Lemma 12.1.6 *Let n and ℓ be positive integers, such that $n \geq \ell$, and consider the set E'' defined above. Then,*

1. *the set E'' contains at most $3n$ edges, and*
2. *given the centroid tree $CT(T)$, the set E'' can be computed in $O(n)$ time.*

Proof. Consider the subtrees $T_1, T_2, \ldots, T_g$ that we get by removing the vertices of CV_ℓ from T. We fix an arbitrary vertex r of CV_ℓ. Using this vertex, we divide the edges of E'' into two groups. Let $\{v, u\}$ be any edge of E'' and assume without loss of generality that $v \notin CV_\ell$ and $u \in CV_\ell$. We call edge $\{v, u\}$ an *upstream edge*, if, in the tree T, u is on the path between v and r. Otherwise, $\{v, u\}$ is called a *downstream edge*. We will count the upstream and downstream edges separately.

Each vertex $v \notin CV_\ell$ is incident on exactly one upstream edge. (Otherwise, the tree T would contain a cycle.) Conversely, each upstream edge of E'' is incident on exactly one vertex that is not in CV_ℓ. Therefore, the set E'' contains exactly $n - |CV_\ell|$ upstream edges.

Let u be any vertex of $CV_\ell \setminus \{r\}$, and let $\{u, u'\}$ be the first edge on the path in T from u to r. If we remove this edge from T, we obtain two trees T' and T'', where T' contains r. Consider two distinct downstream edges $\{v, u\}$ and $\{v', u\}$. Hence, neither of v and v' is a vertex of CV_ℓ. Also, neither of v and v' is a vertex of T''. Let i and j be the indices such that v and v' are vertices of T_i and T_j, respectively. Since u is a border vertex of T_i and v is a vertex of T', the subtree T_i contains u' as a vertex. Similarly, since u is a border vertex of T_j and v' is a vertex of T', the subtree T_j contains u' as a vertex. Therefore, we have $i = j$. It follows that each downstream edge of E'' that is incident on u, is incident on a vertex of T_i. Since T_i has less than ℓ vertices, this implies that there are less than ℓ downstream edges that are incident on u. Since each downstream edge is incident on exactly one vertex of $CV_\ell \setminus \{r\}$, this proves that the number of downstream edges in E'' is bounded from above by $(|CV_\ell| - 1)(\ell - 1)$.

We know from Lemma 12.1.5 that CV_ℓ contains at most $2n/\ell$ vertices. Therefore, the number of edges in E'' is bounded from above by

$$(n - |CV_\ell|) + (|CV_\ell| - 1)(\ell - 1) \leq n + |CV_\ell|\, \ell \leq 3n.$$

This proves the first claim. The proof of the second claim is left as an exercise; see Exercise 12.4. ∎