School of Computer Science

Carleton University

Name: **A.M.** Problem Set: **2**
Course: **COMP 3804** Week: **2**

Problems

Consult [1, 2].

Problem 1. *Let $S = 1 + x + x^2 + \cdots + x^n$. Show the following*

1. *If $x = 1$, $S = n + 1$.*
2. *If $x = 0$, $S = 1$.*
3. *If $x \neq \{0, 1\}$, $S = \frac{1-x^{n+1}}{1-x}$.*
4. *If $0 < x < 1$ and $n \to \infty$, $S = \frac{1}{1-x}$.*
5. *If $0 < x < 1$, $S = \frac{1-x^{n+1}}{1-x} \leq \frac{1}{1-x} = \Theta(1)$.*
6. *If $x > 1$, $S_n = \frac{x^{n+1}-1}{x-1} \geq \frac{x^{n+1}-x^n}{x-1} = x^n$ and*
   $S_n = \frac{x^{n+1}-1}{x-1} \leq \frac{x^{n+1}}{x-1} = \frac{x}{x-1}x^n = O(x^n)$.
7. *For $x > 1$, $S = \Theta(x^n)$,*
   i.e. S is proportional to the last term of the series.

Problem 2. *Show that for positive constants α, β and positive number n, $\alpha^{\log_\beta n} = n^{\log_\beta \alpha}$*

Problem 3. *Consider the recurrence $T(n) = aT\left(\frac{n}{b}\right) + cn^k$, where $a \geq 1$, $b > 1$, $k \geq 0$, and $c > 0$ be constants. Show that $T(n) = O(n^{\log_b a}) + \sum_{i=0}^{\log_b n} a^i c \left(\frac{n}{b^i}\right)^k$. Conclude that*

1. *if $a > b^k$ then $T(n) = \Theta(n^{\log_b a})$*
2. *if $a = b^k$ then $T(n) = \Theta(n^k \log_b n)$*
3. *if $a < b^k$ then $T(n) = \Theta(n^k)$*

Problem 4. *Evaluate following recurrence, where you can assume that $T(1) = O(1)$.*

1. $T(n) = T(\frac{n}{2}) + 1$
2. $T(n) = 2T(\frac{n}{2}) + n$
3. $T(n) = 3T(\frac{n}{2}) + n$
4. $T(n) = 7T(\frac{n}{2}) + n^2$

Problem 5. *Evaluate the recurrence $T(n) = 2T(\frac{n}{2}) + n \log n$, where $T(1) = O(1)$. (Observe that it doesn't fit the pattern of the previous two problems.)*

Problem 6. *Consider the recurrenc $T(n) = T(n/3) + T(2n/3) + n$. We can assume $T(n) = O(1)$ for small values of n. Show that $T(n) = O(n \log n)$. It is best to try the substitution method, i.e., prove using induction on n.*

Problem 7. *Let S be a set of n points on a real line. How fast can you find a pair of points that have the smallest distance? What if the points are in 2-dimensional real plane?*

References

[1] S. DasGupta, C. Papadimitriou, V. Vazirani. *Introduction to Algorithms.* McGraw Hill.

[2] A. Maheshwari. *Notes on Algorithm Design*, Chapter 1, `https://people.scs.carleton.ca/~maheshwa/Notes/DAA/notes.pdf`