School of Computer Science

Carleton University

Name: **A.M.** Problem Set: **6**

Course: **COMP 3804** Week: **1-13**

Consult [1, 2, 3]. The problem set consists of problems on complexity classes **P** and **NP**, polynomial time reducibility, and **NP**-Complete decison problems.

Problem 1. *Formulate a decision problem corresponding to the following optimization problems:*

1. *(Clique) In an undirected graph $G = (V, E)$, find the largest size clique. A set of vertices $K \subseteq V$ is said to form a clique, if for every pair of vertices $u, v \in K$, $uv \in E$.*
2. *(Independent Set) In an undirected graph $G = (V, E)$, find the largest size independent set. A set of vertices $I \subseteq V$ is said to be independent, if for every pair of vertices $u, v \in I$, $uv \notin E$.*
3. *(Vertex Cover) In an undirected graph $G = (V, E)$, find the smallest size vertex cover. A set of vertices $C \subseteq V$ is said to form a cover, if for every edge $e = (u, v) \in E$, $u \in C$ or $v \in C$.*

Problem 2. *For the above three decision problems, state an equivalent formulation in terms of the language of the decision problem.*

Problem 3. *Let $k < n$ be a positive integer. Let us define the language*
$k{-}COMP = \{(G = (V, E), k) | \ G$ *is a simple undirected graph on n vertices containing at most k components*$\}$.
*Is $k - COMP \in$ **P**? Is $k - COMP \in$ **NP**?*

Problem 4. *Are the decision version of the problems stated in Problem 1 in* **NP***?*

Problem 5. *Let $n = |V|$. Show that the decision problems stated in Problem 1 are equivalent with respect to the polynomial time reducibility. I.e, show that*

1. *Clique$(G, k) \leq_P$ Independent-set$(\bar{G}, k)$*
2. *Independent-set$(\bar{G}, k) \leq_P$ Vertex-Cover$(\bar{G}, n - k)$*
3. *Vertex-Cover$(\bar{G}, n - k) \leq_P$ Clique(G, k)*

Problem 6. *Show that each of the decision problems stated in Problem 1 are* **NP***-complete.*

Problem 7. *Given a graph $G = (V, E)$, a dominating set of G is a subset $V' \subseteq V$ such that every vertex of V is either in V' or adjacent to a vertex in V. The DOMINATING-SET problem takes as input a graph G and an integer k and asks if G contains a dominating set of size k. Prove that DOMINATING-SET is* **NP***-complete.*
Hint: Try 3CNF-SAT $\leq_P$ DOMINATING-SET. Suppose ϕ consist of n variables and k clauses. Construct a graph G on $3n + k$ vertices as follows: for each Boolean variable x, create a clique of size 3, consisting of vertices x, $\bar{x}$ and x'. For each clause c, create a vertex c and join it to the corresponding 3 literal vertices from three triangles. Argue that if ϕ has a satisfying assignment, G has a dominating set of size n. Suppose G has a dominating set of size n. Show that we need a vertex in the dominating set from each variable triangle. If it is x, set $x = TRUE$, otherwise set $x = FALSE$. Show that ϕ is satisfied with this assignment.

Alternatively, try to reduce the vertex cover problem. For each edge $e = xy$ in G for the vertex cover problem, add vertex v_{xy} and connect it to x and y to get a new graph G'. Show that G has a vertex cover of size k if and only if G' has a dominating set of size k. To keep it simple, you may assume G has no isolated vertex.

References

[1] T. Cormen, C. Leiserson, R. Rivest, and C. Stein, *Introduction to Algorithms.* 3rd. ed., MIT Press, 2009.

[2] S. DasGupta, C. Papadimitriou, V. Vazirani. *Introduction to Algorithms.* McGraw Hill.

[3] A. Maheshwari. *Notes on Algorithm Design*, Chapter 1, `https://people.scs.carleton.ca/~maheshwa/Notes/DAA/notes.pdf`