Edge Coloring of Graphs
Advanced Algorithms Seminar

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Outline

1. Definitions

2. Misra and Gries’ Edge-Coloring Algorithm
Outline

1 Definitions

2 Misra and Gries’ Edge-Coloring Algorithm
Graph Colorings

- Assignment of labels (colors) to elements of the graph subject to some constraints.
- Most common form: **vertex coloring**: mapping $c : V(G) \rightarrow C$. If for any edge $(u, v) \in E(G)$, $c(u) \neq c(v)$, the coloring is proper.
- Problem: what is the minimum of colors required such that a graph $G$ has a proper vertex coloring? This is a graph parameter denoted $\chi(G)$, the chromatic number of $G$. 
Graph Colorings

**Figure:** Improper coloring

**Figure:** Proper coloring
Graph Colorings

- In this talk: **edge coloring**: mapping $c : E(G) \rightarrow C$. An edge $(x_1, x_2)$ is adjacent to an edge $(y_1, y_2)$ if $x_1 = y_1$ or $x_2 = y_2$. If for any edge $e \in E(G)$, $c(e) \neq c(e') \forall e' : e$ is adjacent to $e'$, the coloring is proper.

- Minimum number of colors such that a graph $G$ has a proper edge coloring is denoted $\chi'(G)$, the chromatic index of $G$. 
Graph Colorings

Figure: Improper edge coloring

Figure: Proper coloring
Edge Colorings

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Edge Colorings

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- Yes: Vizing’s Theorem states that

$$\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1.$$ 

Some graphs have $\chi'(G) = \Delta(G)$ (class 1), some others have $\chi'(G) = \Delta(G) + 1$ (class 2).
Edge Colorings

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- However, it is NP-complete to decide which is right for a general graph.
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However, it is NP-complete to decide which is right for a general graph.

Still, there are polynomial time algorithms to find a $\Delta(G) + 1$ edge coloring of any graph.
Some Interesting Results

- **Erdos & Wilson (1977):** Almost all graphs are of class 1. In the random graph model, if $p(n)$ is the probability that a graph on $n$ vertices is of class 1, $p(n)$ tends to 1 as $n$ goes to infinity.

- **Vizing (1965), Sanders & Zhao (2001):** All planar graphs of minimum degree at least 7 are of class 1. Vizing conjectured that this is true for all planar graphs of minimum degree at least six.
Outline

1. Definitions

2. Misra and Gries’ Edge-Coloring Algorithm
Due to Misra and Gries (1992)

- Produces a proper edge coloring of any graph $G$ on $\Delta(G) + 1$ colors
- Runs in $O(|V||E|)$ time.
- For the next slides, let $n = |V|$, $m = |E|$. 
Free colors

Definition 1

The color $x$ of an edge $(u, v)$ is free on $u$ if $c(u, z) \neq x$ for all $(u, z) \in E(G) : z \neq v$.

Figure: The color of $(u, v)$, red, is free on $u$. 
Definition 2

A fan of a vertex $u$ is a sequence of vertices $F[1 : k]$ that satisfies the following conditions:

1. $F[1 : k]$ is a non-empty sequence of distinct neighbors of $u$
2. $(F[1], u) \in E(G)$ is uncolored
3. The color of $F[i + 1]$ is free on $F[i]$ for $1 \leq i < k$

Definition 3

Given a fan $F$ of $u$, any edge $(F[i], u)$ for $1 \leq i \leq k$ is a fan edge.
Fans

Figure: A fan $F = [x_1, x_2, x_3]$ of $v$ (dashed edges are uncolored), $(v, x_1), (v, x_2), (v, x_3)$ are the fan edges. Note that $F' = [x_1, x_2]$ is also a fan of $v$, but it is not maximal.
**$cd_x$ path**

- Let $c, d$ be colors and $c \in V(G)$. A $cd_x$ path is an edge path that goes through $x$, only contains edges colored $c$ and $d$, and is maximal.

- Given $c, d, x$, the $cd_x$ path is unique.

**Figure:** Examples: $ac, cg, gd$ is a red-green$\ c$ path, $bd, dg$ is a red-orange$\ d$ path, $ac$ is a red-orange$\ a$ path.
Inverting a $cd_x$ path

Given a $cd_u$ path, switch every edge having color $c$ to $d$ and $d$ to $c$ on the path. In this example, inverting the red-green path from the left figure results in the right figure.

![Diagram showing inverting a path](image)

The coloring is still valid, as no other edges with a color in \{c, d\} is adjacent to the path by definition.
Given a fan $F[1 : k]$ of a vertex $X$, the “rotate fan” operation does the following (in parallel):

1. $c(F[i], X) = c(F[i + 1], X)$ for $1 \leq i < k$
2. Uncolor $F[k]$

This operation leaves the coloring valid as the color of $F[i + 1]$ was free on $F[i]$ for each $1 \leq i < k$ (definition of a fan), and we uncolor $F[k]$. 
We rotate the fan $F = [x_1, x_2, x_3]$ in the left figure, and get the coloring in the right figure.
Algorithm

**Input:** A graph $G$

**Output:** A proper edge-coloring $c$ of $G$

1. Let $U \leftarrow E(G)$
2. While $U \neq \emptyset$, do:
   1. Let $(u, v)$ be any edge in $U$.
   3. Let $c$ be a color that is free on $u$ and $d$ be a color that is free on $F[k]$.
   4. Invert the $cd_u$ path
   5. Let $w \in V(G)$ be such that $w \in F$, $F' = [F[1]...w]$ is a fan and $d$ is free on $w$.
   6. Rotate $F'$ and set $c(u, w) = d$.
   7. $U \leftarrow U - \{(u, v)\}$
Proof of correctness

Claim 1

The inversion of the cd path guarantees a vertex \( w \) such that \( w \in F, F' = [F[1]...w] \) is a fan and \( d \) is free on \( w \).

- There are two cases.
First case: The fan has no edge colored $d$.

- $F$ is a maximal fan and $d$ is free on $F[k]$.
- This implies there is no edge with color $d$ adjacent to $u$, otherwise it would follow $F[k]$, as $d$ is free on $f[k]$, but $F$ was maximal.
- Thus, $d$ is free on $u$.
- Since $c$ is also free on $u$, the $cd_u$ path is empty and the inversion has no effect.
- We can set $w = F[k]$.
Second case

Second case: The fan has one edge colored $d$.

- Let $F[x + 1]$ be this edge.
- $x + 1 \neq 1$ since $F[1]$ is uncolored, and $x \neq k$ since the fan has length $k$ but there exists a $F[x + 1]$.
- Thus, $d$ is free on $F[x]$.
- Claim (1): after the inversion, for each $y \in \{1, \cdots, x - 1, x + 1, \cdots, k\}$, the color of $(F[y + 1], u)$ is free on $y$
  - Prior to the inversion, the color of $(u, F[y + 1])$ is not $c$ or $d$ since $c$ is free on $u$ and $(u, F[x + 1])$ has color $d$ and the coloring is valid.
  - The inversion only affects edges that are colored $c$ or $d$, so (1) holds.
Second case cont.

- $F[x]$ can either be in the $cd_u$ path or not.
- If it is not, then the inversion will not affect the set of free colors on $F[x]$, and $d$ will remain free on it. We can set $w = F[x]$.
- Otherwise, we can show that $F$ is still a fan and $d$ remains free on $F[k]$.
- $d$ was free on $F[x]$ before the inversion and $F[x]$ is on the path.
- Thus, $F[x]$ is an endpoint of the $cd_u$ path and $c$ will be free on $F[x]$ after the inversion.
The inversion will change the color of $(u, F[x+1])$ from $d$ to $c$

- As $c$ is now free on $F[x]$ and (1) holds, $F$ remains a fan.
- Also, $d$ remains free on $F[k]$, since $F[k]$ is not on the $cd_u$ path (suppose that it is; since $d$ is free on $F[k]$, then it would have to be an endpoint of the path, but $u$ and $F[x]$ are the endpoints).
- Select $w = F[k]$.

In any case, $F' = [F[1] \cdots , w]$ is a prefix of $F$, so it is a valid fan and $d$ us free on $w$.  

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Proof, cont.

Claim 2

The edge coloring produces by the algorithm is proper.

Proof: (by induction on the number of colored edges)

- Base case: no edge is colored, this is valid.
- Induction step: suppose this was true at the end of the previous iteration.
- In the current iteration, after inverting the path, $d$ will be free on $u$.
- By Claim 1, it will also be free on $w$.
- Rotating $F'$ does not compromises the validity of the coloring.
- Thus, after setting $c(u, w) = d$, the coloring is still valid. □
Proof, cont.

Claim 3

The algorithm requires at most \( \Delta + 1 \) colors.

- In a given step, we need to find colors \( c \) and \( d \).
- \( u \) is adjacent to at least one uncolored edge and its degree is bounded by \( \Delta \).
- This implies that at least one color in \( \{1, \ldots, \Delta\} \) is available for \( c \).
- For \( d \), \( F[k] \) may have degree \( \Delta \) and no uncolored adjacent edge.
- Thus, a color \( \Delta + 1 \) may be required.
Proof, cont.

Theorem 1

The algorithm computes a proper edge coloring on $\Delta + 1$ in $O(|E||V|)$ time.

- At each step, the rotation uncolors $(u, w)$ and colors $(u, v)$, which was previously uncolored.
- Thus, one additional edge gets colored.
- Hence, the loop will run $O(|E|)$ times.
- Finding the maximal fan, the colors $c$ and $d$ and invert the $cd_u$ path can be done in $O(|V|)$ time, finding and removing the edge can be done using a stack in $O(1)$.
- Thus, each iteration of the loop takes $O(|V|)$ time.
- The total running time is $O(|E| + |E||V|) = O(|E||V|)$.
- The rest follows by Claims 2 and 3.
Questions?