1 Lemma 14.3.1 is wrong

Let S be a set of n points in $\mathbb{R}^d$, let G be a connected Euclidean graph with vertex set S, and let p and q be two distinct points of S. We denote by $\delta_2(p,q)$ the length of a second shortest simple path in G between p and q. In order to define $\delta_2(p,q)$ more precisely, recall that a path is called simple, if its vertices are pairwise distinct. Consider all distinct simple paths $P_1, P_2, \ldots, P_m$ between p and q, and let ℓ_i denote the length of P_i, $1 \leq i \leq m$. Note that the paths may have vertices or edges in common. We assume that these paths are numbered such that $\ell_1 \leq \ell_2 \leq \ldots \leq \ell_m$. If $m \geq 2$, then we define $\delta_2(p,q) := \ell_2$, and we call P_2 a *second shortest path* between p and q. If $m = 1$, then there is only one simple path between p and q, and we define $\delta_2(p,q) := \infty$.

Lemma 14.3.1 states the following:

> **Lemma:** Let S be a set of n points in $\mathbb{R}^d$, let $t > 1$ be a real number, and let $G = (S,E)$ be an undirected t-spanner for S. Assume that $\delta_2(p,q) > t|pq|$, for every edge $\{p,q\}$ in E. Then, the edge set E satisfies the t-leapfrog property.

Here is a counter example to this lemma: Let ϵ be a real number with $0 < \epsilon < 1$, let $a = (0,0)$, $b = (1,0)$, and $c = (\epsilon, 0)$, and let $S = \{a,b,c\}$. Let G be the graph with vertex set S and edges $\{a,b\}$ and $\{a,c\}$. Then G is a t-spanner for $t = (1+\epsilon)/(1-\epsilon)$. Since $\delta_2(a,b) = \delta_2(a,c) = \infty$, the graph G satisfies the conditions of Lemma 14.3.1. However, the t-leapfrog property does not hold: Take $k = 2$, $p_1 = p_2 = a$, $q_1 = b$, and $q_2 = c$. If the t-leapfrog property holds, then

$$t|p_1q_1| < |p_2q_2| + t(|p_1p_2| + |q_2q_1|),$$

i.e.,

$$t < \epsilon + t(1-\epsilon),$$

i.e., $t < 1$, which is a contradiction.

2 Lemma 14.3.1 is true for strong spanners

The following definition is from Exercise 4.6: We call a Euclidean graph $G = (S,E)$ a *strong t-spanner* for the point set S, if for any two points p and q of S, there exists a t-spanner path between p and q in G, all of whose edges have length at most $|pq|$. Here is the statement of Lemma 14.3.1 for strong spanners:

> **Lemma:** Let S be a set of n points in $\mathbb{R}^d$, let $t > 1$ be a real number, and let $G = (S,E)$ be an undirected strong t-spanner for S. Assume that $\delta_2(p,q) > t|pq|$, for every edge $\{p,q\}$ in E. Then, the edge set E satisfies the t-leapfrog property.

Proof. Let $k \geq 2$, and let $\{p_i, q_i\}$, $1 \leq i \leq k$, be an arbitrary sequence of pairwise distinct edges in E. We have to show that the t-leapfrog property holds, i.e.,

$$t|p_1q_1| < \sum_{i=2}^{k} |p_iq_i| + t\left(|p_1p_2| + \sum_{i=2}^{k-1} |q_ip_{i+1}| + |q_kq_1|\right). \tag{1}$$

Let P_1 be a t-spanner path in G between p_1 and p_2, all of whose edges have length at most $|p_1p_2|$. For each i with $2 \leq i \leq k-1$, let P_i be a t-spanner path in G between q_i and p_{i+1}, all of whose edges have length at most $|q_ip_{i+1}|$. Finally, let P_k be a t-spanner path in G between q_k and q_1, all of whose edges have length at most $|q_kq_1|$.

Let P be the (possibly non-simple) path

$$P_1, \{p_2, q_2\}, P_2, \{p_3, q_3\}, P_3, \ldots, P_{k-1}, \{p_k, q_k\}, P_k$$

between p_1 and q_1, and let P' be a simple path between p_1 and q_1 obtained by removing all cycles from P. We distinguish two cases.

Case 1: The path P' consists of at least two edges.

In this case, we have

$$\begin{aligned}
|P'| &\leq |P| \\
&= \sum_{i=1}^{k} |P_i| + \sum_{i=2}^{k} |p_iq_i| \\
&\leq t\left(|p_1p_2| + \sum_{i=2}^{k-1} |q_ip_{i+1}| + |q_kq_1|\right) + \sum_{i=2}^{k} |p_iq_i|.
\end{aligned}$$

Since $\{p_1, q_1\}$ is an edge of E, and since P' contains at least two edges, we have $\delta_2(p_1, q_1) \leq |P'|$. Then, the assumption of the lemma implies that

$$|P'| \geq \delta_2(p_1, q_1) > t|p_1q_1|.$$

Hence, the required inequality (1) holds.

Case 2: The path P' consists of exactly one edge.

In this case, the non-simple path P contains the edge $\{p_1, q_1\}$. In fact, there is an index j with $1 \leq j \leq k$, such that the path P_j contains this edge. If $2 \leq j \leq k-1$, then (1) follows from the fact that $|p_1q_1| \leq |q_jp_{j+1}|$. If $j = 1$, then (1) follows from the fact that $|p_1q_1| \leq |p_1p_2|$. If $j = k$, then (1) follows from the fact that $|p_1q_1| \leq |q_kq_1|$. ∎

3 A modified version of Lemma 14.3.1

By replacing the leapfrog property by the generalized leapfrog property of Section 14.9, Lemma 14.3.1 is true:

Lemma: Let S be a set of n points in $\mathbb{R}^d$, let $t > 1$ be a real number, and let $G = (S, E)$ be an undirected t-spanner for S. Assume that $\delta_2(p, q) > t|pq|$, for every edge $\{p, q\}$ in E. Then, the edge set E satisfies the (t, t^2)-leapfrog property.

Proof. Let $k \geq 2$, and let $\{p_i, q_i\}$, $1 \leq i \leq k$, be an arbitrary sequence of pairwise distinct edges in E. We have to show that the (t, t^2)-leapfrog property holds, i.e.,

$$t|p_1q_1| < \sum_{i=2}^{k} |p_iq_i| + t^2 \left(|p_1p_2| + \sum_{i=2}^{k-1} |q_ip_{i+1}| + |q_kq_1| \right). \tag{2}$$

Let P_1 be a shortest path in G between p_1 and p_2. For each i with $2 \leq i \leq k-1$, let P_i be a shortest path in G between q_i and p_{i+1}. Finally, let P_k be a shortest path in G between q_k and q_1. Since G is a t-spanner, the length $|P_i|$ of each path P_i is less than or equal to t times the distance between its endpoints. Let P be the (possibly non-simple) path

$$P_1, \{p_2, q_2\}, P_2, \{p_3, q_3\}, P_3, \ldots, P_{k-1}, \{p_k, q_k\}, P_k$$

between p_1 and q_1, and let P' be a simple path between p_1 and q_1 obtained by removing all cycles from P. We distinguish two cases.

Case 1: The path P' consists of at least two edges.

In this case, we have

$$\begin{aligned} |P'| &\leq |P| \\ &= \sum_{i=1}^{k} |P_i| + \sum_{i=2}^{k} |p_iq_i| \\ &\leq t \left(|p_1p_2| + \sum_{i=2}^{k-1} |q_ip_{i+1}| + |q_kq_1| \right) + \sum_{i=2}^{k} |p_iq_i| \\ &\leq t^2 \left(|p_1p_2| + \sum_{i=2}^{k-1} |q_ip_{i+1}| + |q_kq_1| \right) + \sum_{i=2}^{k} |p_iq_i|. \end{aligned}$$

Since $\{p_1, q_1\}$ is an edge of E, and since P' contains at least two edges, we have $\delta_2(p_1, q_1) \leq |P'|$. Then, the assumption of the lemma implies that

$$|P'| \geq \delta_2(p_1, q_1) > t|p_1q_1|.$$

Hence, the required inequality (2) holds.

Case 2: The path P' consists of exactly one edge.

In this case, the non-simple path P contains the edge $\{p_1, q_1\}$. In fact, there is an index j with $1 \leq j \leq k$, such that the path P_j contains this edge. Hence, we have

$$|p_1q_1| \leq |P_j| \leq \sum_{i=1}^{k} |P_i| \leq t \left(|p_1p_2| + \sum_{i=2}^{k-1} |q_ip_{i+1}| + |q_kq_1| \right),$$

from which the inequality (2) follows. ∎

4 Corrected version of Lemma 15.2.15

By changing the parameters for the leapfrog property in Lemma 15.2.15, we obtain the following lemma:

> **Lemma:** Let S be a set of n points in $\mathbb{R}^d$ and let $t > 1$, $t' > 1$, $\alpha > 0$, and $\mu > 1$ be real numbers, such that $t > t'$ and
>
> $$\alpha \leq \frac{\sqrt{t} - \sqrt{t'}}{2\sqrt{t} + 18\sqrt{t'}}.$$
>
> Let $G = (S, E)$ be the output of algorithm $\textsc{FastPathGreedy}(S, t, t', \alpha, \mu)$. The edge set $E \setminus E_0$ satisfies the (t', tt')-leapfrog property.

Proof. Let $k \geq 2$, and let $\{p_i, q_i\}$, $1 \leq i \leq k$, be an arbitrary sequence of pairwise distinct edges in $E \setminus E_0$. We have to show that the (t', tt')-leapfrog property holds, i.e.,

$$t'|p_1q_1| < \sum_{i=2}^{k} |p_iq_i| + tt' \left(|p_1p_2| + \sum_{i=2}^{k-1} |q_ip_{i+1}| + |q_kq_1| \right). \tag{3}$$

Let P_1 be a shortest path in G between p_1 and p_2. For each i with $2 \leq i \leq k-1$, let P_i be a shortest path in G between q_i and p_{i+1}. Finally, let P_k be a shortest path in G between q_k and q_1. Since G is a t-spanner, the length $|P_i|$ of each path P_i is less than or equal to t times the distance between its endpoints. Let P be the (possibly non-simple) path

$$P_1, \{p_2, q_2\}, P_2, \{p_3, q_3\}, P_3, \ldots, P_{k-1}, \{p_k, q_k\}, P_k$$

between p_1 and q_1, and let P' be a simple path between p_1 and q_1 obtained by removing all cycles from P. We distinguish two cases.

Case 1: The path P' consists of at least two edges.

In this case, we have

$$\begin{aligned} |P'| &\leq |P| \\ &= \sum_{i=1}^{k} |P_i| + \sum_{i=2}^{k} |p_iq_i| \\ &\leq t \left(|p_1p_2| + \sum_{i=2}^{k-1} |q_ip_{i+1}| + |q_kq_1| \right) + \sum_{i=2}^{k} |p_iq_i| \\ &\leq tt' \left(|p_1p_2| + \sum_{i=2}^{k-1} |q_ip_{i+1}| + |q_kq_1| \right) + \sum_{i=2}^{k} |p_iq_i|. \end{aligned}$$

Since $\{p_1, q_1\}$ is an edge of $E \setminus E_0$ and since P' contains at least two edges, we have $\delta_{G,2}(p_1, q_1) \leq |P'|$ and, by Lemma 15.2.14, $\delta_{G,2}(p_1, q_1) > t'|p_1q_1|$. It follows that

$$|P'| \geq \delta_{G,2}(p_1, q_1) > t'|p_1q_1|.$$

Hence, the required inequality (3) holds.

Case 2: The path P' consists of exactly one edge.

In this case, the non-simple path P contains the edge $\{p_1, q_1\}$. In fact, there is an index j with $1 \leq j \leq k$, such that the path P_j contains this edge. Hence, we have

$$|p_1 q_1| \leq |P_j| \leq \sum_{i=1}^{k} |P_i| \leq t \left(|p_1 p_2| + \sum_{i=2}^{k-1} |q_i p_{i+1}| + |q_k q_1| \right),$$

from which the inequality (3) follows. ∎