Plane Geometric Spanners

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1 PROBLEM DEFINITION

Let S be a set of n points in the plane and let G be an undirected graph with vertex set S, in which each edge (u, v) has a weight, which is equal to the Euclidean distance $|uv|$ between the points u and v. For any two points p and q in S, we denote their shortest-path distance in G by $\delta_G(p, q)$. If $t \geq 1$ is a real number, then we say that G is a *t-spanner* for S if $\delta_G(p, q) \leq t|pq|$ for any two points p and q in S. Thus, if t is close to 1, then the graph G contains close approximations to the $\binom{n}{2}$ Euclidean distances determined by the pairs of points in S. If, additionally, G consists of $O(n)$ edges, then this graph can be considered a sparse approximation to the complete graph on S. The smallest value of t for which G is a t-spanner is called the *stretch factor* (or *dilation*) of G. For a comprehensive overview of geometric spanners, see the book by Narasimhan and Smid [16].

We assume that each edge (u, v) of G is embedded as the straight-line segment between the points u and v. We say that the graph G is *plane* if edges only intersect at their common vertices.

In this entry, we consider the following two problems:

Problem 1. *Determine the smallest real number $t > 1$ for which the following is true: For every set S of n points in the plane, there exists a plane graph with vertex set S, which is a t-spanner for S. Moreover, design an efficient algorithm that constructs such a plane t-spanner.*

Problem 2. *Determine the smallest positive integer D for which the following is true: There exists a constant t, such that for every set S of n points in the plane, there exists a plane graph with vertex set S and maximum degree at most D, which is a t-spanner for S. Moreover, design an efficient algorithm that constructs such a plane t-spanner.*

2 KEY RESULTS

Let S be a finite set of points in the plane that is in *general position*, i.e., no three points of S are on a line and no four points of S are on a circle. The *Delaunay triangulation* of S is the plane graph with vertex set S, in which (u, v) is an edge if and only if there exists a circle through u and v that does not contain any point of S in its interior. (Since S is in general position, this graph is a triangulation.) The Delaunay triangulation of a set of n points in the plane can be constructed in $O(n \log n)$ time. Dobkin, Friedman and Supowit [10] were the first to show that the stretch factor of the Delaunay triangulation is bounded by a constant: They proved that the Delaunay triangulation is a t-spanner for $t = \pi(1 + \sqrt{5})/2$. The currently best known upper bound on the stretch factor of this graph is due to Keil and Gutwin [12]:

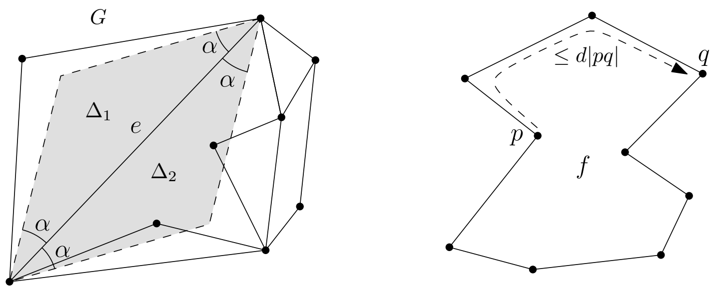

Figure 1: *On the left, the α-diamond property is illustrated. At least one of the triangles Δ_1 and Δ_2 does not contain any point of S in its interior. On the right, the d-good polygon property is illustrated. p and q are two vertices on the same face f which can see each other. At least one of the two paths between p and q along the boundary of f has length at most $d|pq|$.*

Theorem 1. *Let S be a finite set of points in the plane. The Delaunay triangulation of S is a t-spanner for S, for $t = 4\pi\sqrt{3}/9$.*

A slightly stronger result was proved by Bose *et al.* [3]. They proved that for any two points p and q in S, the Delaunay triangulation contains a path between p and q, whose length is at most $(4\pi\sqrt{3}/9)|pq|$ and all edges on this path have length at most $|pq|$.

Levcopoulos and Lingas [14] generalized the result of Theorem 1: Assume that we are given the Delaunay triangulation of the set S. Then, for any real number $r > 0$, a plane graph G with vertex set S can be constructed in $O(n)$ time, such that G is a t-spanner for S, where $t = (1+1/r)4\pi\sqrt{3}/9$, and the total length of all edges in G is at most $2r + 1$ times the weight of a minimum spanning tree of S.

The Delaunay triangulation can alternatively be defined to be the dual of the *Voronoi diagram* of the set S. By considering the Voronoi diagram for a metric other than the Euclidean metric, we obtain a corresponding Delaunay triangulation. Chew [7] has shown that the Delaunay triangulation based on the Manhattan-metric is a $\sqrt{10}$-spanner (we remark that in this spanner, path-lengths are measured in the Euclidean metric). The currently best result for Problem 1 is due to Chew [8]:

Theorem 2. *Let S be a finite set of points in the plane, and consider the Delaunay triangulation of S that is based on the convex distance function defined by an equilateral triangle. This plane graph is a 2-spanner for S (where path-lengths are measured in the Euclidean metric).*

Das and Joseph [9] have generalized the result of Theorem 1 in the following way (refer to Figure 1). Let G be a plane graph with vertex set S and let α be a real number with $0 < \alpha < \pi/2$. For any edge e of G, let Δ_1 and Δ_2 be the two isosceles triangles with base e and base angle α. We say that e satisfies the *α-diamond property*, if at least one of the triangles Δ_1 and Δ_2 does not contain any point of S in its interior. The plane graph G is said to satisfy the *α-diamond property*, if every edge e of G satisfies this property. For a real number $d \geq 1$, we say that G satisfies the *d-good polygon property*, if for every face f of G, and for every two vertices p and q on the boundary of f, such that the line segment joining them is completely inside f, the shortest path between p and q along the boundary of f has length at most $d|pq|$. Das and Joseph [9] proved that any plane graph satisfying both the α-diamond property and the d-good polygon property is a t-spanner, for some real number t that depends only on α and d. A slight improvement on the value of t was obtained by Lee [13]:

Theorem 3. *Let $\alpha \in (0, \pi/2)$ and $d \geq 1$ be real numbers, and let G be a plane graph that satisfies the α-diamond property and the d-good polygon property. Then, G is a t-spanner for the vertex set of G, where*

$$t = \frac{8(\pi - \alpha)^2 d}{\alpha^2 \sin^2(\alpha/4)}.$$

To give some examples, it is not difficult to show that the Delaunay triangulation satisfies the α-diamond property with $\alpha = \pi/4$. Drysdale *et al.* [11] have shown that the minimum weight triangulation satisfies the α-diamond property with $\alpha = \pi/4.6$. Finally, Lee [13] has shown that the greedy triangulation satisfies the α-diamond property with $\alpha = \pi/6$. Of course, any triangulation satisfies the d-good polygon property with $d = 1$.

We now turn to Problem 2, that is, the problem of constructing plane spanners whose maximum degree is small. The first result for this problem is due to Bose *et al.* [2]. They proved that the Delaunay triangulation of any finite point set contains a subgraph of maximum degree at most 27, which is a t-spanner (for some constant t). Li and Wang [15] improved this result, by showing that the Delaunay triangulation contains a t-spanner of maximum degree at most 23. Given the Delaunay triangulation, the subgraphs in [2, 15] can be constructed in $O(n)$ time. The currently best result for Problem 2 is by Bose *et al.* [6]:

Theorem 4. *Let S be a set of n points in the plane. The Delaunay triangulation of S contains a subgraph of maximum degree at most 17, which is a t-spanner for S, where*

$$t = \frac{4\pi\sqrt{3}}{9}\left(\pi + 9 + 9\pi\sqrt{3}/2\right).$$

Given the Delaunay triangulation of S, this subgraph can be constructed in $O(n)$ time.

In fact, the result in [6] is more general:

Theorem 5. *Let S be a set of n points in the plane, let $\alpha \in (0, \pi/2)$ be a real number, and let G be a triangulation of S that satisfies the α-diamond property. Then, G contains a subgraph of maximum degree at most $14 + \lceil 2\pi/\alpha \rceil$, which is a t-spanner for S, where*

$$t = \frac{8(\pi - \alpha)^2}{\alpha^2 \sin^2(\alpha/4)}\left(1 + \frac{2(\pi - \alpha)}{\alpha \sin(\alpha/4)} \cdot \max\{1, 2\sin(\alpha/2)\}\right).$$

Given the triangulation G, this subgraph can be constructed in $O(n)$ time.

3 APPLICATIONS

Plane spanners have applications in on-line path-finding and routing problems that arise in, for example, geographic information systems and communication networks. In these application areas, the complete environment is not known, and routing has to be done based only on the source, the destination, and the neighborhood of the current position. Bose and Morin [4, 5] have shown that, in this model, good routing strategies exist for plane graphs, such as the Delaunay triangulation and graphs that satisfy both the α-diamond property and the d-good polygon property. These strategies are competitive, in the sense that the paths computed have lengths that are within a constant factor of the Euclidean distance between the source and destination. Moreover, these routing strategies use only a limited amount of memory.

4 OPEN PROBLEMS

None of the results for Problems 1 and 2 that are mentioned in Section 2 seem to be optimal. We mention the following open problems:

1. Determine the smallest real number t, such that the Delaunay triangulation of any finite set of points in the plane is a t-spanner. It is widely believed that $t = \pi/2$. By Theorem 1, we have $t \leq 4\pi\sqrt{3}/9$.

2. Determine the smallest real number t, such that a plane t-spanner exists for any finite set of points in the plane. By Theorem 2, we have $t \leq 2$. By taking S to be the set of four vertices of a square, it follows that t must be at least $\sqrt{2}$.

3. Determine the smallest integer D, such that the Delaunay triangulation of any finite set of points in the plane contains a t-spanner (for some constant t) of maximum degree at most D. By Theorem 4, we have $D \leq 17$. It follows from results in Aronov *et al.* [1] that the value of D must be at least 3.

4. Determine the smallest integer D, such that a plane t-spanner (for some constant t) of maximum degree at most D exists for any finite set of points in the plane. By Theorem 4 and results in [1], we have $3 \leq D \leq 17$.

5 CROSS REFERENCES

Who takes care of this?

Mention the other two entries by Gudmundsson, Narasimhan and Smid.

Mention entries on Delaunay triangulations and Voronoi diagrams.

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