Assignment 6

Usual Instructions

Fall 2017

1. Assume all edges in a simple undirected connected weighted graph $G$ have distinct cost. Show that the edge with the maximum cost in any cycle in $G$ cannot be in the Minimum Spanning Tree of $G$. Can you use this to design an algorithm for computing MST of $G$ by deletion of edges, and what will be its complexity?

2. Let $F$ be a spanning tree of simple connected weighted graph $G$. Show that $F$ is a minimum spanning tree of $G$ if and only if every edge in $E \setminus F$ is $F$-heavy.

3. Solve the running time recurrence for the randomized MST algorithm. Running time $T(|V|,|E|)$ of the algorithm on a graph $G = (V,E)$ is expressed as $T(|V|,|E|) = O(|V| + |E|) + T(|V|/8,|E|/2) + T(|V|/8,|V|/4)$.
Show that this recurrence solves to $O(|V| + |E|)$.

4. Assume that you have a set $P$ of $n$ distinct numbers. Form a sequence $S$ of these numbers by a taking a random permutation of elements of $P$. We compute the minimum element, $MIN(P)$, of this set by an incremental algorithm as follows:

(a) $MIN(P) := S[1]$;

(b) For $i := 2$ to $n$
if $S[i] < MIN(P)$, $MIN[P] := S[i]$ 

What is the expected number of times that $MIN(P)$ will be updated in Step (b) of the above algorithm?

5. Let $X$ be the total number of heads obtained in a sequence of $n$ independent flips of a fair coin. We know that the expected value of $X$ is $\frac{n}{2}$. Using Chernoff bounds compute the following probabilities:

(a) $Pr(|X - \frac{n}{2}| > \frac{1}{2}\sqrt{6n\ln n})$

(b) $Pr(|X - \frac{n}{2}| > \frac{n}{4})$

(c) and evaluate the above expressions for different values of $n$ and make some remarks on the values you obtain.

6. You are given a large integer $n$ and processes $P_1, P_2, \ldots, P_n$, each of which tries to access a single shared database. We assume that time is being divided into discrete rounds. The database can be accessed by at most one process in a single round.
Thus, if two or more processes try to access the database simultaneously, then all processes are “locked out” for the duration of that round.

Consider the following algorithm (where $T$ is a large integer):

- In round $t$, where $t$ runs from 1 to $T$:
  - Each process $P_i$ ($1 \leq i \leq n$) flips a coin that comes up head with probability $p = 1/n$. Let $f_i$ denote the outcome (“head” or “tail”) of $P_i$’s coin.
  - If there is exactly one $i$ such that $f_i$ is “head”, then the corresponding process $P_i$ accesses the database.

Answer the following:

(a) Consider a fixed index $i$ and a fixed round $t$. Let $A_{it}$ be the event $A_{it}$: process $P_i$ accesses the database in round $t$.

Determine $\Pr(A_{it})$.

(b) Consider a fixed index $i$. Let $NA_i$ be the event $NA_i$: process $P_i$ does not access the database in any of the rounds 1, 2, $\ldots$, $T$.

Prove that $\Pr(NA_i) \leq e^{-T/(en)}$.

(Hint: $1 - x \leq e^{-x}$ and for large $n$, $(1 - 1/n)^{n-1} \geq 1/e$.)

(c) Let $T = 2en \ln n$ and let $E$ be the event $E$: each process $P_i$ accesses the database at least once during the rounds 1, 2, $\ldots$, $T$.

Prove that $\Pr(E) \geq 1 - 1/n$.

7. You are given a list $L$ of elements and want to choose a random element in this list. Each element of $L$ should have the same probability of being chosen. Unfortunately, you do not know the number of elements in $L$. You are allowed to make only one pass over the list. Consider the following algorithm:

\begin{verbatim}
Algorithm ChooseRandomElement(L):
  u = first element of L;
  i = 1;
  while u exists do
    with probability $1/i$, set $x = u$;
    u = successor of u in L;
    i = i + 1
  endwhile;
  return x
\end{verbatim}

Prove that the output $x$ of this algorithm is indeed a random element of $L$. In other words, prove the following: Let $v$ be an arbitrary element of $L$. Then, the probability that $x = v$ after ChooseRandomElement($L$) has terminated is equal to $1/n$, where $n$ is the number of elements in $L$. 

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