Randomized Linear Programming in low dimension

By Katie Duong
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What is a linear program?

- Given a linear function.
- Given linear constraints.
- Compute the maximum of the function, subject to the constraints.
Maximize \[ c_1 x_1 + c_2 x_2 + \cdots + c_d x_d \]

Subject to \[ a_{1,1} x_1 + \cdots + a_{1,d} x_d \leq b_1 \]

\[ a_{2,1} x_1 + \cdots + a_{2,d} x_d \leq b_2 \]

\[ \vdots \]

\[ a_{n,1} x_1 + \cdots + a_{n,d} x_d \leq b_n \]

\[ d : \text{number of variables.} \]

\[ \text{dimension of the linear program} \]

\[ n: \text{number of constraints} \]
Example: Given linear function $2x - y$
We want to maximize it.

Subject to:
- $-x + y \leq 2$
- $-3x - y \leq 3$
- $-y \leq 0$
- $x + y \leq 2$
- $-x \leq 3$

Equivalent
- $y \leq x + 2$
- $y \geq -3x - 3$
- $y \geq 0$
- $y \leq -x + 2$
- $x \geq -3$

Called Constraints
Max $2x-y$

Goal: Find optimal vertex without computing all vertices

Optimal vertex
After rotation: Given linear function $y$. We want to maximize it, subject to $n$ linear constraints.

In feasible region: find the highest point.
Constraints 1 and 2

Assume we know highest vertex $\text{opt}_i$ for constraints 1, ..., $i$

now add constraint $i+1$

How to compute the next highest vertex $\text{opt}_{i+1}$?
If $\text{opt}_i$ satisfies constraint $i+1$: $\text{opt}_{i+1} = \text{opt}_i$

good case: constant running time
If $\text{opt}_i$ does not satisfy constraint $i+1$: $\text{opt}_{i+1}$ on boundary of constraint $i+1$
opti+1 on line that bounds constraint i+1

Compute opti+1 \(O(i)\) time
Step $i+1$: \[ \begin{cases} O(1) & \text{if } \text{opt}_i \text{ satisfies constraint } i+1 \\ O(i) & \text{Otherwise} \end{cases} \]

Total time ( $i=2,3,4,\ldots,n-1$): 
Worst-case: $2+3+4+\ldots+(n-1) = O(n^2)$
Step $i+1$: 
\[
\begin{cases} 
O(1) & \text{if } \text{opt}_i \text{ satisfies constraint } i+1 \\
O(i) & \text{Otherwise}
\end{cases}
\]

Total time ($i=2, 3, 4, \ldots, n-1$): 
Total time: $1 + 1 + \ldots + 1 = O(n)$
Running time depends on numbering of constraints.

Take random numbering (out of n! possible numberings)
$X = \text{Total running time}$

$$X_i = \begin{cases} 
1 & \text{if opt}_i \text{ does not satisfies constraint } i+1 \\
0 & \text{if opt}_i \text{ satisfies constraint } i+1 
\end{cases}$$

$$X = O(n) + \sum_{i=2}^{n-1} X_i \cdot O(i)$$

$$E(X) = O(n) + \sum_{i=2}^{n-1} E(X_i) \cdot O(i)$$

$$E(X_i) = \Pr(X_i = 1)$$
opti

add constraint i+1

opti+1

delete constraint i+1

\[ \Pr(X_i = 1) = \frac{2}{i+1} \]
\[ x = O(n) + \sum_{i=2}^{n-1} X_i \cdot O(i) \]

\[ E(x) = O(n) + \sum_{i=2}^{n-1} E(X_i) \cdot O(i) \]

\[ E(X_i) = \Pr(X_i = 1) \]

\[ E(x) = O(n) + \sum_{i=2}^{n-1} \frac{2}{i+1} \cdot O(i) \]

\[ E(x) = O(n) + O(n) = O(n) \]