Hamilton cycles in the random geometric graph

Nick Wormald

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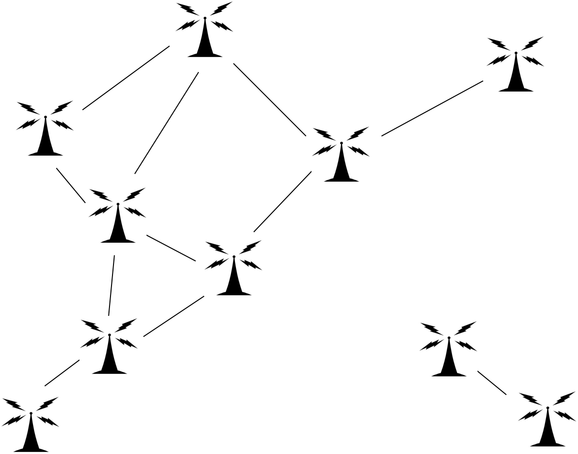
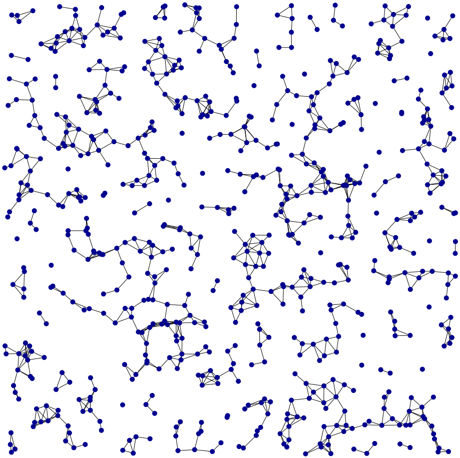
joint work with Tobias Müller and *Xavier Pérez Giménez

(*also contributed to presentation)

Wireless networks



Random geometric graph



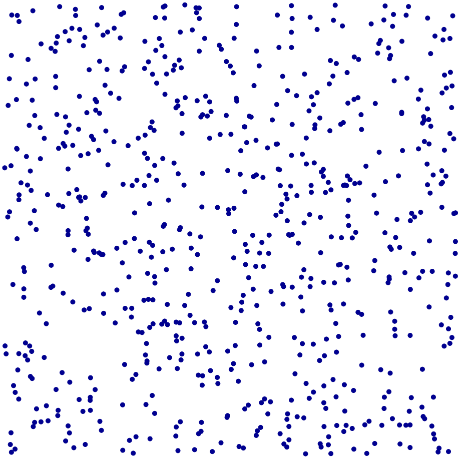
(Gilbert 1961)

n vertices

radius $r = r(n)$

$n \to \infty$

Random process: $0 \leq r \leq \sqrt{2}$



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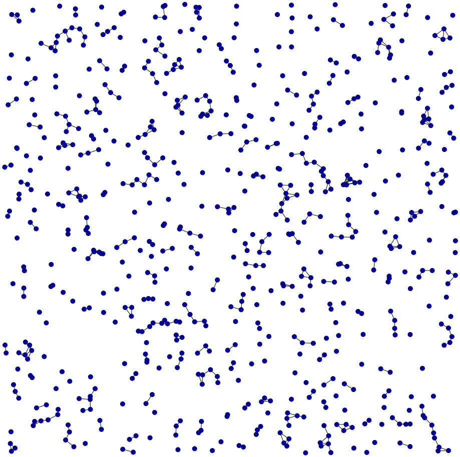

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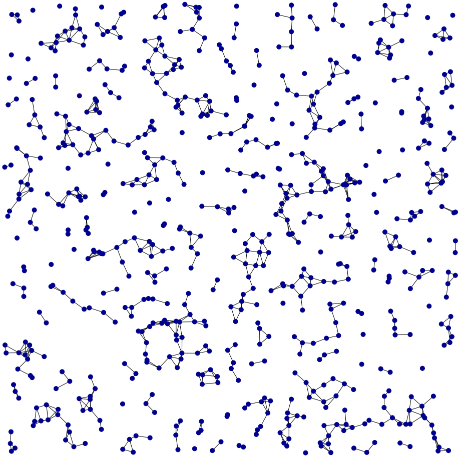

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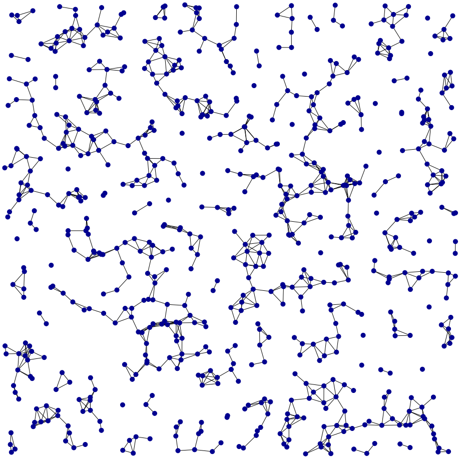
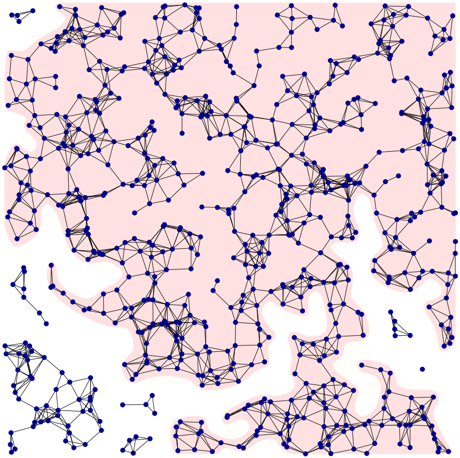


no giant component yet

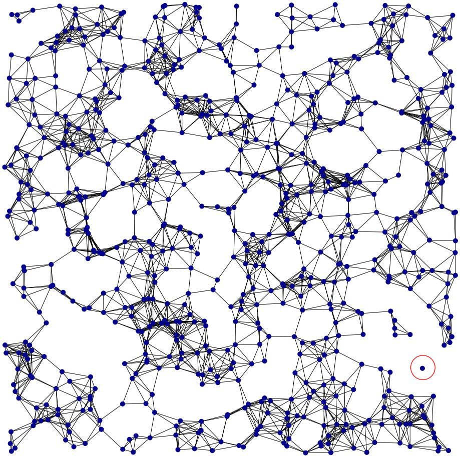
Random process: $0 \le r \le \sqrt{2}$



$$r \sim \sqrt{C/n}$$

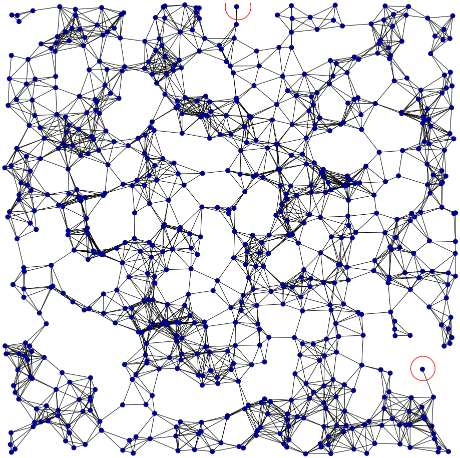
giant component!

Random process: $0 \le r \le \sqrt{2}$



still disconnected!

Random process: $0 \leq r \leq \sqrt{2}$



connected

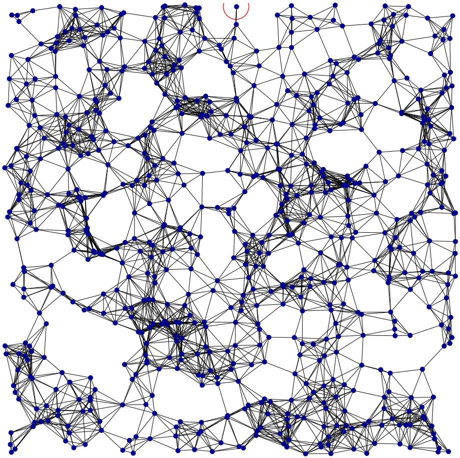
=

no isolated vertices

(a.a.s.)

$$r = \sqrt{\frac{\log n + O(1)}{\pi n}}$$

Random process: $0 \leq r \leq \sqrt{2}$



2-connected

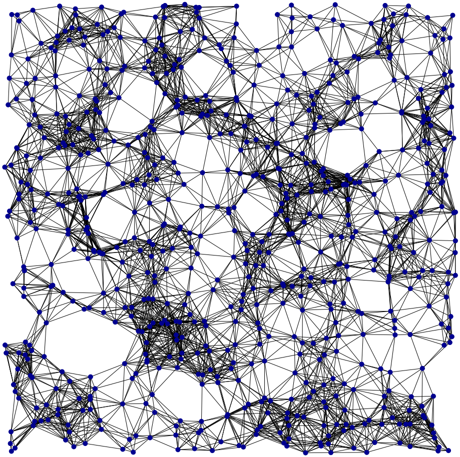
=

no deg. 1 vertices

(a.a.s.)

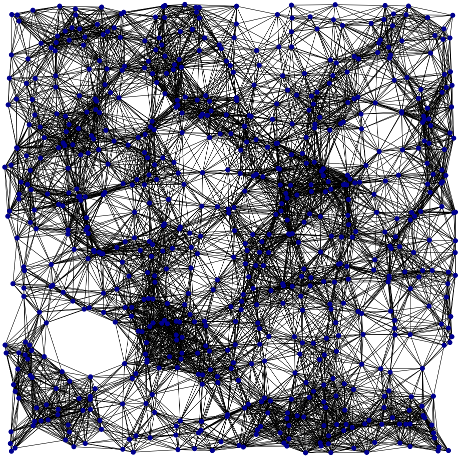
$$r = \sqrt{\frac{\log n + \log\log n + O(1)}{\pi n}}$$

Random process: $0 \le r \le \sqrt{2}$



higher connectivity

Random process: $0 \leq r \leq \sqrt{2}$

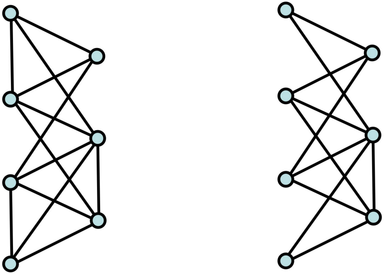


still large diameter:

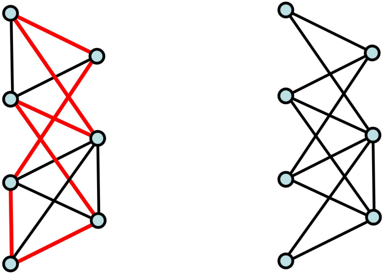
$\Theta(1/r)$

bad expansion

What about hamilton cycles?



What about hamilton cycles?



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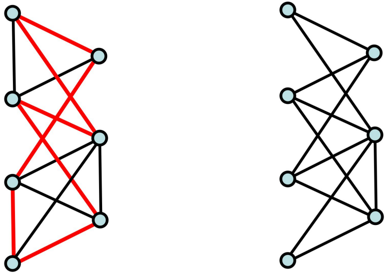
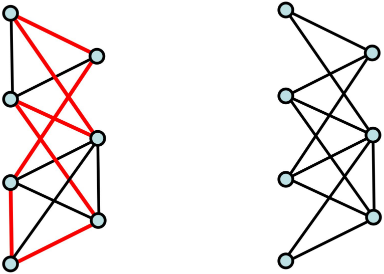


Necessary conditions:

min. deg. ≥ 2
2-connected

What about hamilton cycles?



Necessary conditions:

min. deg. ≥ 2
2-connected

Are they sufficient for the RGG?

Hamilton cycles in random graphs

$\mathcal{G}(n, m)$ is the random graph with n vertices and m edges chosen randomly ...

... a snapshot of the random graph process at time m.

Thm (Bollobás 1984)

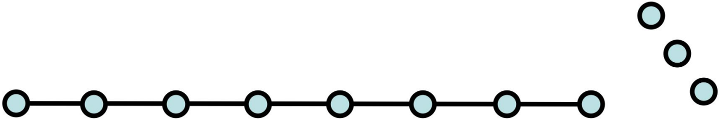
Asymptotically almost surely, the first edge to give the graph min degree 2 also gives it a Hamilton cycle.

Thm (Bollobás and Frieze 1985)

Asymptotically almost surely, the first edge to give the graph min degree k also gives it $k/2$ edge-disjoint Hamilton cycles.

Proof technique for random graphs

Based on Pósa's idea from 1976.



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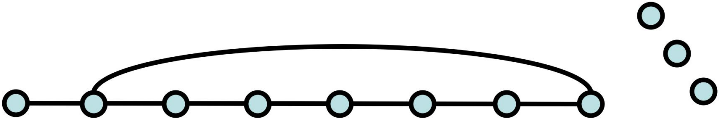

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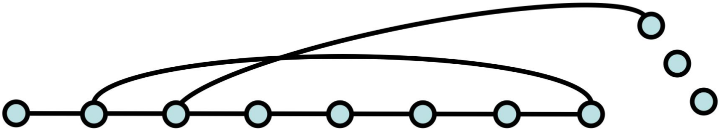

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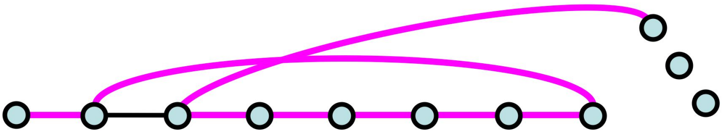
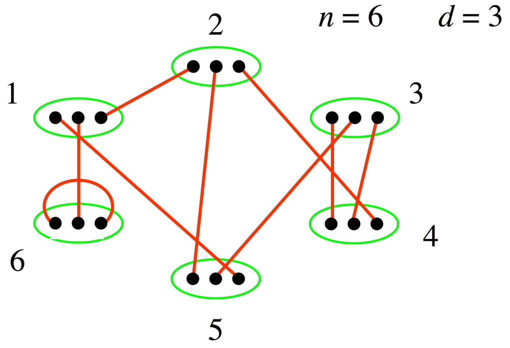


Hamilton cycles in random regular graphs

$\mathcal{G}_{n,d}$: d-regular graph on n vertices chosen uniformly at random.

Pairing model

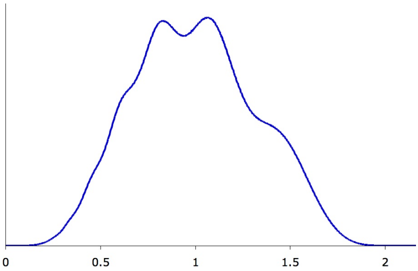


Hamilton cycles in random regular graphs

Let Y_n be number of Hamilton cycles in $\mathcal{G}_{n,3}$.

Then $\mathbf{E}\, Y_n \sim e\sqrt{\frac{\pi}{2n}}\left(\frac{4}{3}\right)^{n/2}$.

Density of $Y_n/\mathbf{E}\,Y_n$:



Earlier results on RGG

In RGG, edges are added in increasing length.

Thm (Penrose 1999)

Asymptotically almost surely, the edge making the RGG have minimum degree k also makes it k-connected, and this happens for $r \sim \sqrt{(\log n)/\pi n}$.

Thm (Petit 2001)

The RGG with $r = \sqrt{\omega(\log n)/n}$ a.a.s. has a Hamilton cycle.

Thm (Díaz, Mitsche & Pérez Giménez 2007)

For any $\epsilon > 0$, the RGG with $r \geq (1+\epsilon)\sqrt{\frac{\log n}{\pi n}}$ a.a.s. has a Hamilton cycle.

(And extensions to general ℓ_p norm.)

Recent results

Thm (Balogh, Bollobás, Krivelevich, Müller, Pérez Giménez, Walters & W. 2010)

In the RGG process:
Hamiltonian $\iff$ min. deg. ≥ 2 (a.a.s.)
(extension to general dimension and ℓ_p norm)

Thm (Balogh, Bollobás & Walters 2010)

Weaker analogue for the k-Nearest Neighbour Graph.

Thm (Krivelevich & Müller 2010)

Pancyclic $\iff$ min. deg. ≥ 2 (a.a.s.)

Recent results

Thm (Balogh, Bollobás, Krivelevich, Müller, Pérez Giménez, Walters & W. 2010)

In the RGG process:
Hamiltonian $\iff$ min. deg. ≥ 2 (a.a.s.)
(extension to general dimension and ℓ_p norm)

Thm (Müller, Pérez Giménez & W. 2010)

$k/2$ disjoint Hamilton cycles $\iff$ min. deg. $\geq k$ (a.a.s.)
(extension to general dimension and ℓ_p norm)

For k odd there is an additional disjoint perfect matching.

Proof for disjoint Hamilton cycles

Preliminaries

From Penrose (2003):
Let r_k be the smallest r such that RGG is k-connected.

Then

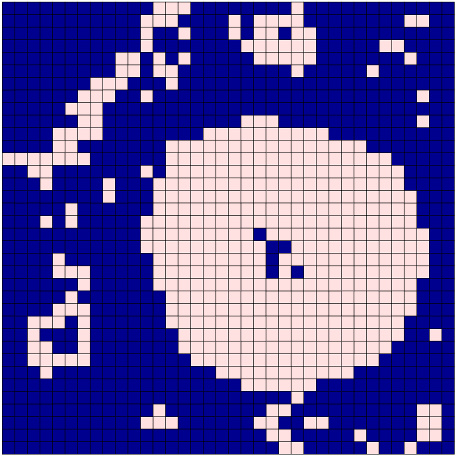
$$\pi n r_k^2 - \log n - (2k-3)\log\log n$$

is bounded in probability.

Relevant r:

$$r = \sqrt{\frac{\log n + m \log\log n + \lambda}{\pi n}}$$

First step: tesselation

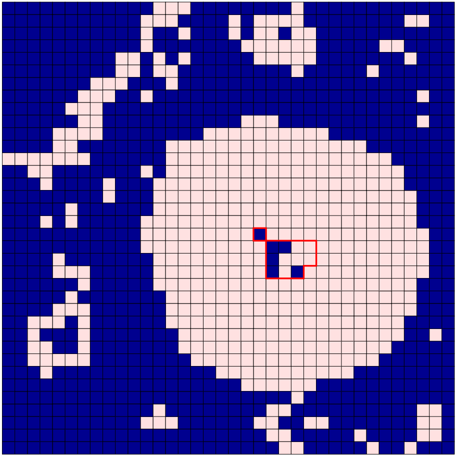


$$r = \sqrt{\frac{\log n + O(\log\log n)}{\pi n}}$$

$\square \updownarrow \delta r$

- dense ($\geq M$ points)
- sparse ($< M$ points)

First step: tesselation



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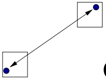
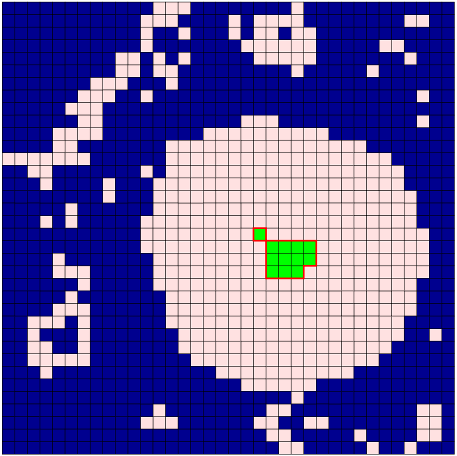
sparse ($< M$ points)



dist. $\leq r$

First step: tesselation

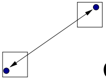


$$r = \sqrt{\frac{\log n + O(\log\log n)}{\pi n}}$$

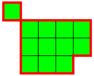
$\updownarrow \; \delta r$

dense ($\geq M$ points)

sparse ($< M$ points)



dist. $\leq r$



bad cells

Simple computations:

P(a given cell is sparse)

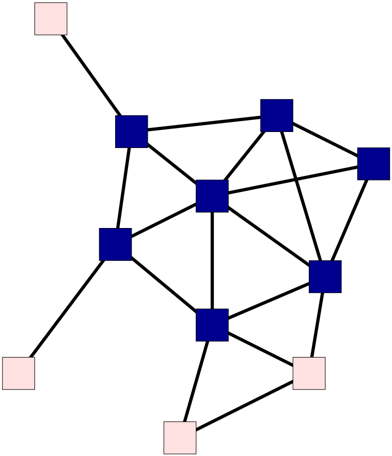
$$= \sum_{t=0}^{M-1} \binom{n}{t} (\delta^2 r^2)^t (1 - \delta^2 r^2) n - t$$

$$= (\Theta(\delta^2 \log n))^{M-1} (\log n)^{O(1)} n^{-\pi\delta^2}$$

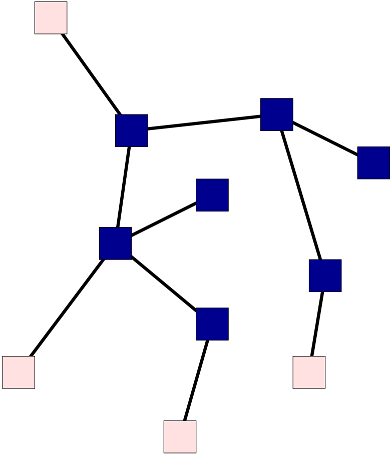
... after some computations ...

Every bad component is "small".

Hamilton cycles: large-scale template



Hamilton cycles: large-scale template



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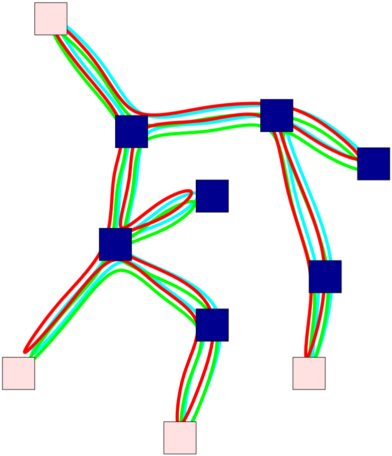
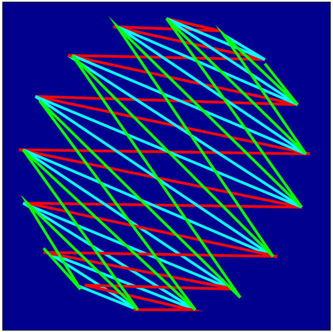


Rerouting at dense cells



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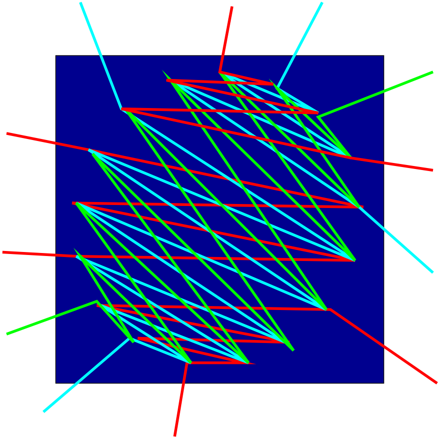
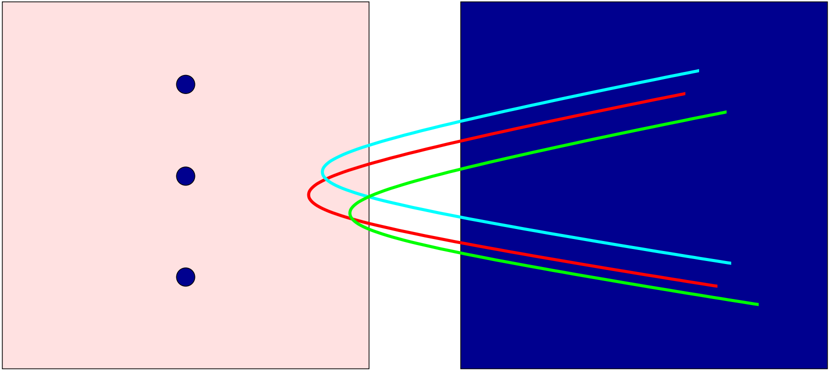


Extension into sparse good cells



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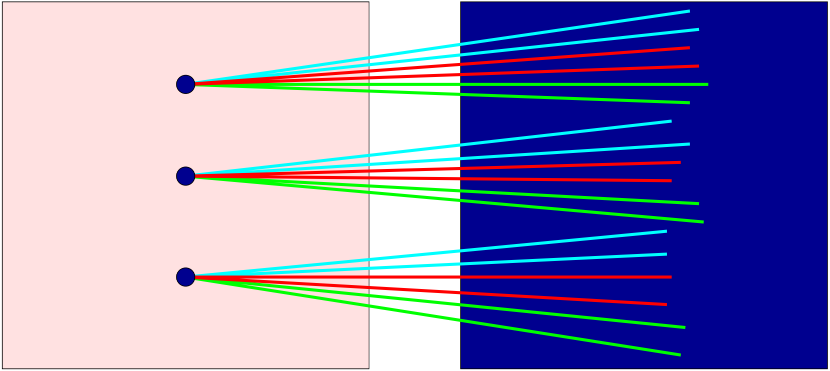

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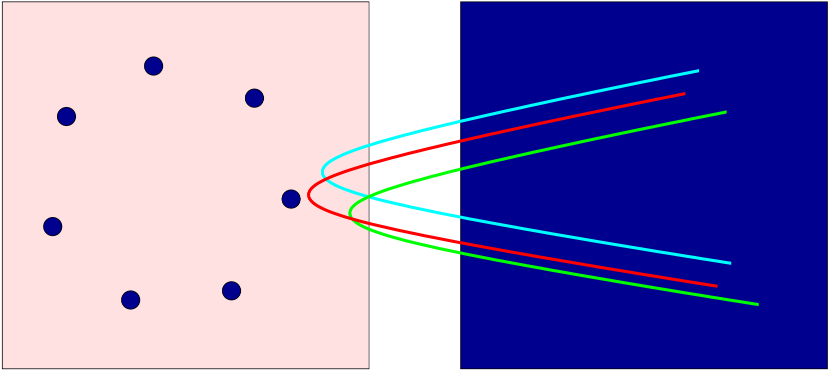

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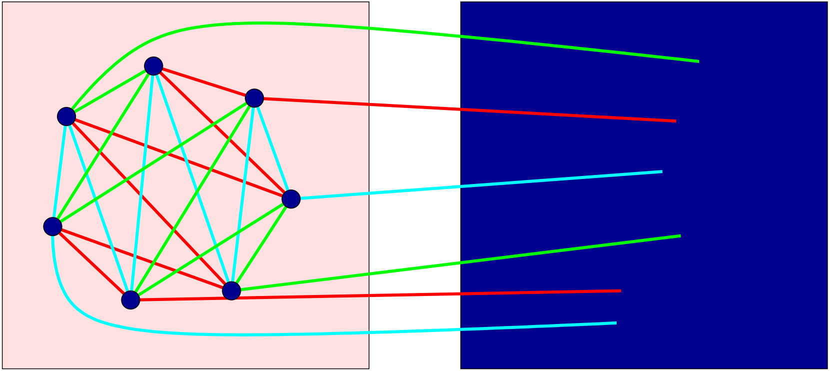
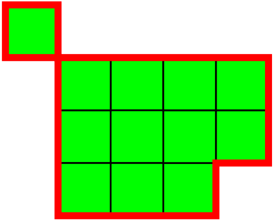
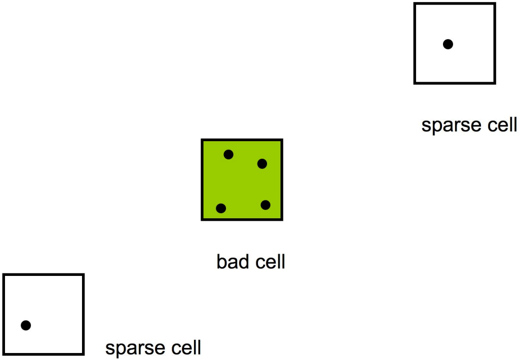


Extension into bad cells



a lot harder!

$k = 4$



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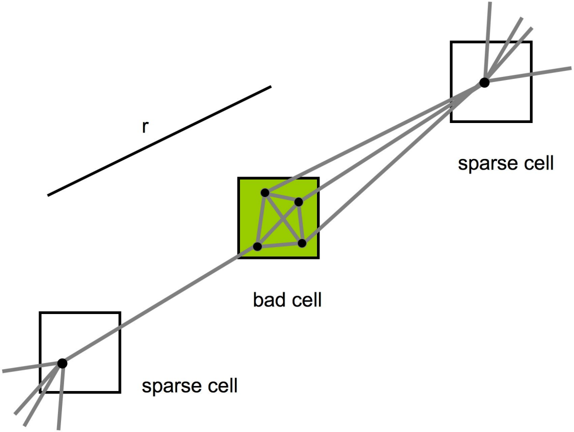
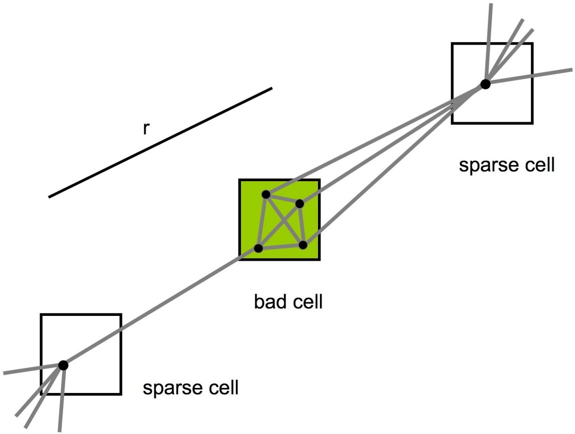


$k = 4$



But not 4-connected.

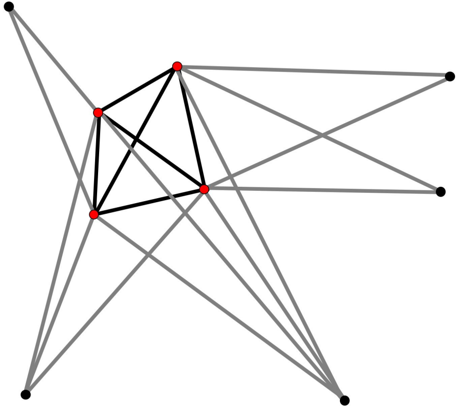
First solution for bad cells

Let graph G consist of a clique on vertex set J, $|J| = j$, and a bipartite graph H with parts J and B, where

- each vertex in J has degree at least k;
- for each v, $v' \in J$, $|N_G(v) \cup N_G(v') \setminus \{u, v\}| \geq k$;
- some vertex in J has degree at least $k + 1$.

Then G contains a packing of $k/2$ edge-disjoint linear forests, with each vertex in J of degree 2 in each forest.

$k = 6$



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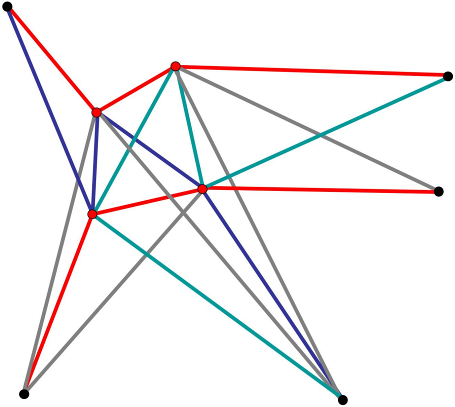

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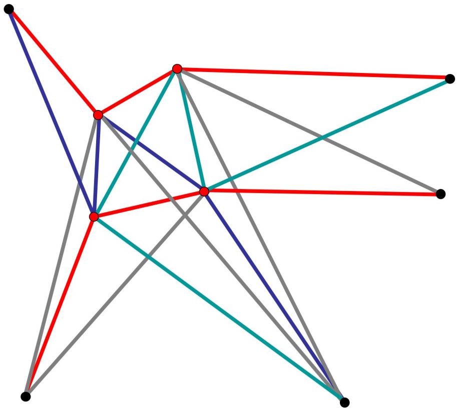

[Conjecture that ‘degree $\geq k + 1$’ condition unnecessary.]

Second solution for bad cells

For sufficiently small $\eta > 0$ and relevant *r*, every set of $j \geq 2$ vertices in a circle of radius ηr (satisfying a certain max degree condition) has *k* common neighbours.

Open question

What if k is not fixed? In particular:

Are there a.a.s. $\lfloor \frac{\delta(RGG)}{2} \rfloor$ edge disjoint Hamilton cycles?