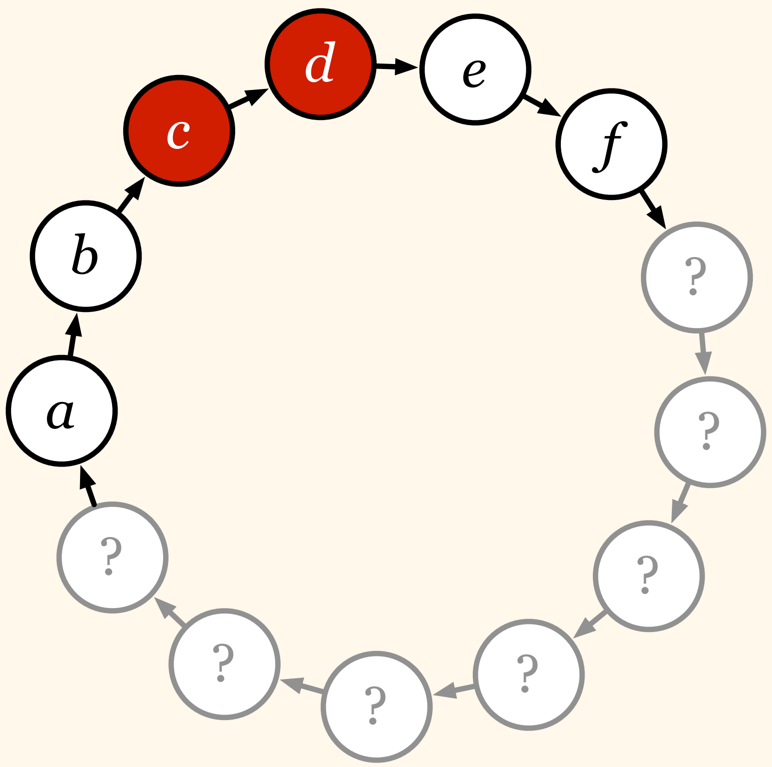
Bad News

- Example: A is a local algorithm with $r = 2$, outputs from $\{1, 2, \ldots, k\}$
  - Assume that $M = \{a, b, c, d, e, f\}$ is a monochromatic subset, $a < b < c < d < e < f$
  - $C(\{a, b, c, d, e\}) = C(\{b, c, d, e, f\})$
  - $A(a, b, c, d, e) = A(b, c, d, e, f)$



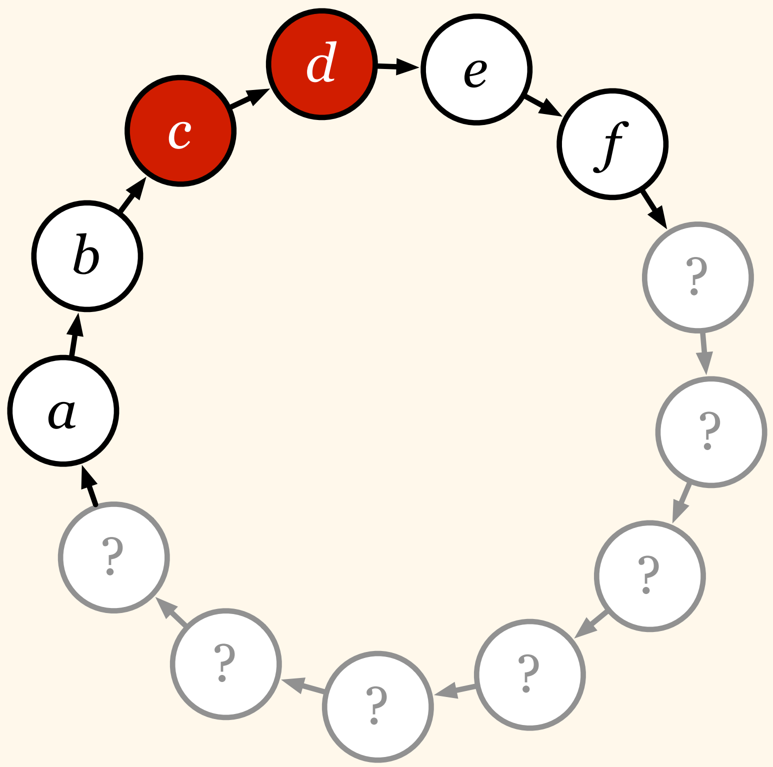
Bad News

- Example: A is a local algorithm with $r = 2$, outputs from $\{1, 2, \ldots, k\}$
    - We have found a “bad input”: nodes with identifiers c and d are adjacent and they produce the same output
    - We already proved that A cannot produce a valid graph colouring!



Bad News

- Example: A is a local algorithm with $r = 2$, outputs from $\{1, 2, \ldots, k\}$
    - We can apply the same idea for any value of r
    - And we can "boost" the argument and show that *almost all nodes* will produce the same output



Bad News

- For
  - any local algorithm A that finds an independent set,
  - any constant $\varepsilon > 0$, and
  - sufficiently large n,

we can choose unique identifiers
in an n-cycle so that A outputs
an independent set with only εn nodes

Bad News

- For
  - any local algorithm A that finds a vertex cover,
  - any constant $\varepsilon > 0$, and
  - sufficiently large n,

we can choose unique identifiers
in an n-cycle so that A outputs
a vertex cover with at least $(1 - \varepsilon)n$ nodes

Dealing with Bad News

- Three traditional escapes:
  - Randomised algorithms
  - Geometric information
  - “Almost local” algorithms

Dealing with Bad News

- Three traditional escapes:
  - *Randomised algorithms*
  - Geometric information
  - “Almost local” algorithms

Randomised Algorithms

- Nodes can *roll dice*

Randomised Algorithms

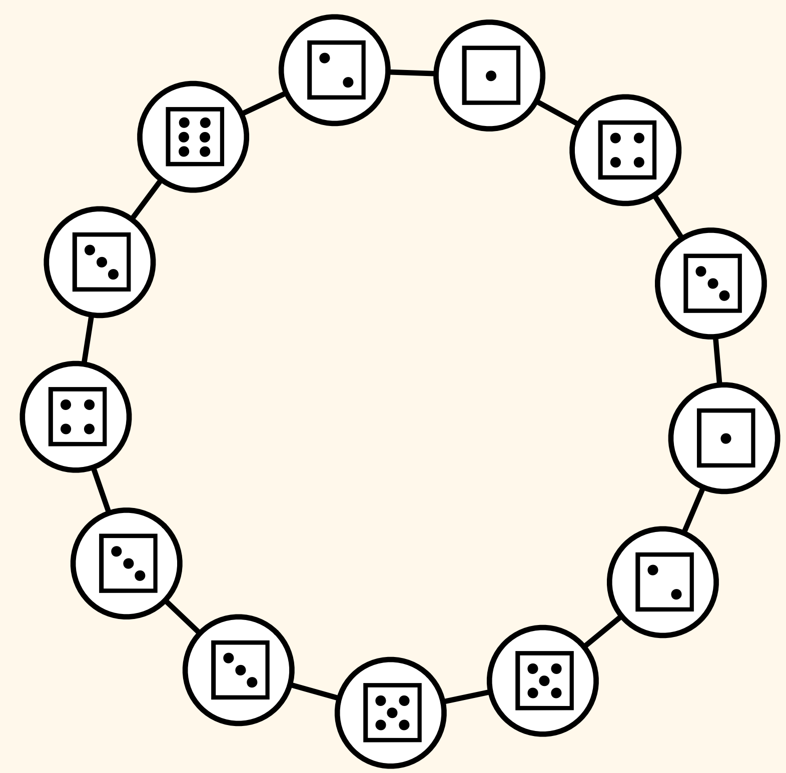
- Nodes can *roll dice* or *toss coins*

Randomised Algorithms

- Nodes can *roll dice* or *toss coins*
- We cannot guarantee that we find a good solution
  - Worst case: all coin tosses equal, no new information
- But we can find a good solution *with high probability* or *in expectation*

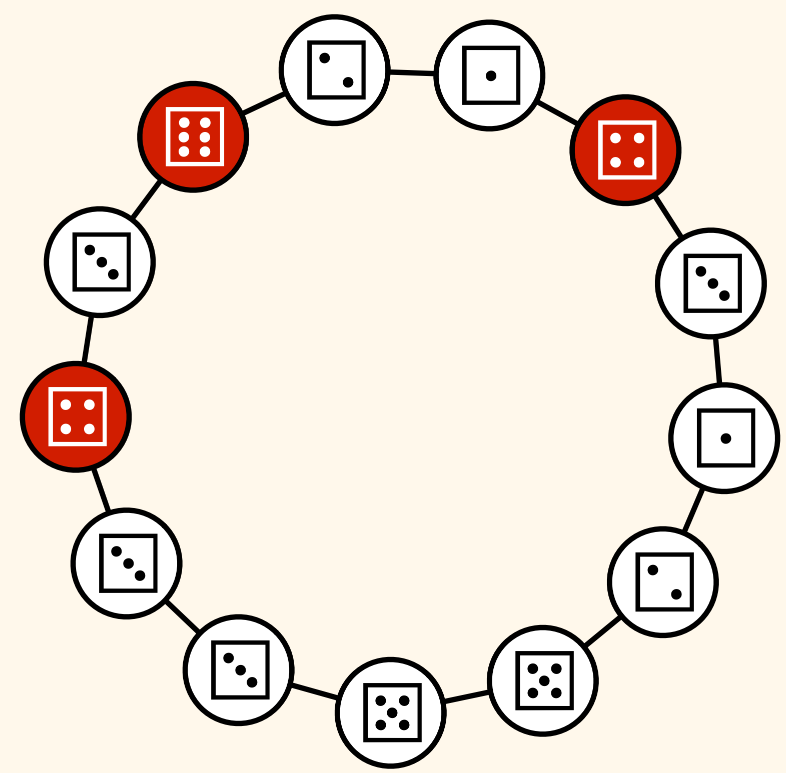
Randomised Algorithms

- *Example:* finding an independent set I
  - Each node v picks uniformly at random $X(v)$ = ⚀, ⚁, ⚂, ⚃, ⚄, ⚅



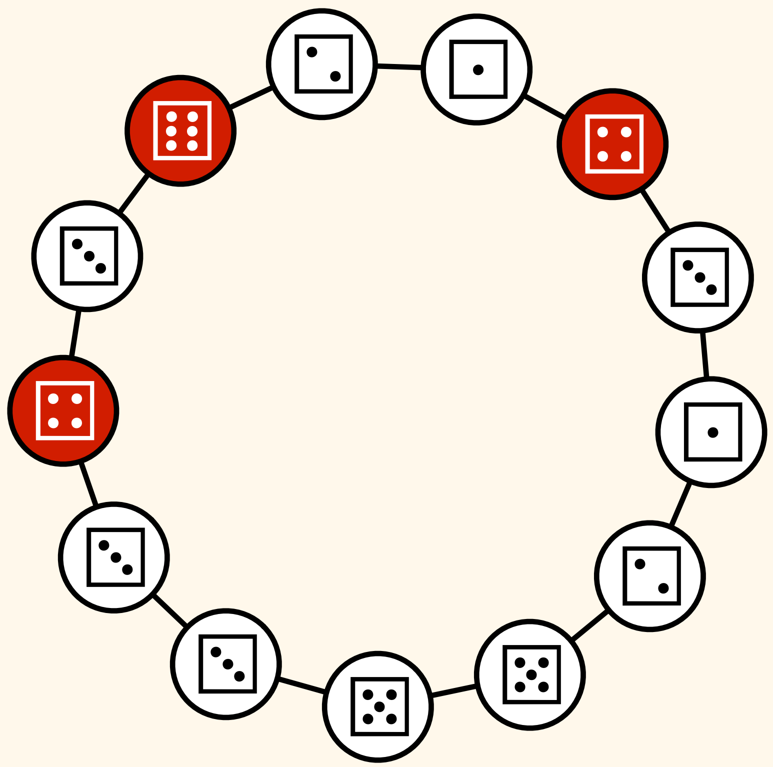
Randomised Algorithms

- *Example:* finding an independent set I
  - Each node v picks uniformly at random $X(v)$ = ⚀, ⚁, ⚂, ⚃, ⚄, ⚅
  - Node v joins I if $X(v)$ is (strict) *local maximum*



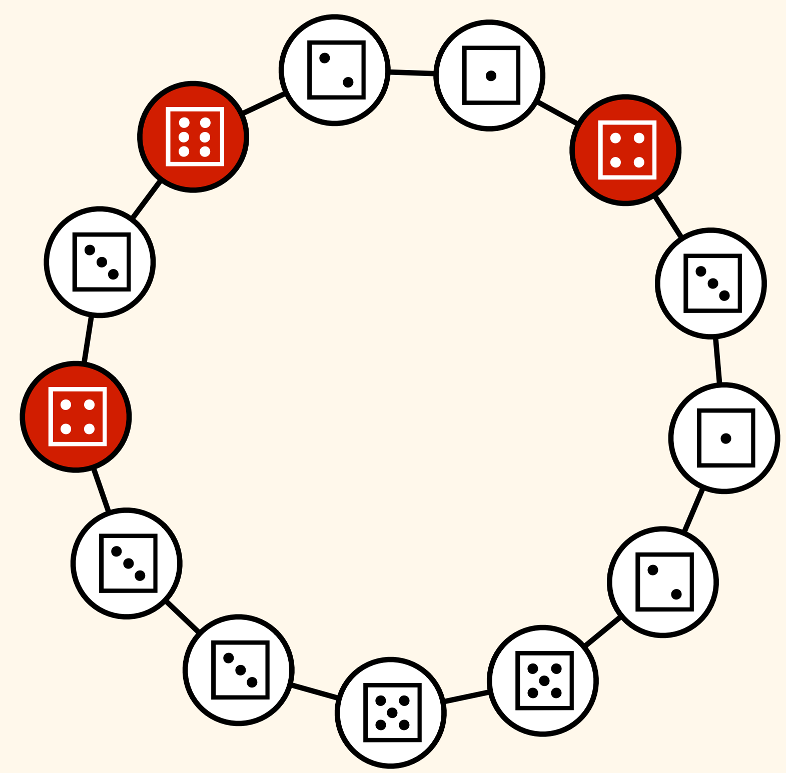
Randomised Algorithms

- *Example:* finding an independent set I
  - Each node v picks uniformly at random $X(v)$ = ⚀, ⚁, ⚂, ⚃, ⚄, ⚅
  - Node v joins I if $X(v)$ is (strict) local maximum
- By construction, I is an *independent set*



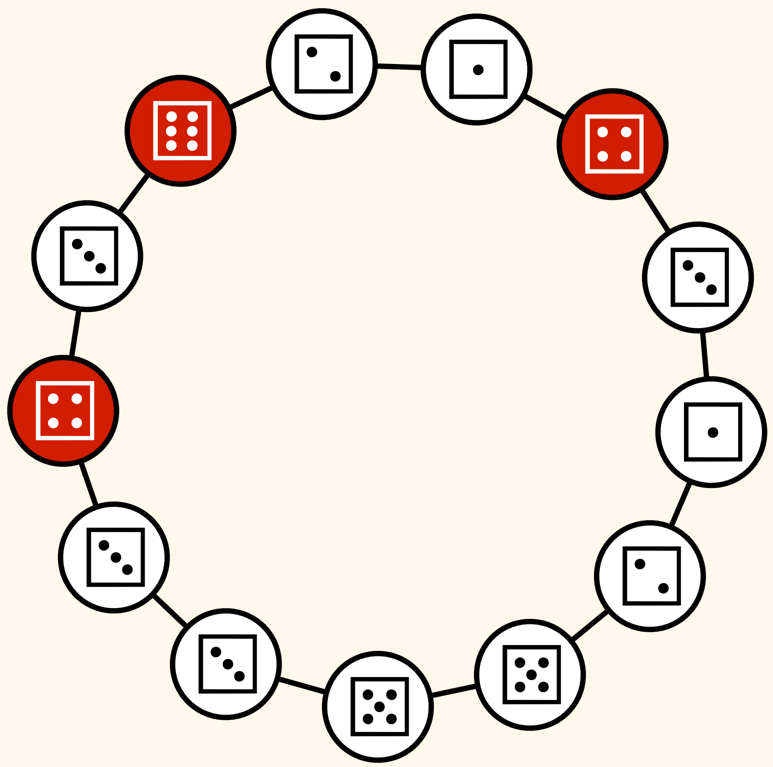
Randomised Algorithms

- *Example:* finding an independent set I
  - Each node v picks uniformly at random $X(v)$ = ⚀, ⚁, ⚂, ⚃, ⚄, ⚅
  - Node v joins I if $X(v)$ is (strict) local maximum
- Expected size of I is *reasonably large*



Randomised Algorithms

- *Example:* finding an independent set I
    - A local randomised algorithm can find a large independent set
    - Approximation algorithm (in expectation)
    - However, we cannot find *maximum* independent set or *maximal* independent set

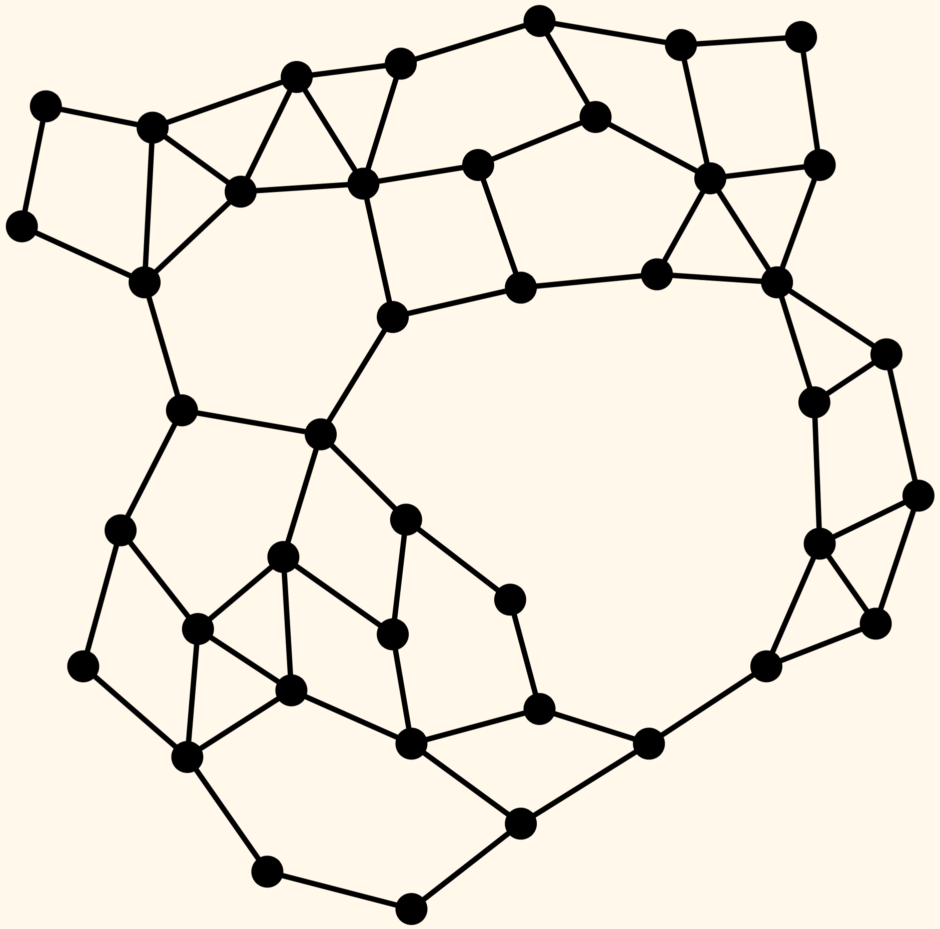


Dealing with Bad News

- Three traditional escapes:
  - Randomised algorithms
  - *Geometric information*
  - “Almost local” algorithms

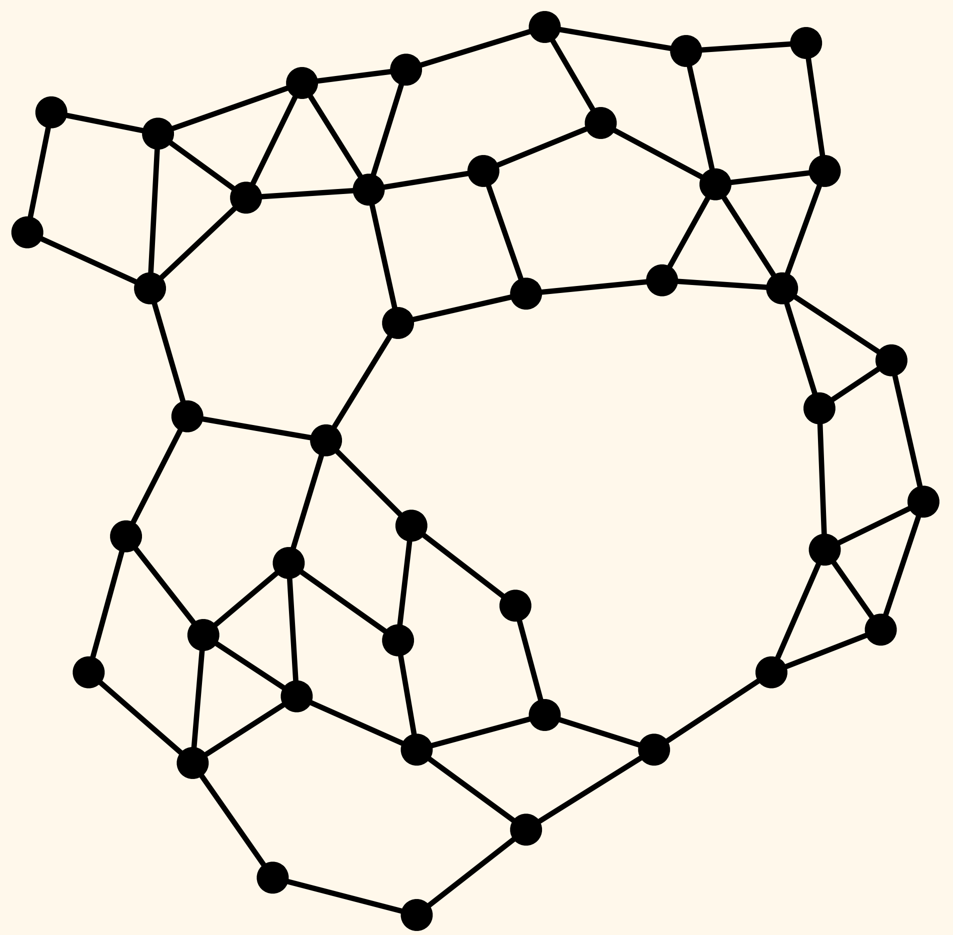
Geometric Graphs

- Assume that nodes are *points in the plane*
- Assume "reasonable" embedding



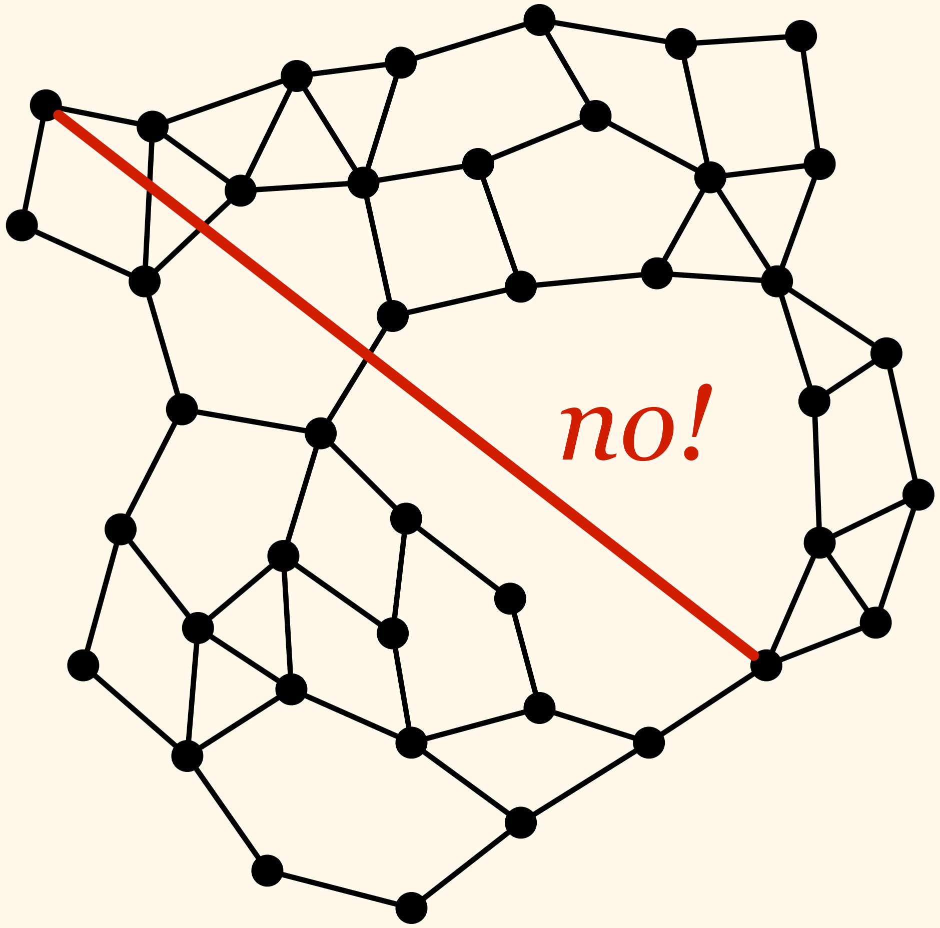
Geometric Graphs

- Assume that nodes are points in the plane
- Assume “reasonable” embedding
  - *Civilised graph*



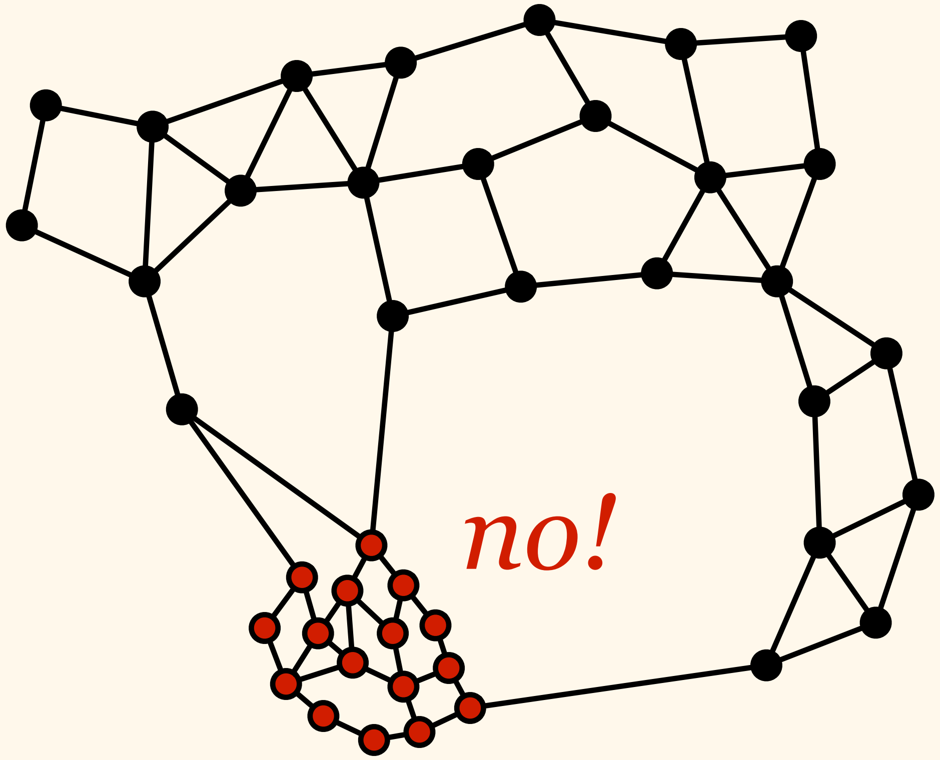
Geometric Graphs

- Assume that nodes are points in the plane
- Assume "reasonable" embedding
    - Civilised graph: *edges not too long*...



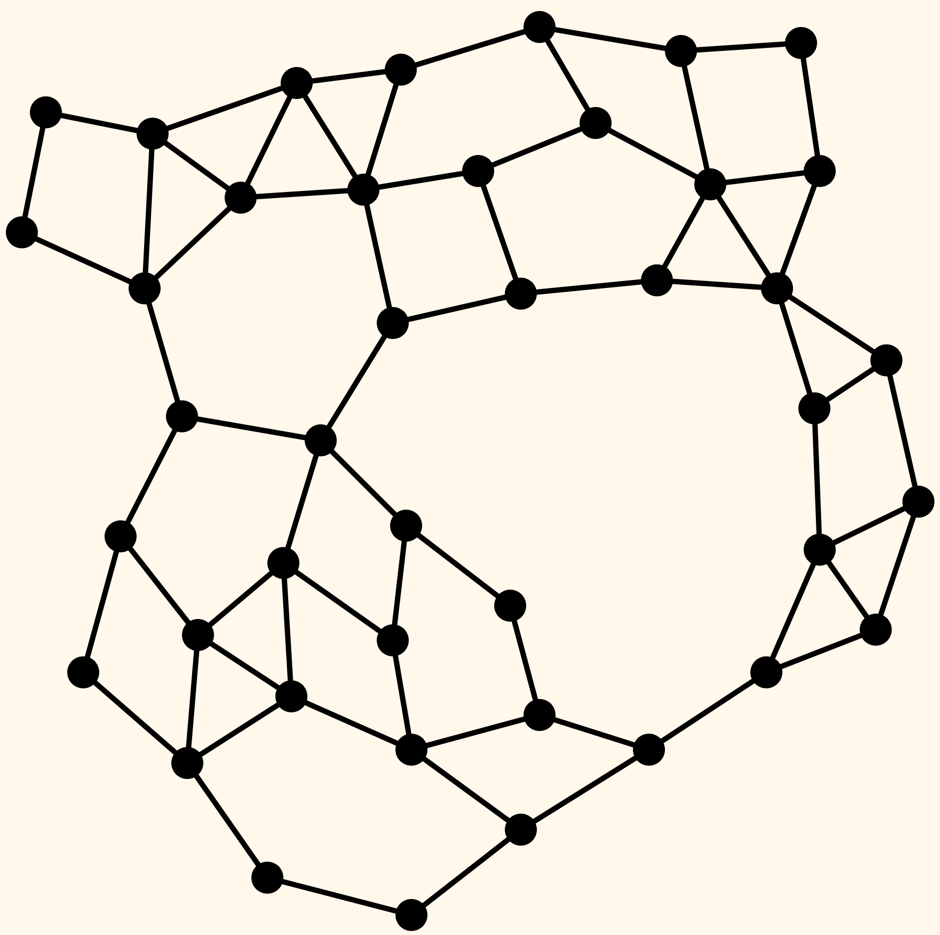
Geometric Graphs

- Assume that nodes are points in the plane
- Assume "reasonable" embedding
  - Civilised graph:
    edges not too long,
    nodes not in too dense



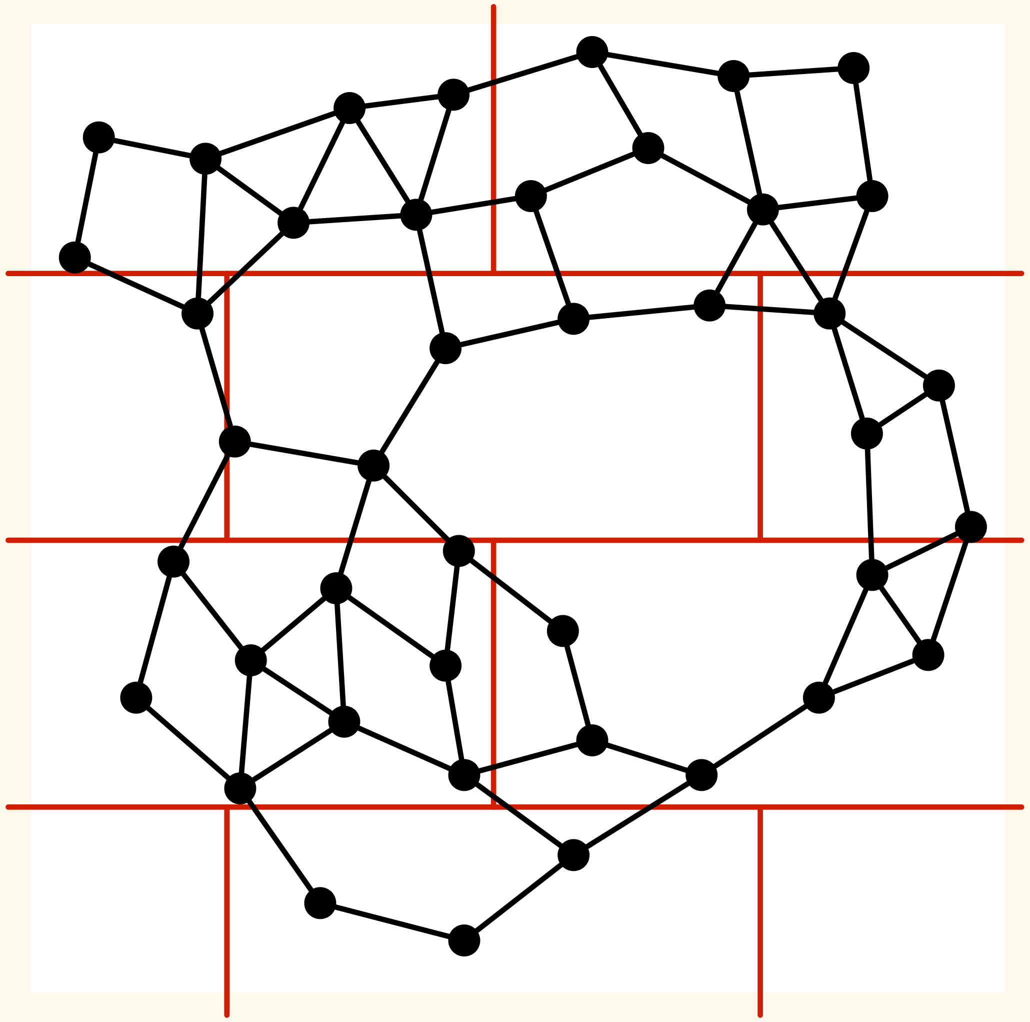
Geometric Graphs

- Assume that nodes are points in the plane
- Assume "reasonable" embedding
    - Civilised graph
    - Unit disk graph
    - Quasi unit disk graph…



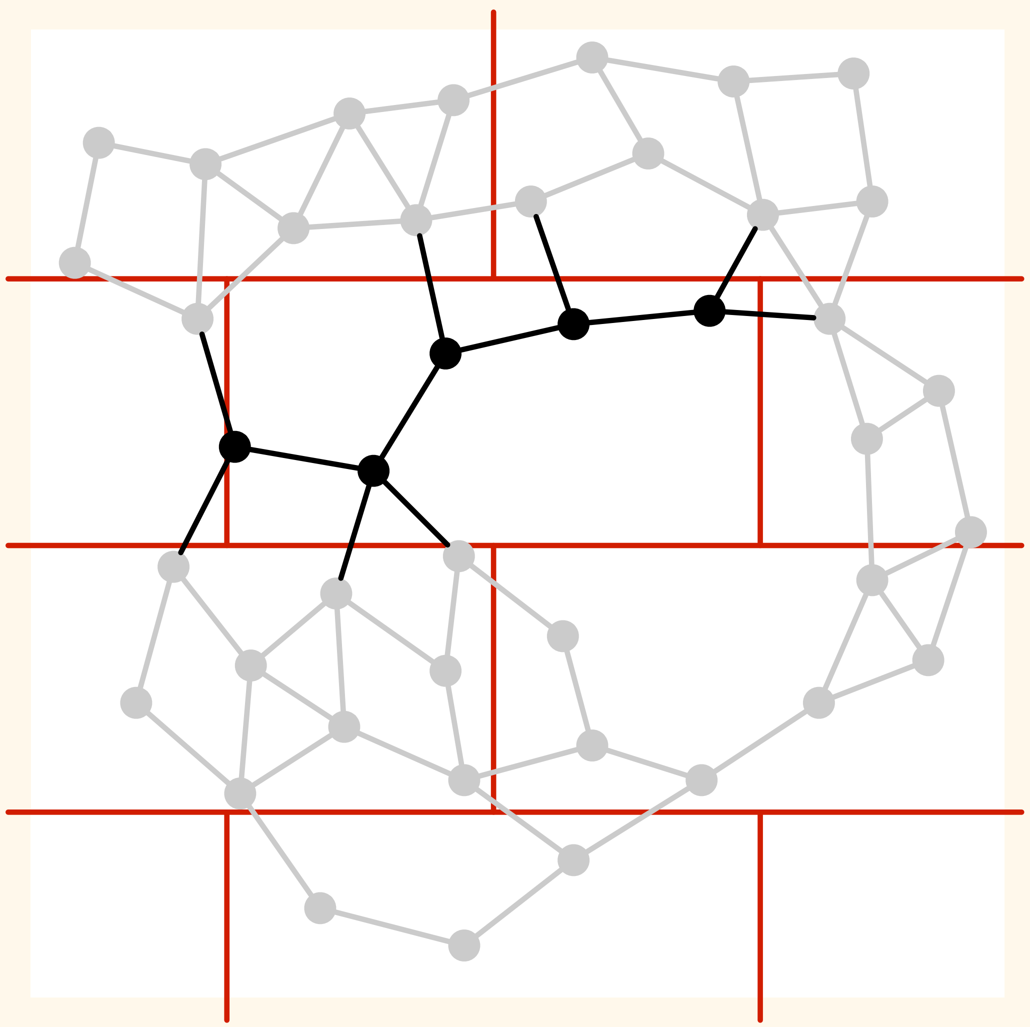
Geometric Graphs

- Exploit coordinates
  - a simple approach: *divide-and-conquer*
  - e.g., partition the plane in rectangular *tiles*



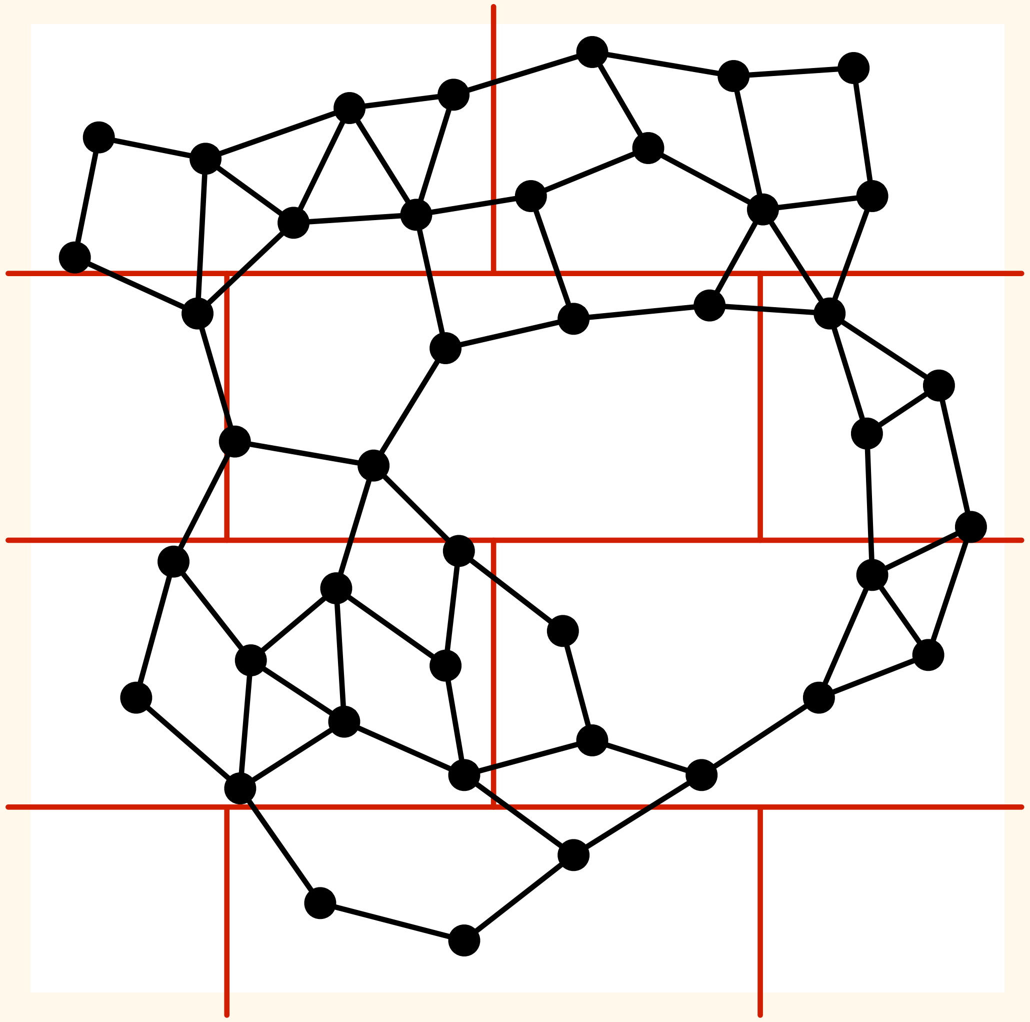
Geometric Graphs

- Exploit coordinates
    - each tile defines a *constant-size subproblem*
    - solve the subproblem locally within each tile (in parallel for all tiles)



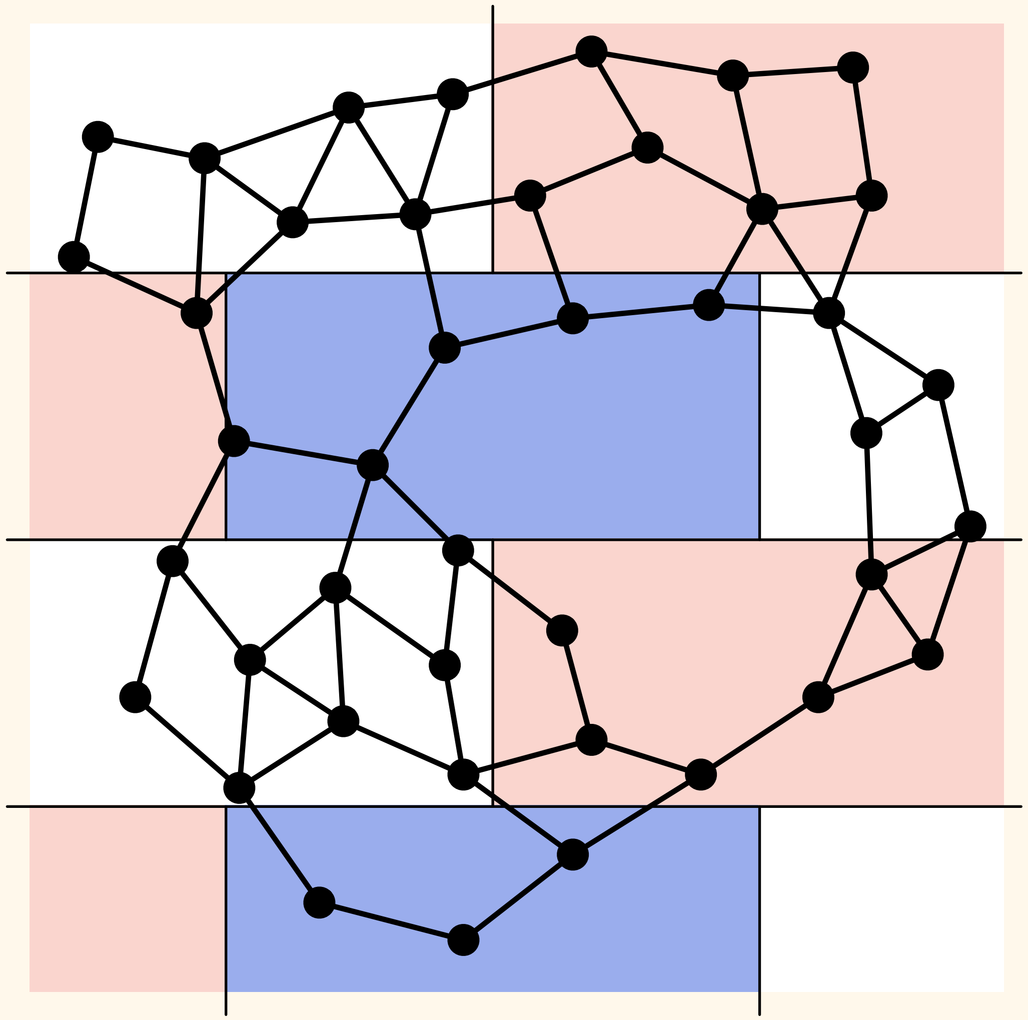
Geometric Graphs

- Exploit coordinates
    - each tile defines a constant-size subproblem
    - solve the subproblem locally within each tile
    - *merge* the solutions of the subproblems



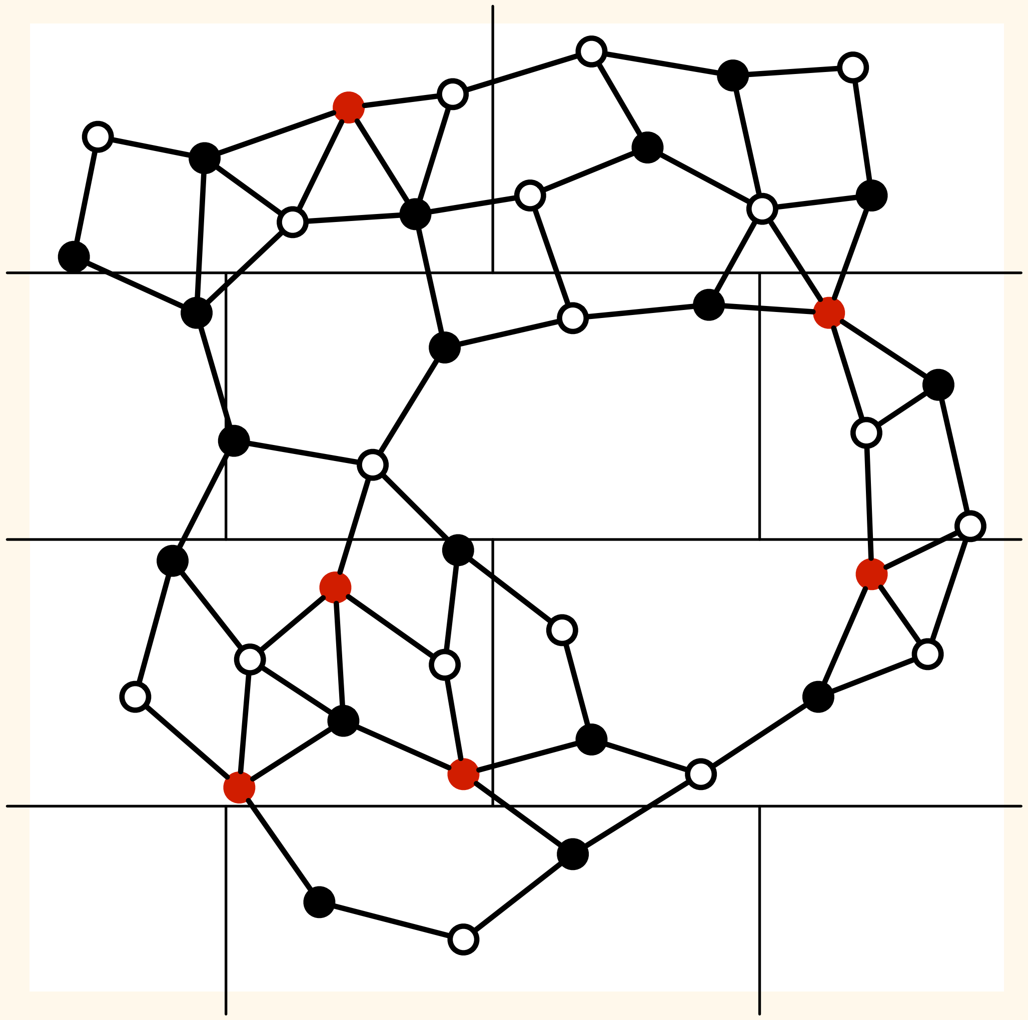
Geometric Graphs

- Graph colouring:
  - f = 3-colouring of tiles
    - all edges are short
    - there is no edge that joins e.g. a blue tile and another blue tile



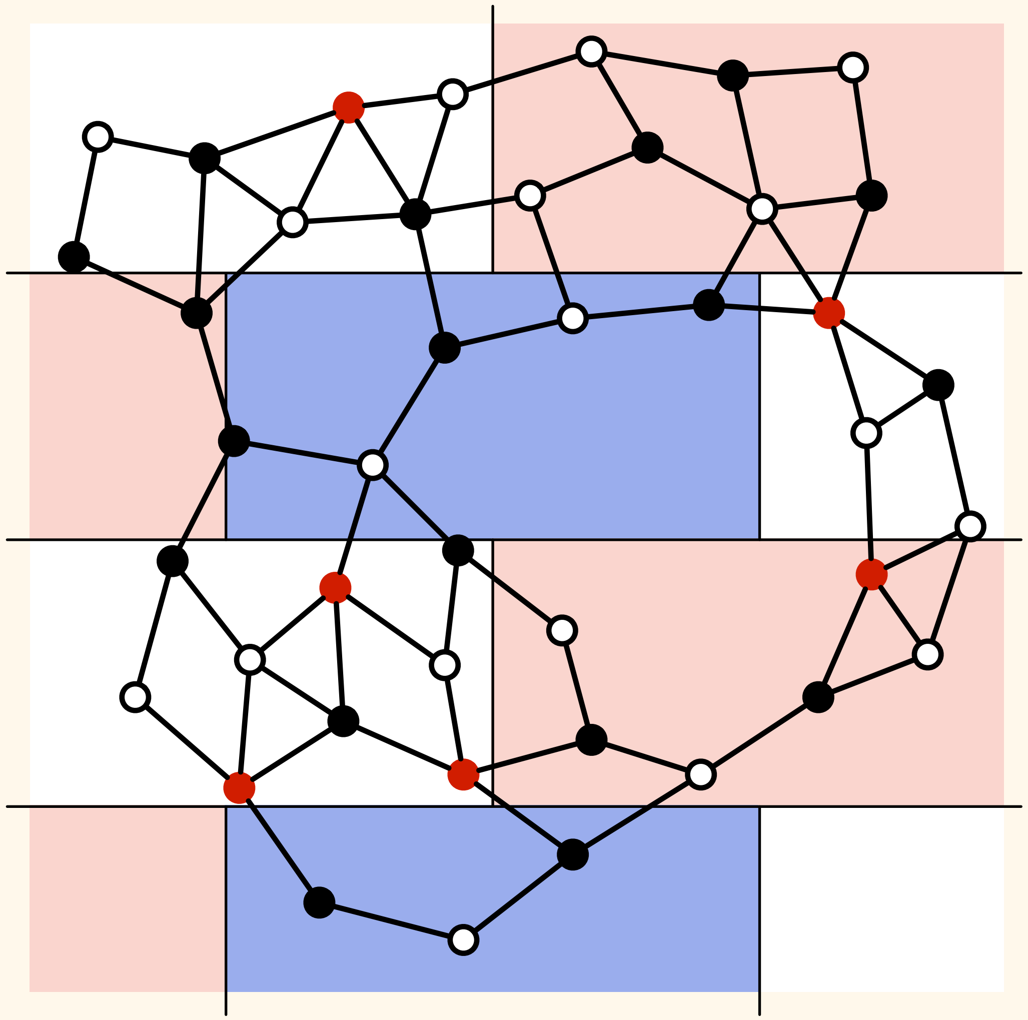
Geometric Graphs

- Graph colouring:
  - f = 3-colouring of tiles
  - g = k-colouring that is valid *inside* each tile
    - can be solved by brute force



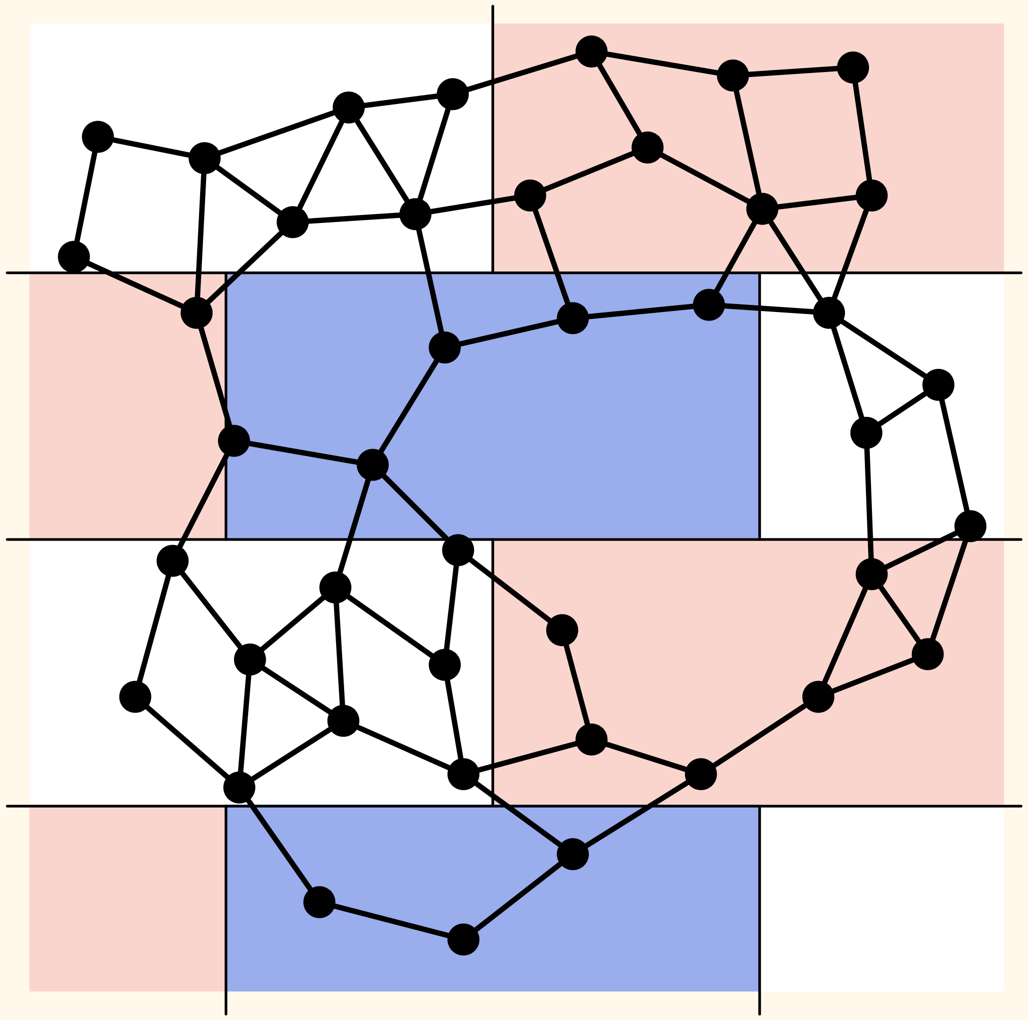
Geometric Graphs

- Graph colouring:
  - f = 3-colouring of tiles
  - g = k-colouring that is valid *inside* each tile
  - Output: (f, g)
  - Valid $3k$-colouring!



Geometric Graphs

- Simple local algorithms:
  - *maximal* matchings, independent sets, ...
  - *approximation algorithms* for vertex covers, dominating sets, colourings, ...



Dealing with Bad News

- Three traditional escapes:
  - Randomised algorithms
  - Geometric information
  - *“Almost local” algorithms*

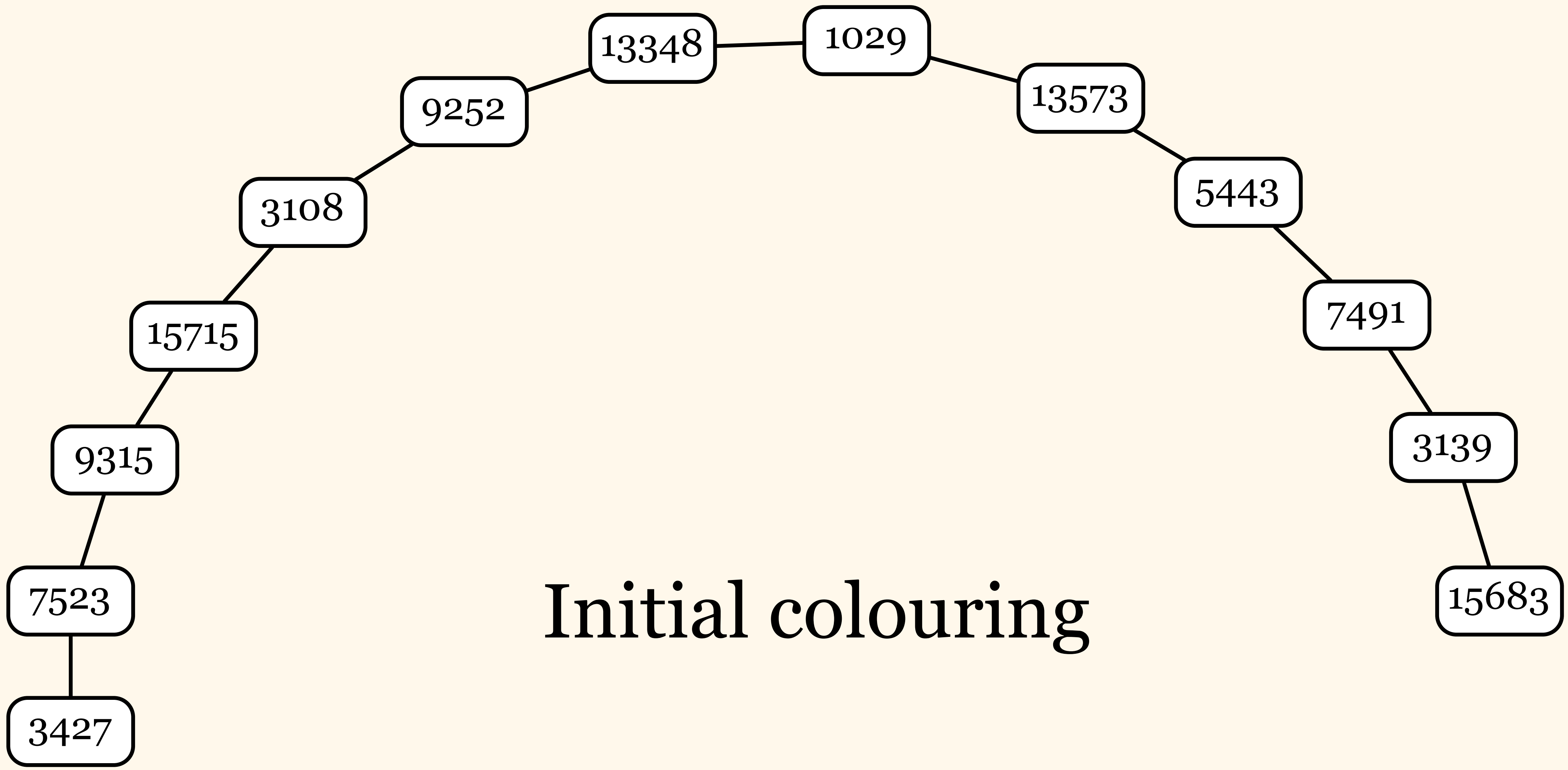
Almost Local Algorithms

- We cannot find non-trivial solutions in a cycle in $O(1)$ rounds
- But we can do it in $O(\log^* n)$ rounds!
  - $\log^* n$ = iterated logarithm
  - $0 \le \log^* n \le 7$ for all real-world values of n
  - Good enough?

Almost Local Algorithms

- Main tool: colour reduction
  - *Cole & Vishkin* (1986)
  - *Goldberg, Plotkin & Shannon* (1988)
- Bit manipulation trick:
  - From k colours to $O(\log k)$ colours in one step
  - Initially poly(n) colours: unique identifiers
  - Iterate $O(\log^* n)$ times until $O(1)$ colours

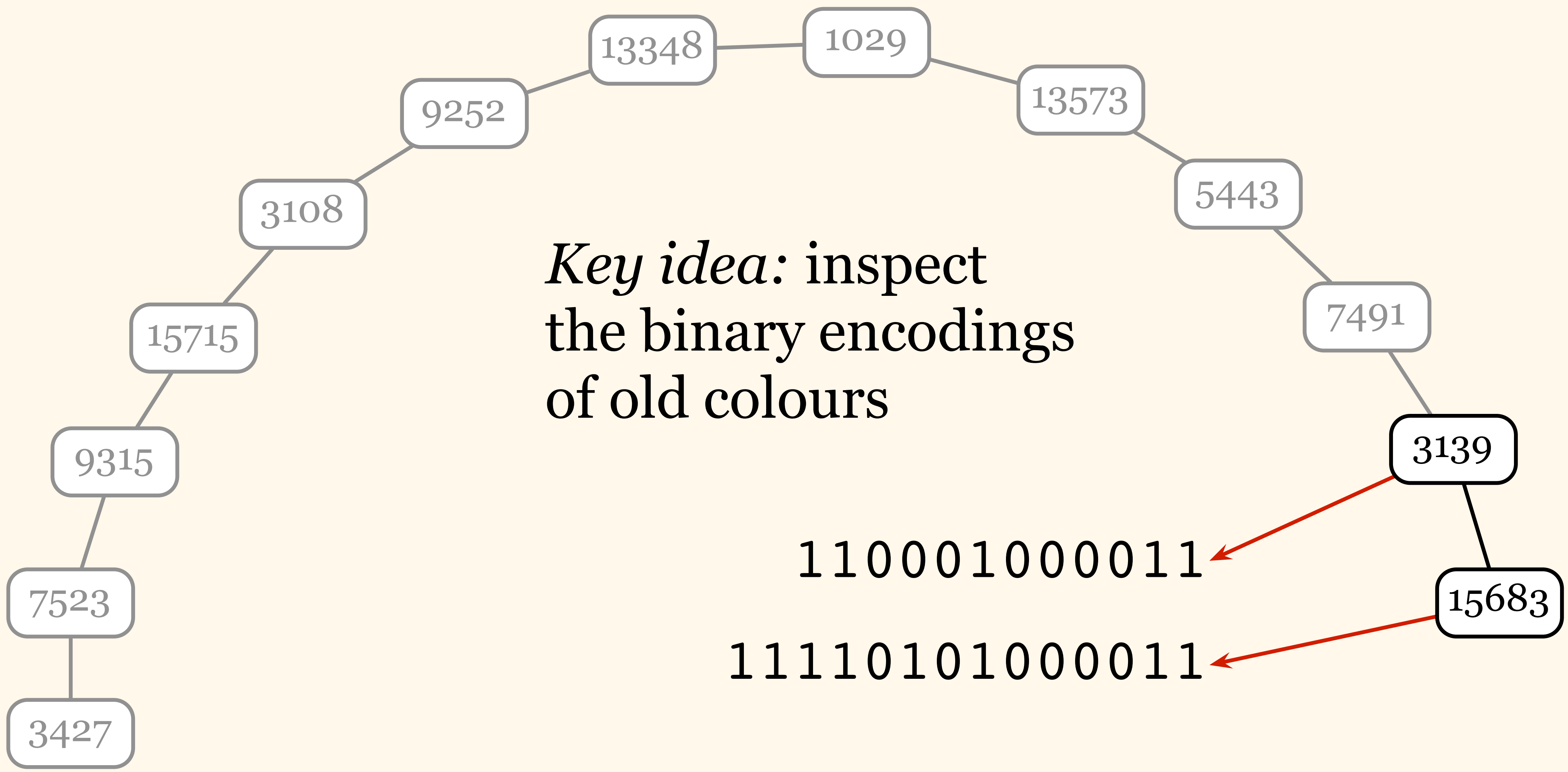
Almost Local Algorithms



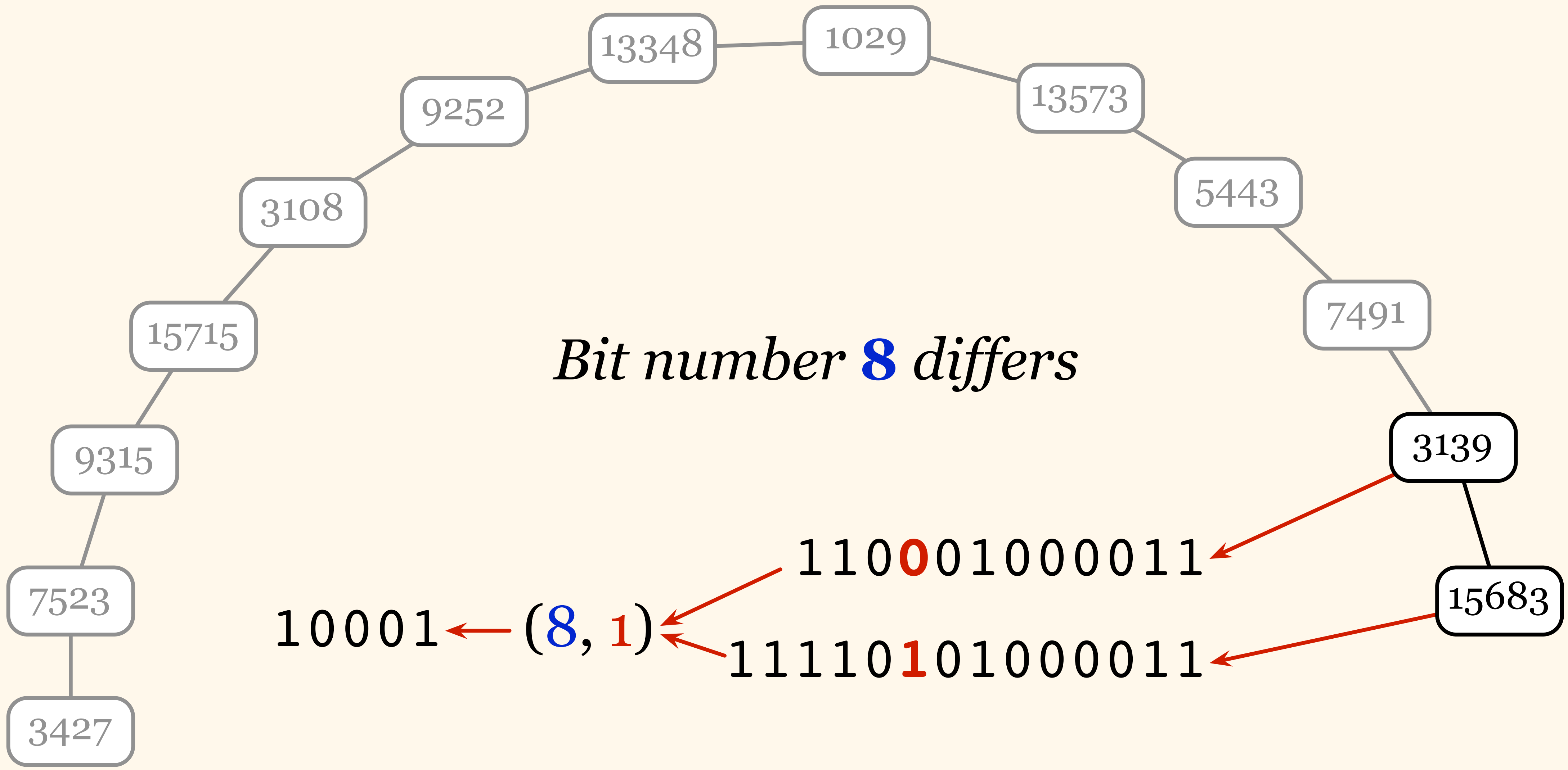
Initial colouring

Almost Local Algorithms

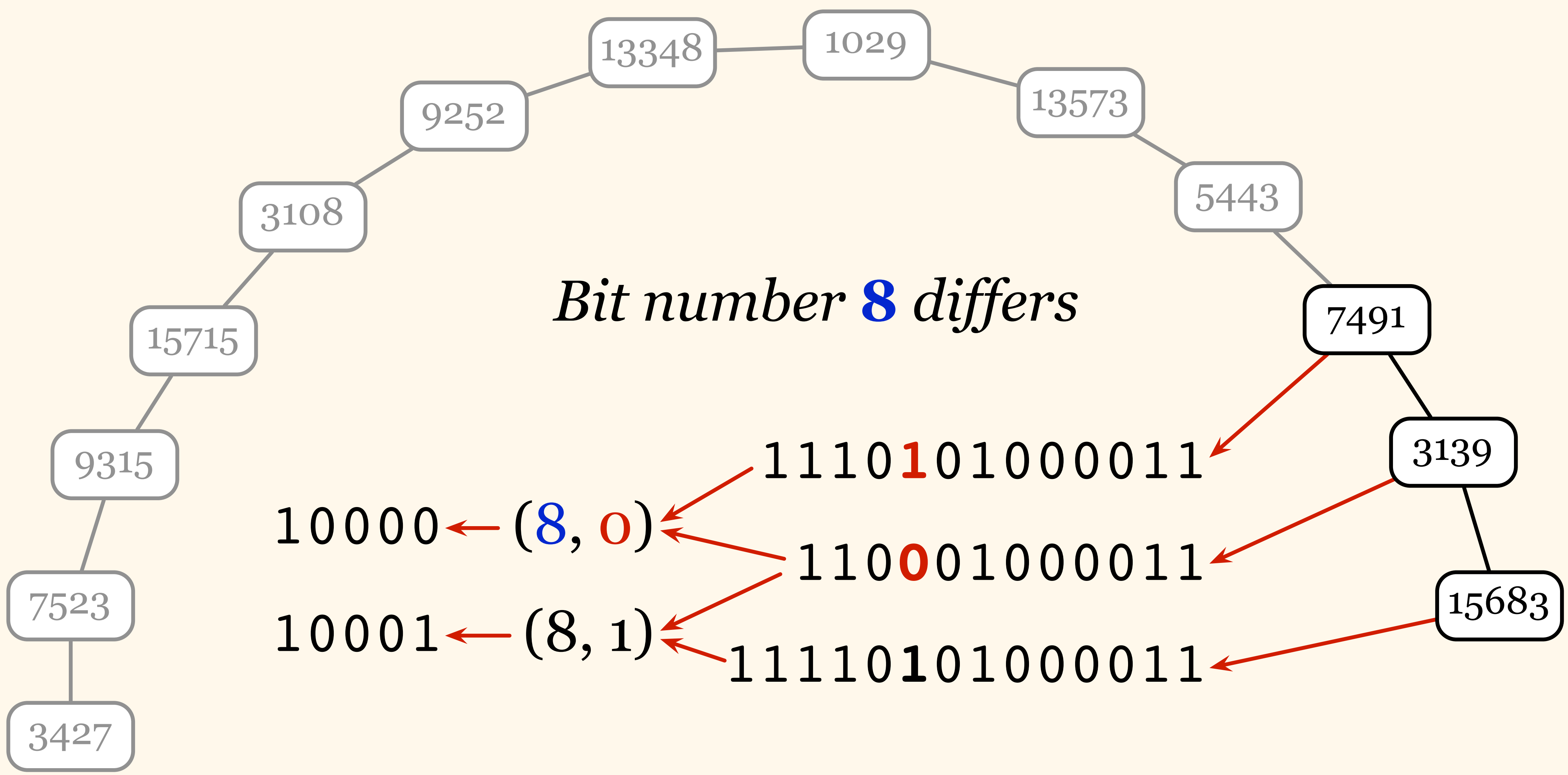
Key idea: inspect the binary encodings of old colours



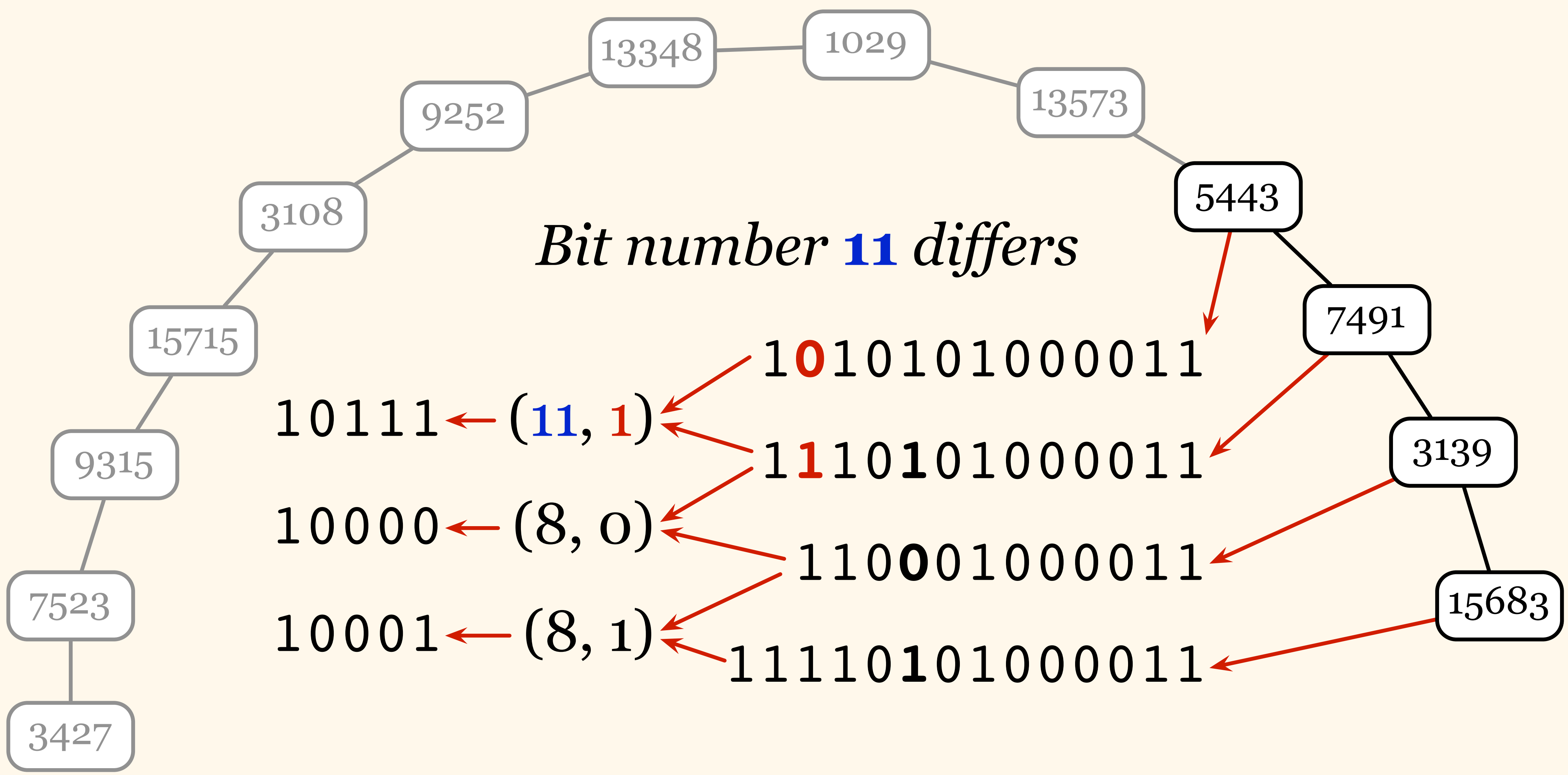
Almost Local Algorithms



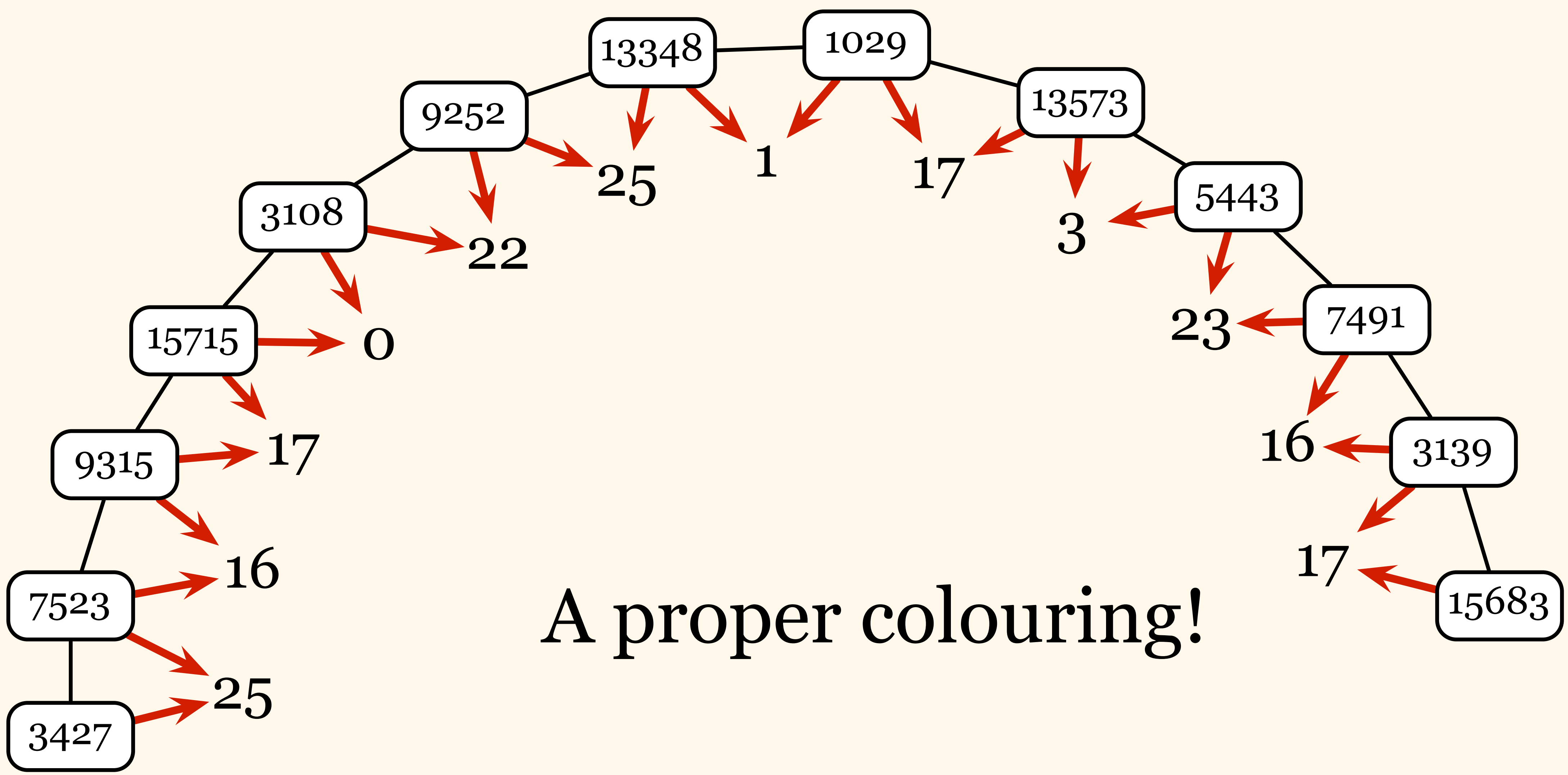
Almost Local Algorithms



Almost Local Algorithms

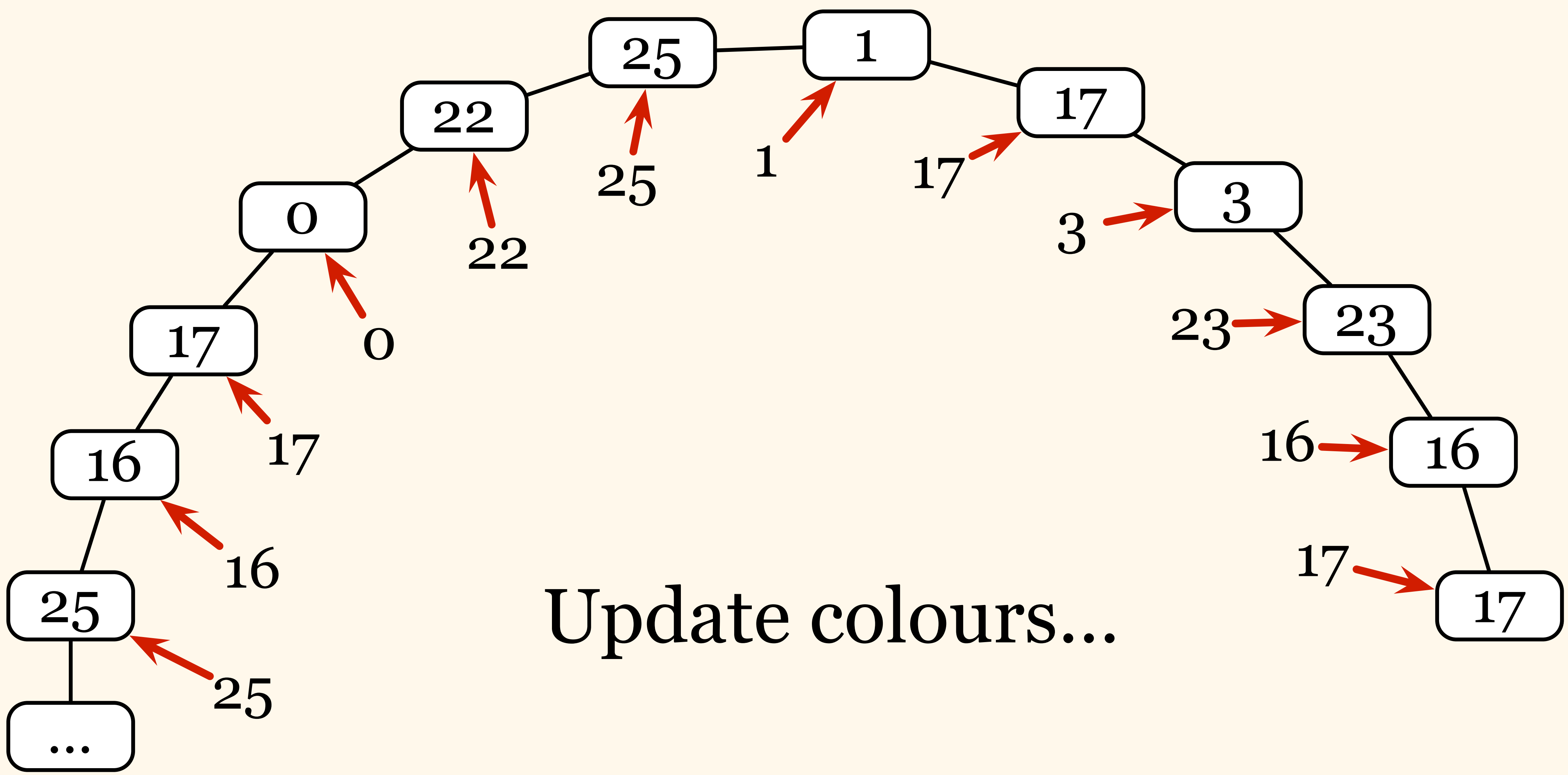


Almost Local Algorithms



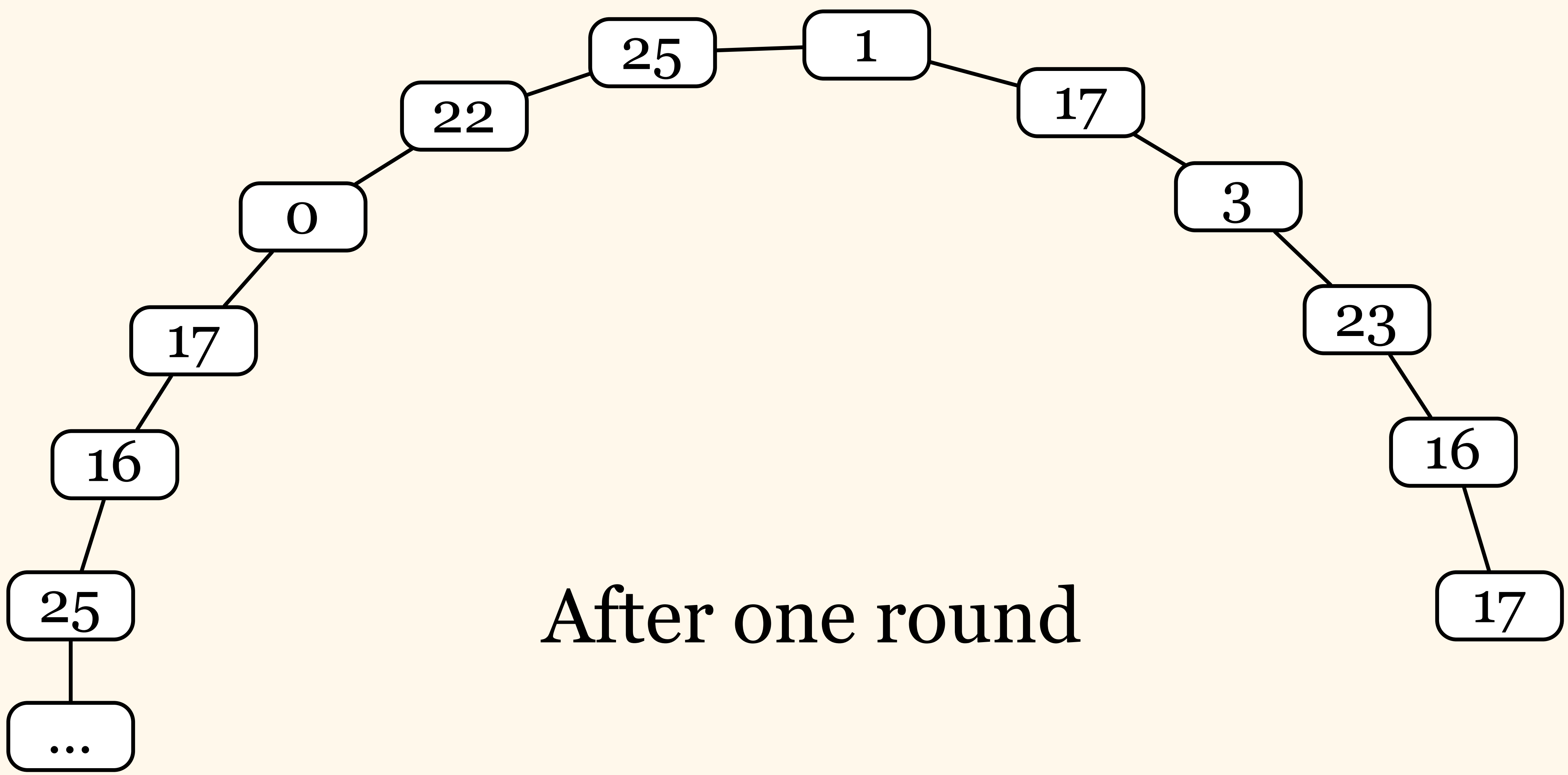
A proper colouring!

Almost Local Algorithms



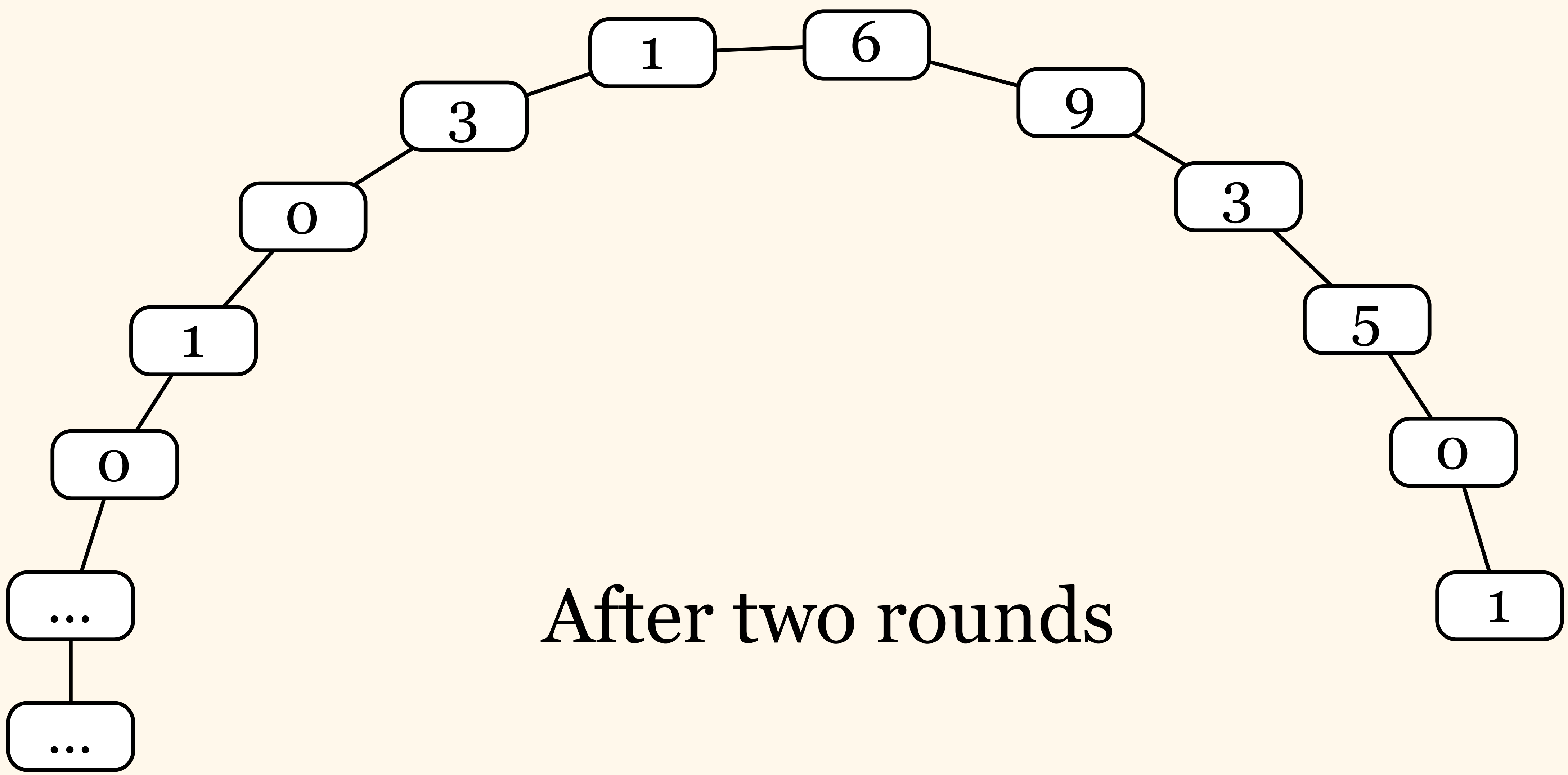
Update colours...

Almost Local Algorithms



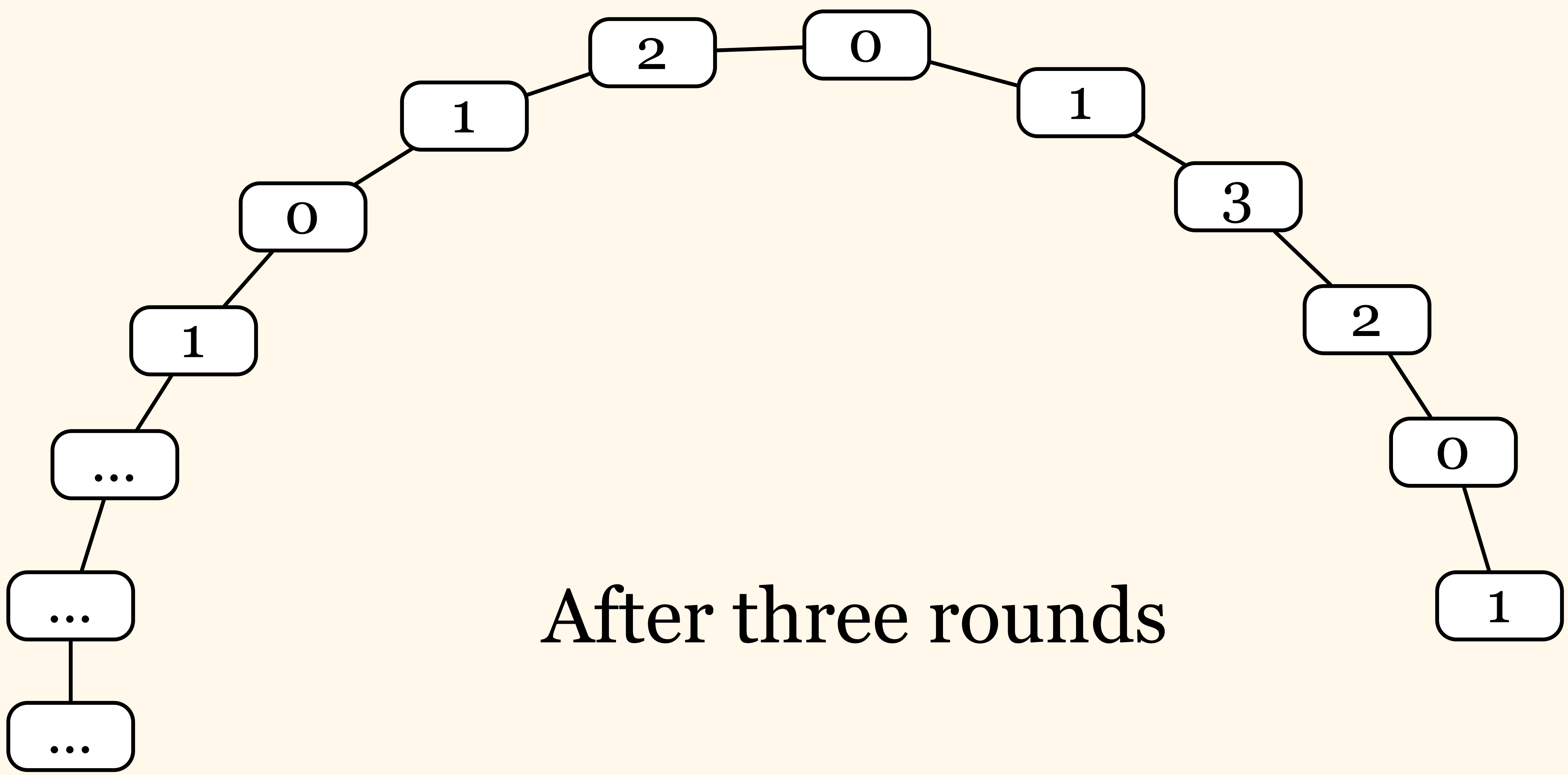
After one round

Almost Local Algorithms



After two rounds

Almost Local Algorithms



After three rounds

Almost Local Algorithms

- Graph colouring in $O(\log^* n)$ rounds
  - Paths or cycles, 3-colouring
- Generalisations:
  - Trees, bounded-degree graphs, …
  - Graphs of maximum degree Δ:
    $(\Delta+1)$-colouring in $O(\Delta + \log^* n)$ rounds

Almost Local Algorithms

- Graph colouring in $O(\log^* n)$ rounds
- Many applications:
  - Maximal independent set:
    first try to add nodes of colour 0 (in parallel),
    then try to add nodes of colour 1 (in parallel), ...
  - Maximal matching
  - Greedy algorithm for dominating sets

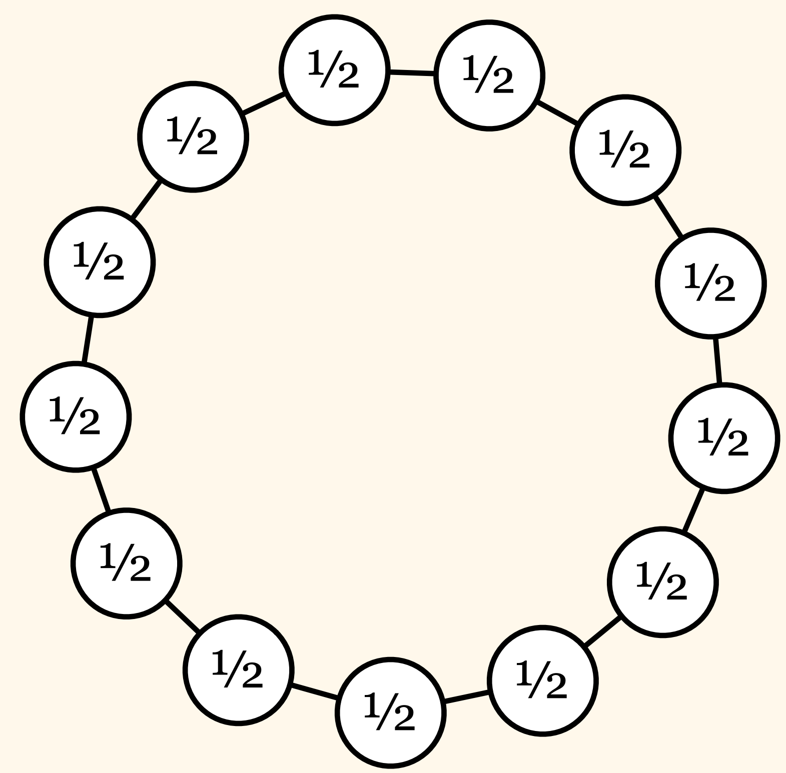
Almost Local Algorithms

- Graph colouring in $O(\log^* n)$ rounds
- Many applications
- Fast, but not strictly local
    - And inherently depends on the existence of small, unique, numerical identifiers

Past: Summary

- Bad news:
  - Cannot break symmetry in cycles
- Three traditional escapes:
  - Randomised algorithms
  - Geometric information
  - “Almost local” algorithms

Present

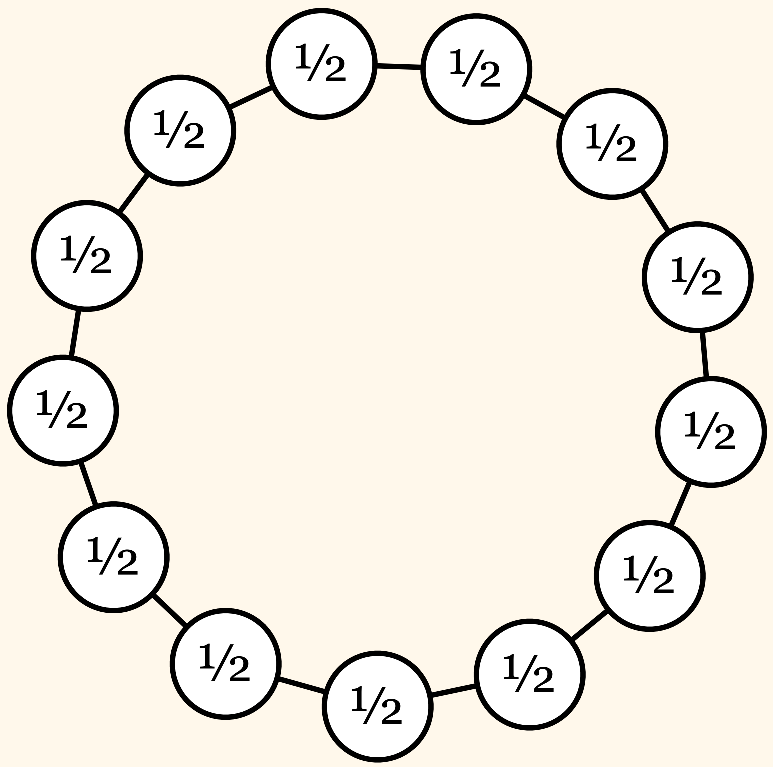


Dealing with Bad News

- You cannot break symmetry in cycles…
- Which problems *do not require* symmetry breaking in cycles?

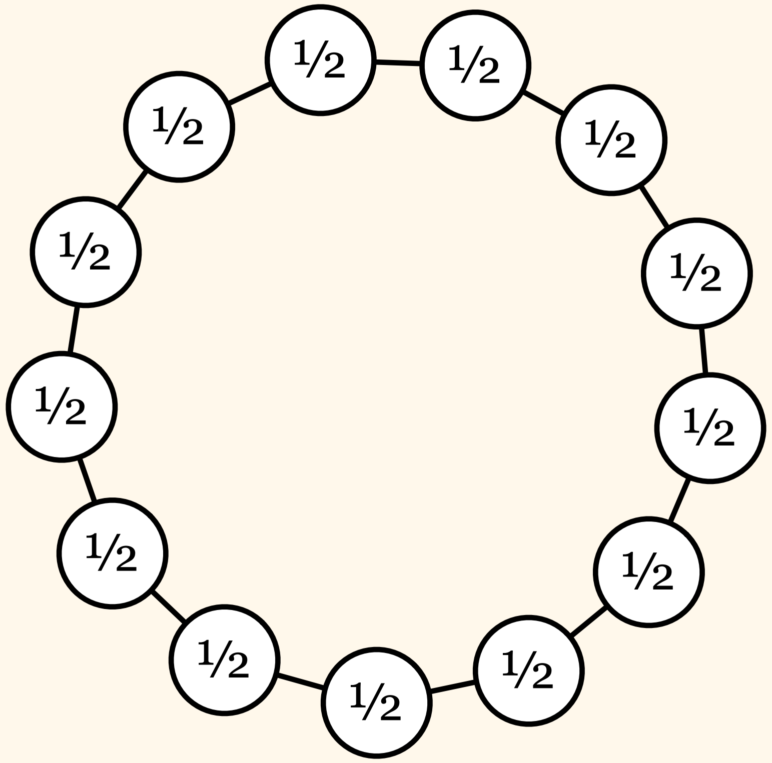
Tractable Problems

- Linear programs (LPs)
  - Many resource-allocation problems can be modelled as LPs
  - If the input is symmetric, a trivial solution is an optimal solution!
  - *Only non-symmetric inputs are challenging*…



Tractable Problems

- Linear programs (LPs)
  - Approximation scheme for packing and covering LPs
  - Local algorithm
  - *Kuhn, Moscibroda & Wattenhofer* (2006)

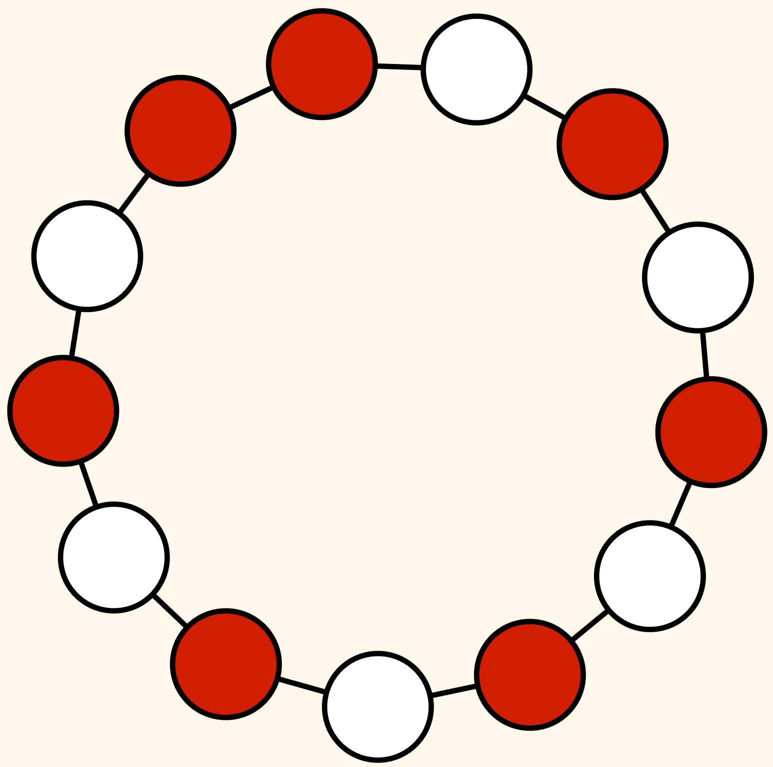


Tractable Problems

- Vertex covers
  - 2-approximation is the best that we can find with *centralised polynomial-time algorithms*
    - Nobody knows how to find 1.9999-approximation efficiently
  - Hence if we could find a 2-approximation with *local algorithms*, it would be amazing!

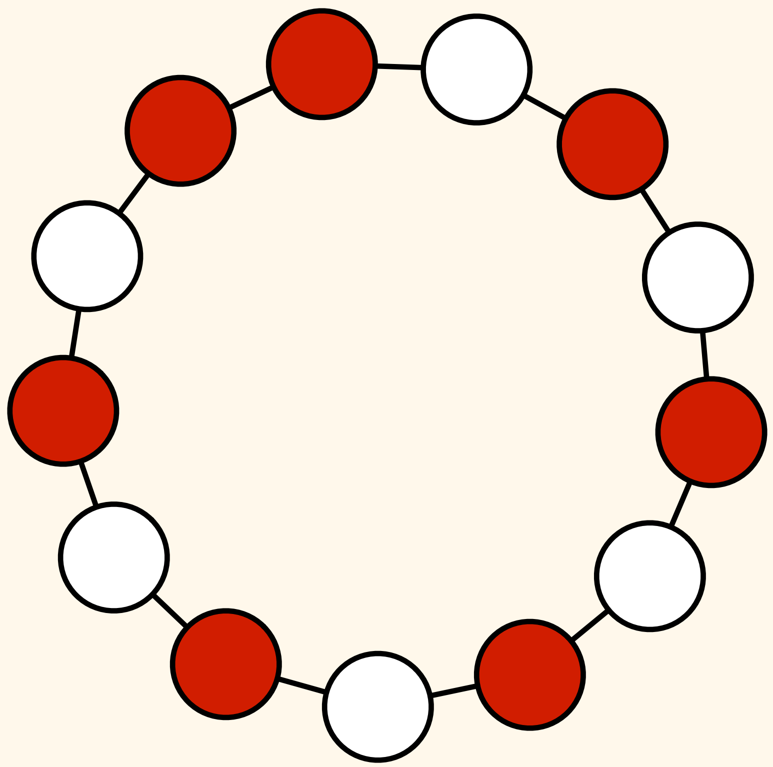
Tractable Problems

- Vertex covers
  - 2-approximation does not require symmetry breaking
  - In a regular graph, trivial solution (all nodes) is 2-approximation
  - Again, *only non-symmetric inputs are challenging*…



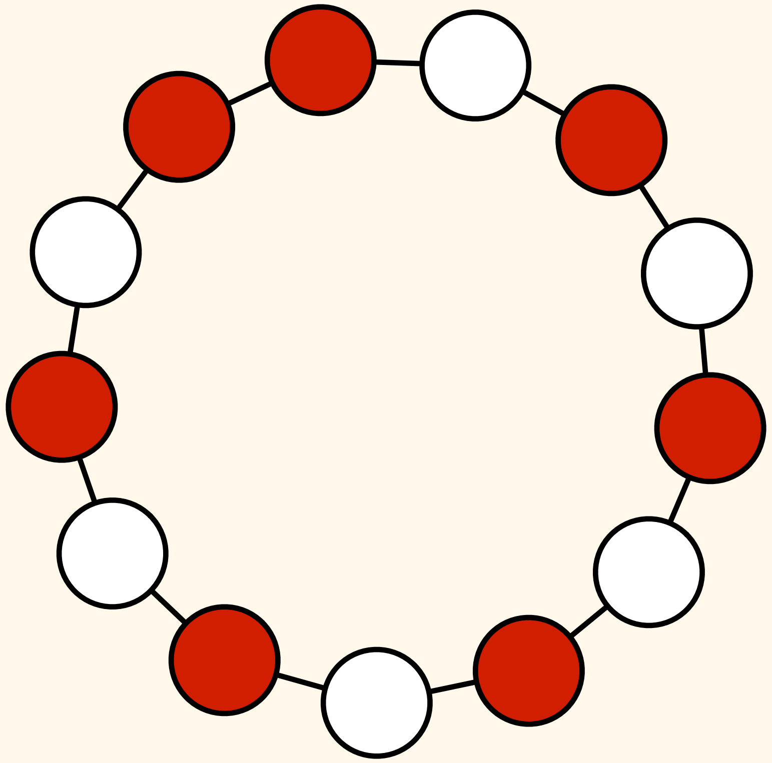
Tractable Problems

- Vertex covers
    - 2-approximation of vertex cover in bounded-degree graphs
    - Local algorithm
    - *Åstrand & Suomela* (2010)



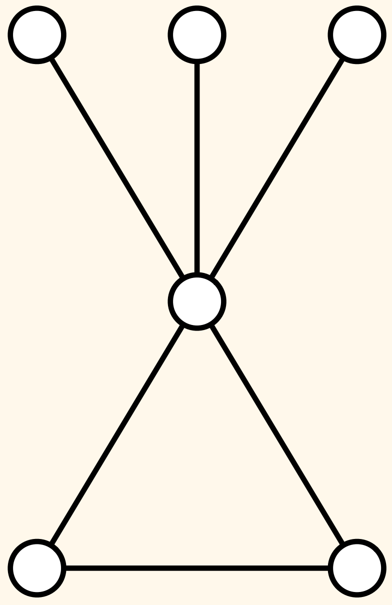
Tractable Problems

- Vertex covers
  - 2-approximation of vertex cover in bounded-degree graphs
  - Local algorithm
  - A bit complicated…
  - Let's have a look at a simpler local algorithm: *3-approximation* of vertex cover



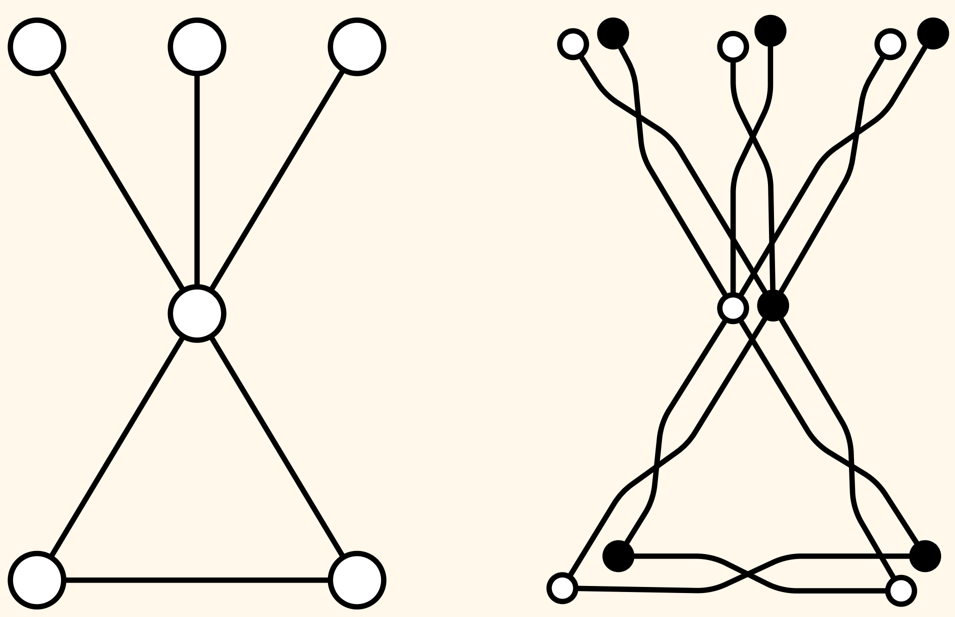
Vertex Cover

A simple local algorithm:
3-approximation of minimum vertex cover



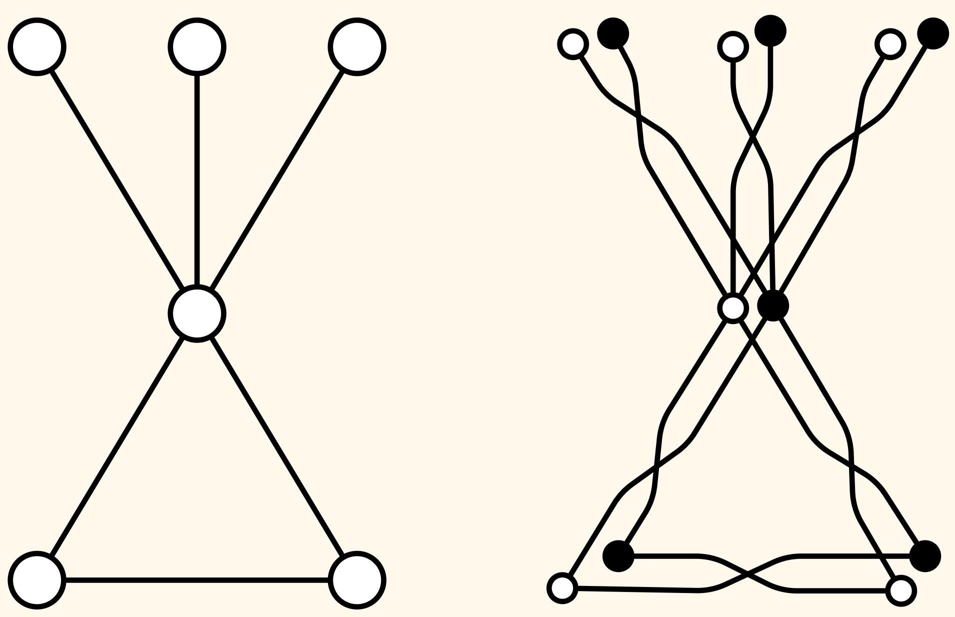
Vertex Cover

Construct a *virtual graph*:
two copies of each node; edges across



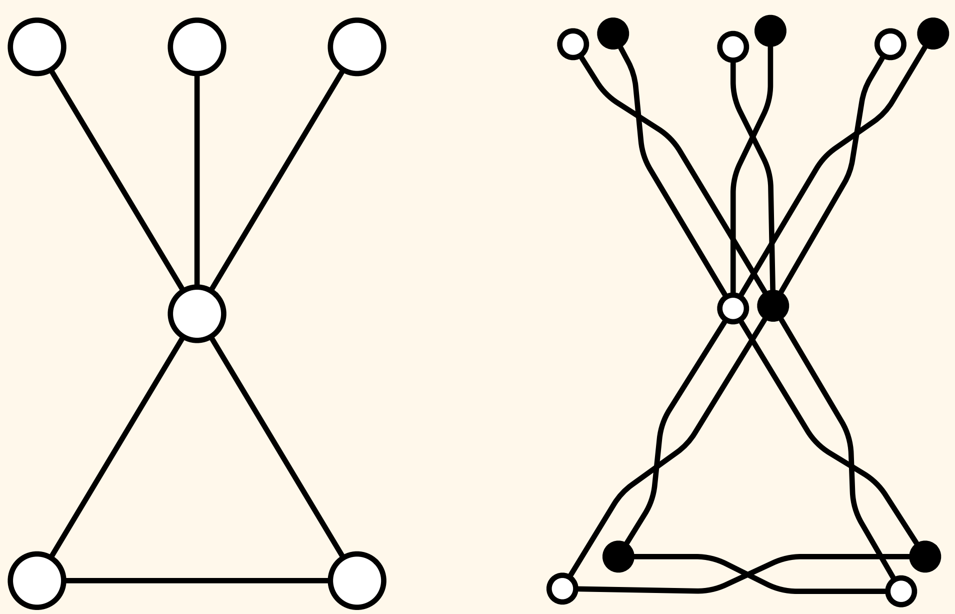
Vertex Cover

The virtual graph is *2-coloured*:
all edges are from white to black



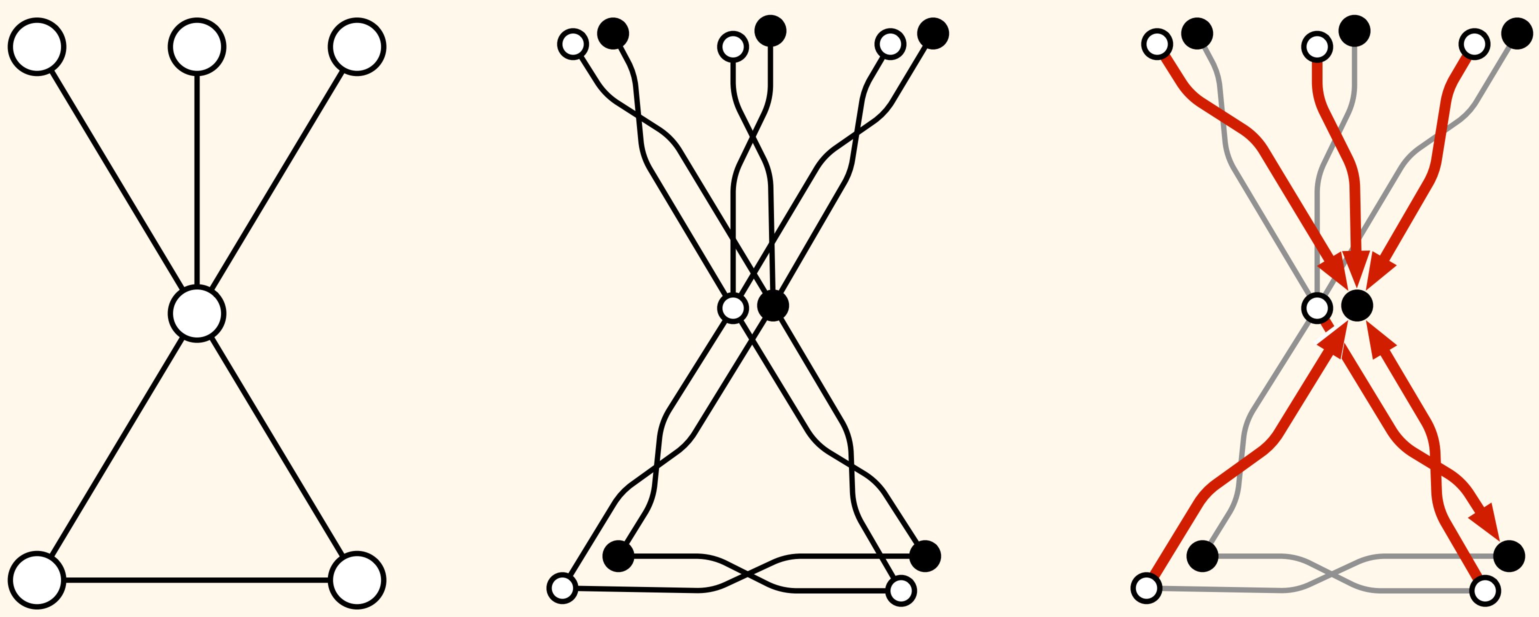
Vertex Cover

The virtual graph is 2-coloured –
– therefore we can find a *maximal matching*!



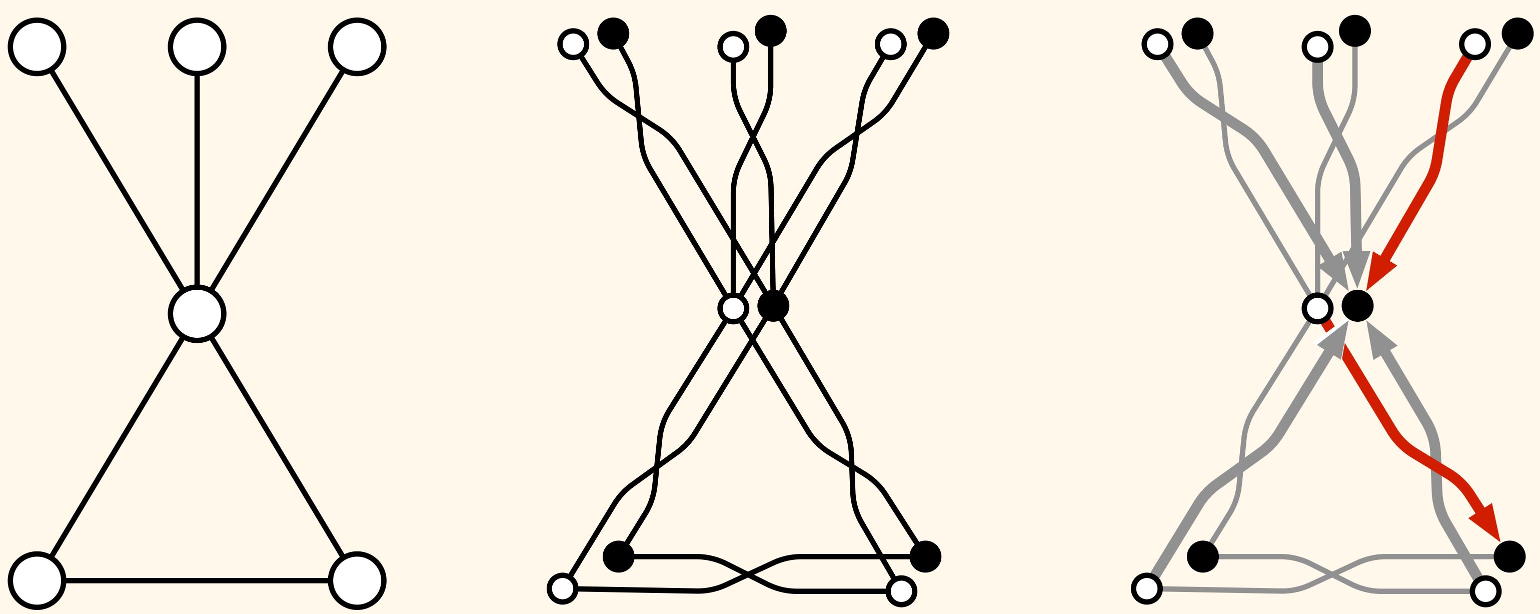
Vertex Cover

White nodes send *proposals* to their black neighbours



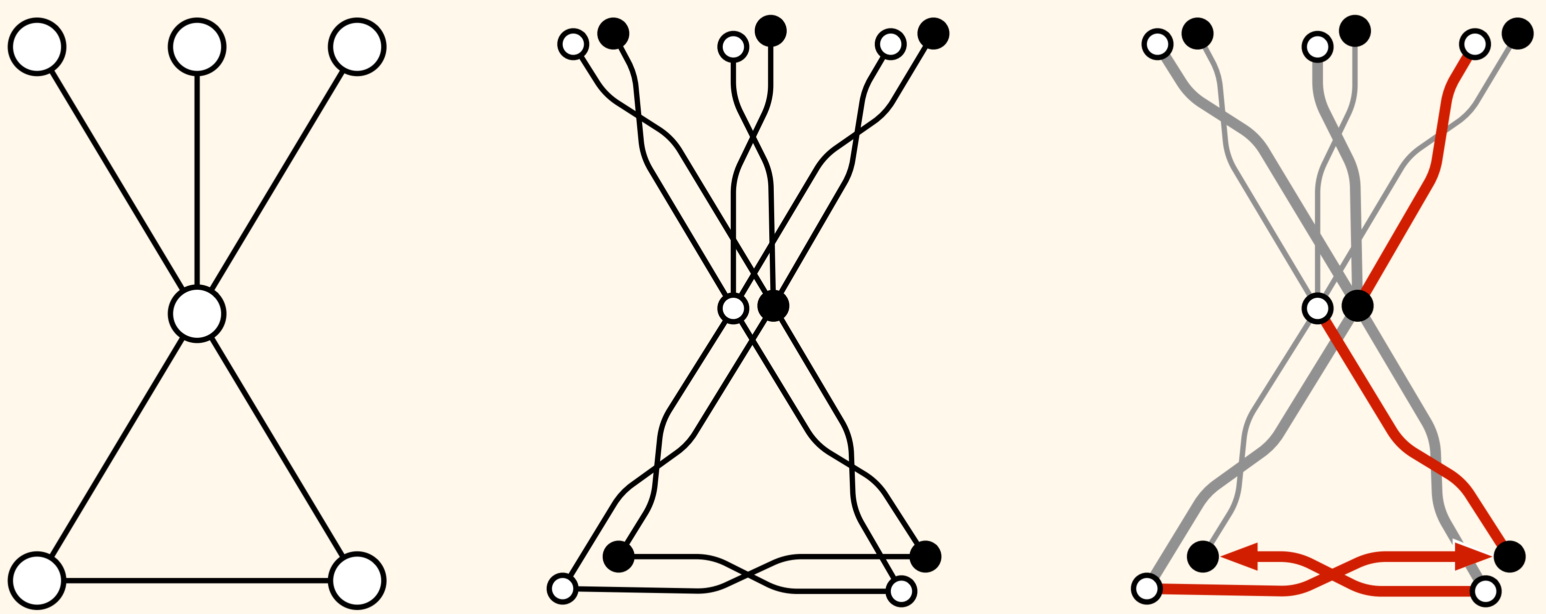
Vertex Cover

Black nodes *accept* one of the proposals



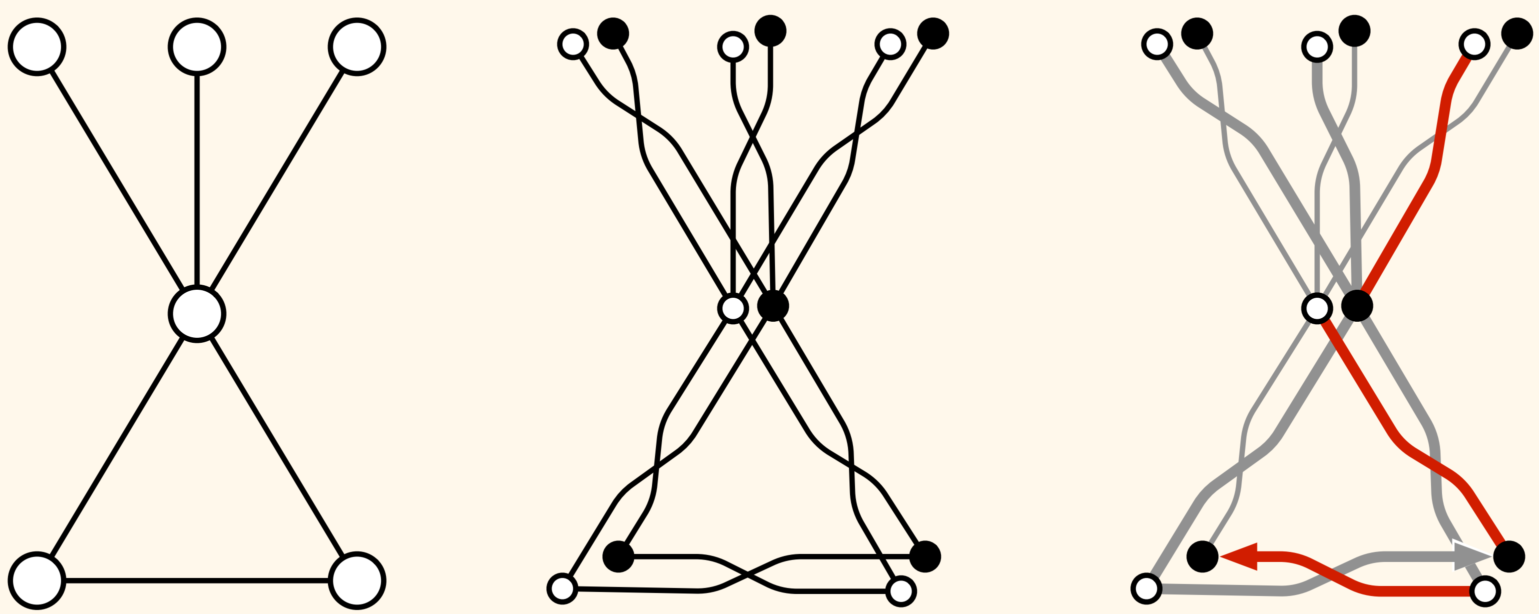
Vertex Cover

White nodes send *proposals* to another black neighbour if they were rejected



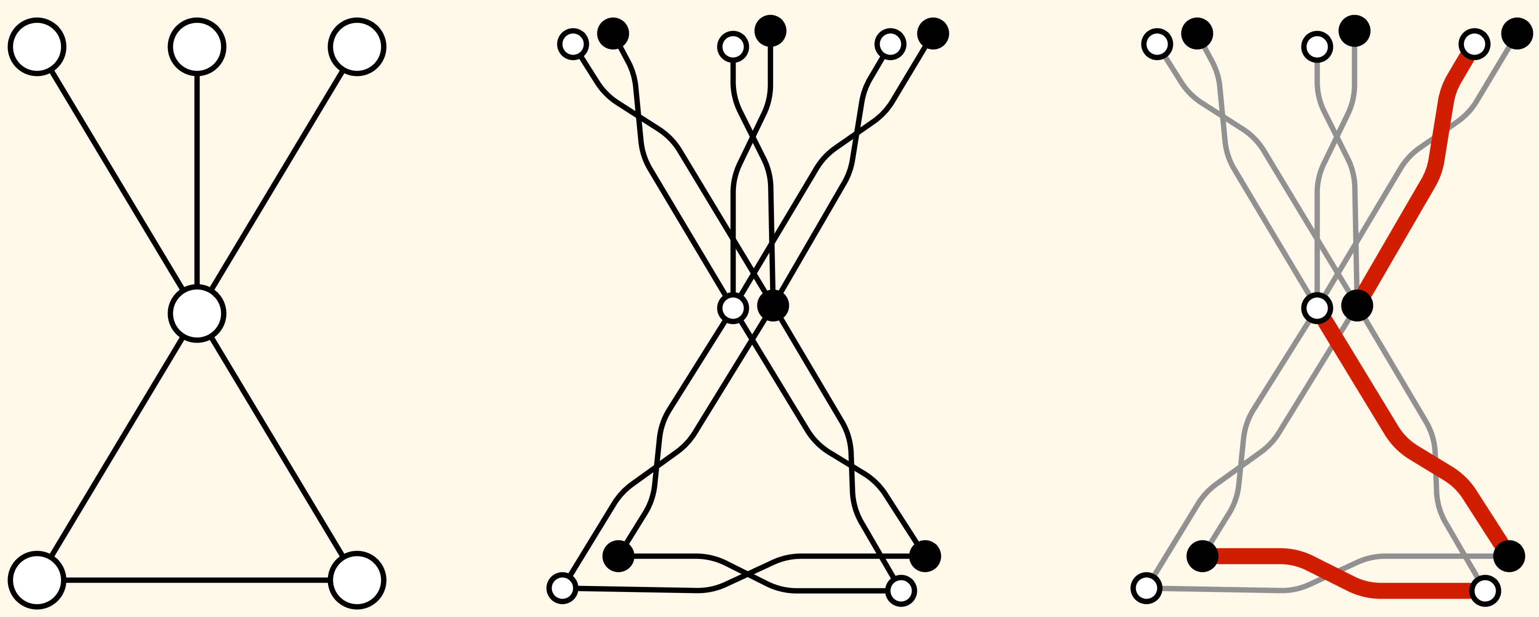
Vertex Cover

Again, black nodes *accept* one proposal – unless they were already matched



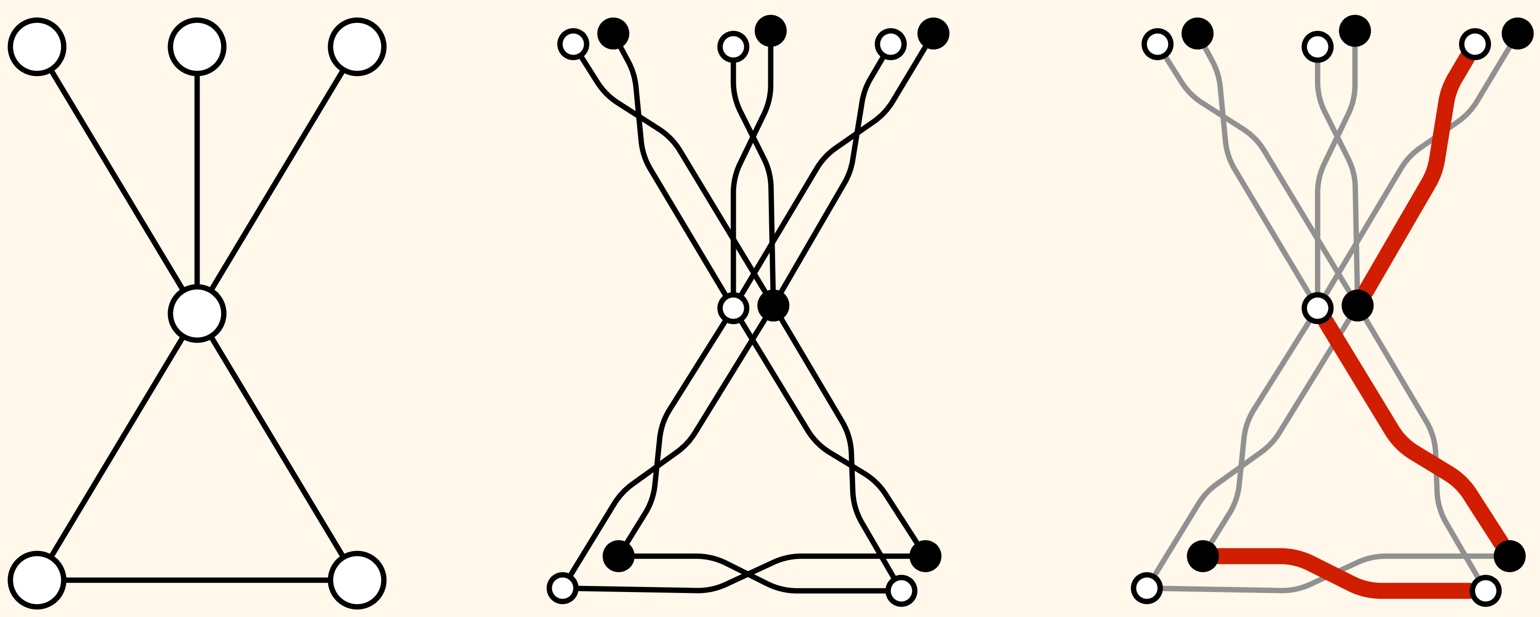
Vertex Cover

Continue until all white nodes are matched – or they are rejected by all black neighbours



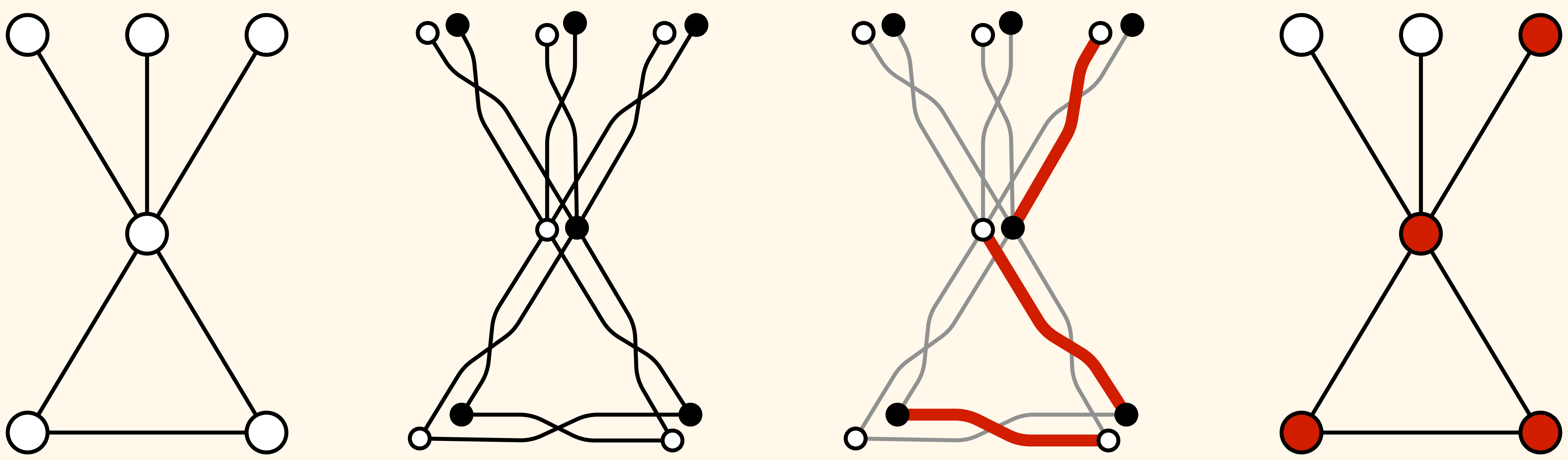
Vertex Cover

End result: a *maximal matching*
in the virtual graph



Vertex Cover

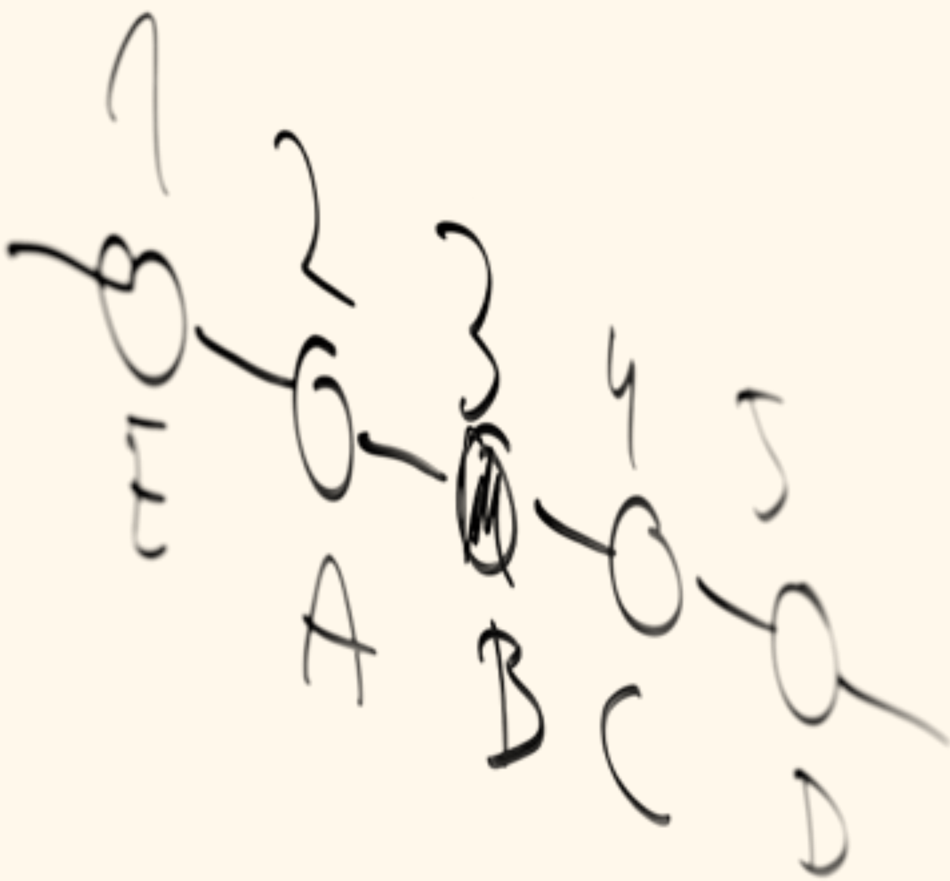
Take all original nodes that were matched –
3-approximation of minimum vertex cover!



Present: Summary

- You cannot break symmetry in cycles…
- But we can study problems
  that *do not require* symmetry breaking!
  - *Linear programs*: local approximation schemes
  - *Vertex covers*: local 2-approximation algorithm
  - *Edge dominating sets*: local approximation algorithm
  - …

Future

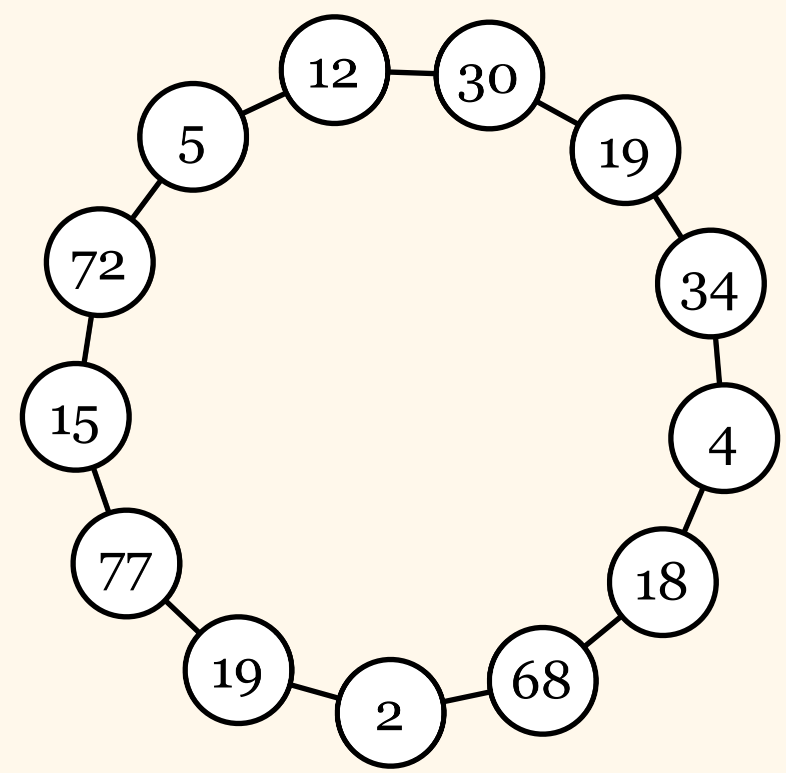


Dealing with Bad News

- Let's have a fresh look at the lower bounds!
  - Exactly what was proved?

Lower Bounds

- Only trivial solutions in cycles
- *Assumption*:
  constant-size output
  - Each node outputs
    constant number of bits
- Innocuous?



Output Size

- Vertex cover, independent set, dominating set, cut: 1 *bit per node*
- Matching, edge dominating set, edge cover: 1 *bit per edge*
    - In a cycle, this is $O(1)$ bits per node

Output Size

- Graph colouring:
  - $O(1)$ colours should be enough in a cycle
  - Hence *$O(1)$ bits per node* is enough to encode the solution
- Linear programs:
  - For a near-optimal solution, we can use *finite-precision rational numbers*

Output Size

- Natural problems seem to have constant-size output
- Hence the negative results apply
  - Unique identifiers do not help in cycles
  - We can only produce trivial solutions in cycles
  - We can only solve problems that do not require symmetry-breaking

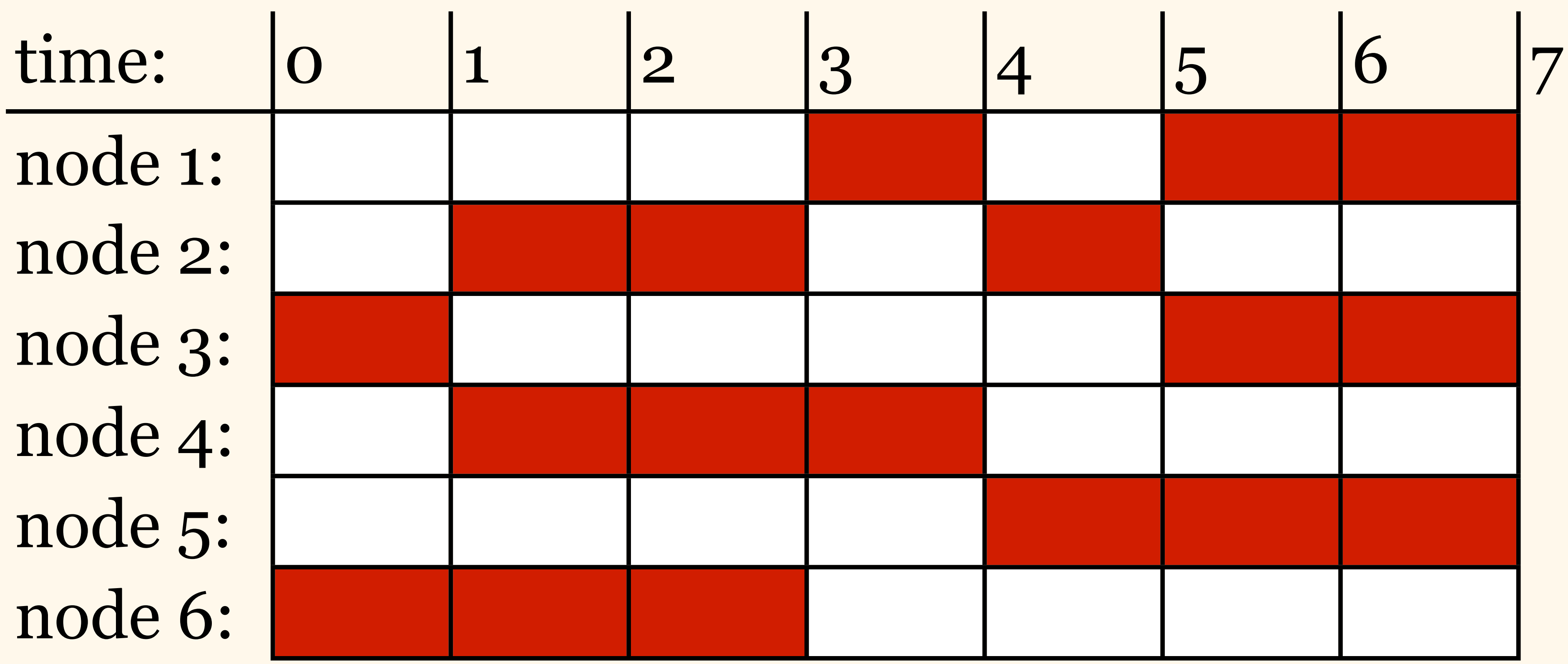
Output Size

- Natural problems seem to have constant-size output
- Hence the negative results apply

- *Did we miss anything?*

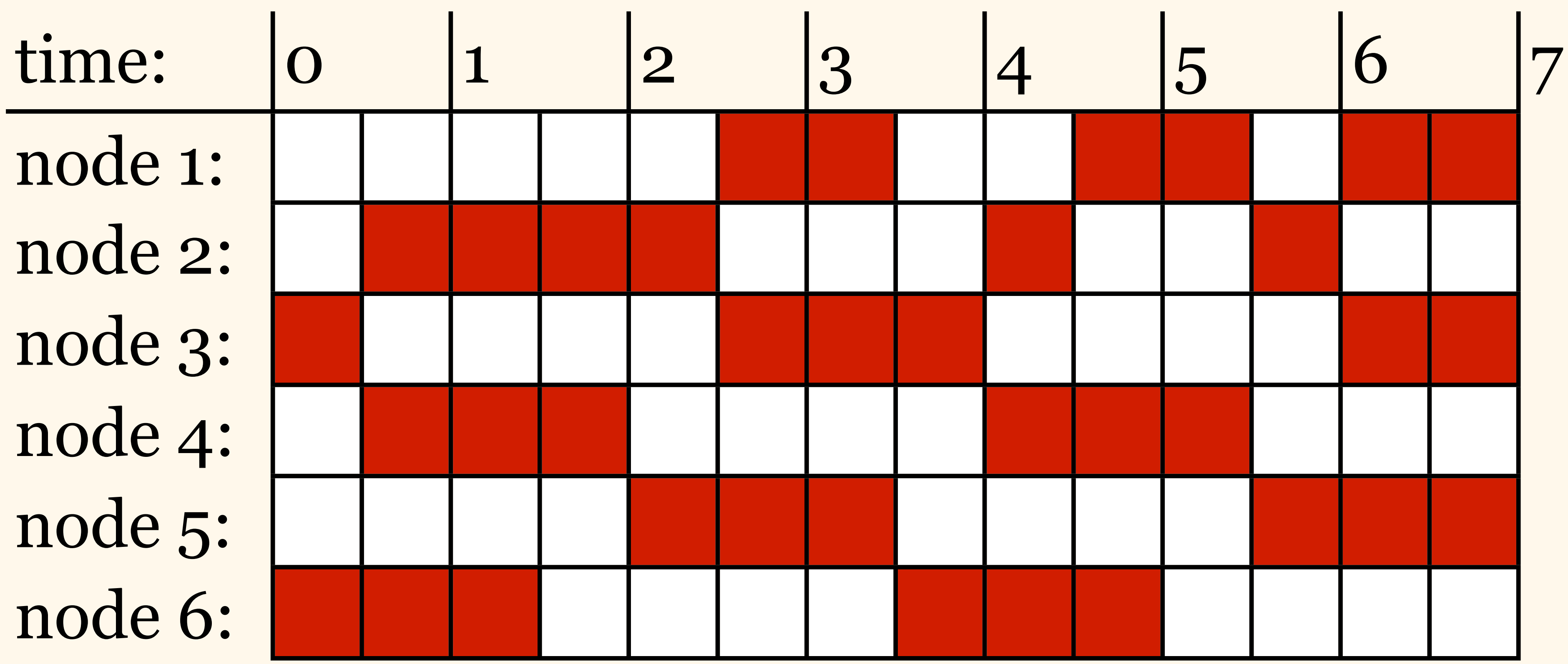
Scheduling Problems

- Neighbours not working simultaneously, everyone must get 3 units of work done



Scheduling Problems

- *Time is continuous*, we can use a more fine-grained schedule if it helps



Scheduling Problems

- Time is continuous, we can use a more fine-grained schedule if it helps
  - A good solution does not necessarily have a constant-size description
  - Existing lower bounds do not apply

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  - Proof artefact? Uninteresting technicality? Just derive a bit stronger negative result?

Scheduling Problems

- Time is continuous, we can use a more fine-grained schedule if it helps
  - A good solution does not necessarily have a constant-size description
  - Existing lower bounds do not apply
  - Proof artefact? Uninteresting technicality? Just derive a bit stronger negative result?
  - Wrong! *There is a local approximation algorithm!*

Scheduling Problems

- Local approximation algorithms
  - Scheduling problems:
    fractional graph colouring,
    fractional domatic partition, ...
  - First example of a local algorithm that
    actually requires unique numerical identifiers
  - *Hasemann, Hirvonen, Rybicki & Suomela*
    (work in progress)

More New Directions

- Deterministic local algorithm
  - cf. deterministic Turing machine – class P
- Randomised local algorithm
  - cf. probabilistic Turing machine – class BPP, etc.
- *Nondeterministic* local algorithm
  - cf. nondeterministic Turing machine – class NP

Decision Problems

- Back to very basics: *decision problems*
  - Is this graph bipartite? Acyclic? Hamiltonian? Eulerian? Connected? 3-colourable? Symmetric?
  - Decision problems form the foundation of classical complexity theory…

Decision Problems

- Decision problems in distributed setting:
  - *yes*-instance: all nodes happy
  - *no*-instance: at least one node raises alarm
- Few decision problems can be solved with deterministic local algorithms
  - But now we have a very natural extension…

Decision Problems

- *Nondeterministic* local algorithms
    - *Yes*-instances have a compact certificate that can be verified with a local algorithm
        - *"locally checkable proof"*
- Cf. class NP:
    - *Yes*-instances have a compact certificate that can be verified in P

Locally Checkable Proofs

- Key question: what is the size of the proof?
  - *How many bits per node are needed?*
  - For example, it is easy to show that a graph is bipartite: just give a 2-colouring, 1 bit per node
  - How do you prove that a graph is *not* bipartite?

Locally Checkable Proofs

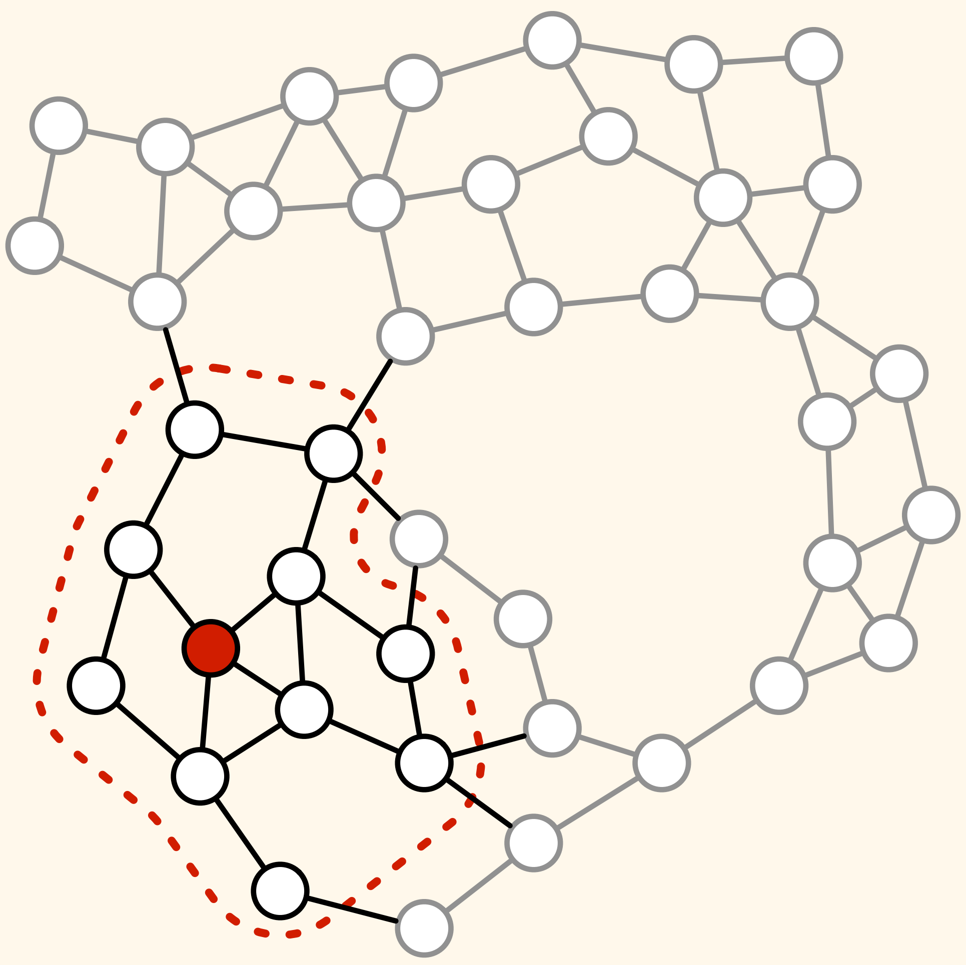
- Key question: what is the size of the proof?
  - *How many bits per node are needed?*
  - For example, it is easy to show that a graph is bipartite: just give a 2-colouring, 1 bit per node
  - How do you prove that a graph is *not* bipartite?
    - Find an odd cycle, and prove that it exists
    - $O(\log n)$ bits is enough, $\Omega(\log n)$ bits necessary

Locally Checkable Proofs

- Natural hierarchy of proof complexities:
  - 2-colourable graphs: $\Theta(1)$ bits per node
  - Non-2-colourable graphs: $\Theta(\log n)$ bits per node
  - Non-3-colourable graphs: $\mathrm{poly}(n)$ bits per node
  - *Göös & Suomela* (2011)

Summary

- Local algorithms
- Strong lower bounds
  - Nevertheless,
    a lot of progress!
- Latest hot topics
  - Scheduling problems
  - Nondeterministic models



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