Consult [H2 3]. The problem set consists of problems on complexity classes $\text{P}$ and $\text{NP}$, polynomial time reducibility, and $\text{NP}$-Complete decision problems.

**Problem 1.** Formulate a decision problem corresponding to the following optimization problems:

1. *(Clique)* In an undirected graph $G = (V, E)$, find the largest size clique. A set of vertices $K \subseteq V$ is said to form a clique, if for every pair of vertices $u, v \in K$, $uv \in E$.

2. *(Independent Set)* In an undirected graph $G = (V, E)$, find the largest size independent set. A set of vertices $I \subseteq V$ is said to be independent, if for every pair of vertices $u, v \in I$, $uv \notin E$.

3. *(Vertex Cover)* In an undirected graph $G = (V, E)$, find the smallest size vertex cover. A set of vertices $C \subseteq V$ is said to form a cover, if for every edge $e = (u, v) \in E$, $u \in C$ or $v \in C$.

**Problem 2.** For the above three decision problems, state an equivalent formulation in terms of the language of the decision problem.

**Problem 3.** Let $k < n$ be a positive integer. Let us define the language $k-\text{COMP} = \{(G = (V, E), k) \mid G$ is a simple undirected graph on $n$ vertices containing at most $k$ components$\}$. Is $k-\text{COMP} \in \text{P}$? Is $k-\text{COMP} \in \text{NP}$?

**Problem 4.** Are the decision version of the problems stated in Problem 1 in $\text{NP}$?

**Problem 5.** Let $n = |V|$. Show that the decision problems stated in Problem 1 are equivalent with respect to the polynomial time reducibility. I.e., show that

1. $\text{Clique}(G, k) \leq_p \text{Independent-set}(\overline{G}, k)$

2. $\text{Independent-set}(\overline{G}, k) \leq_p \text{Vertex-Cover}(\overline{G}, n - k)$

3. $\text{Vertex-Cover}(\overline{G}, n - k) \leq_p \text{Clique}(G, k)$

**Problem 6.** Show that each of the decision problems stated in Problem 1 are $\text{NP}$-complete.

**Problem 7.** Given a graph $G = (V, E)$, a dominating set of $G$ is a subset $V' \subseteq V$ such that every vertex of $V$ is either in $V'$ or adjacent to a vertex in $V'$. The DOMINATING-SET problem takes as input a graph $G$ and an integer $k$ and asks if $G$ contains a dominating set of size $k$. Prove that DOMINATING-SET is $\text{NP}$-complete.

Hint: Try $3\text{CNF-SAT} \leq_p \text{DOMINATING-SET}$. Suppose $\phi$ consist of $n$ variables and $k$ clauses. Construct a graph $G$ on $3n + k$ vertices as follows: for each Boolean variable $x$, create a clique of size 3, consisting of vertices $x$, $\overline{x}$ and $x'$. For each clause $c$, create a vertex $c$ and join it to the corresponding 3 literal vertices from three triangles. Argue that if $\phi$ has a satisfying assignment, $G$ has a dominating set of size $n$. Suppose $G$ has a dominating set of size $n$. Show that we need a vertex in the dominating set from each variable triangle. If it is $x$, set $x = \text{TRUE}$, otherwise set $x = \text{FALSE}$. Show that $\phi$ is satisfied with this assignment.

Alternatively, try to reduce the vertex cover problem. For each edge $e = xy$ in $G$ for the vertex cover problem, add vertex $v_{xy}$ and connect it to $x$ and $y$ to get a new graph $G'$. Show that $G$ has a vertex cover of size $k$ if and only if $G'$ has a dominating set of size $k$. To keep it simple, you may assume $G$ has no isolated vertex.
References

