A probabilistic construction of a dense bipartite graph with high girth

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The *girth* of a graph G is the minimum number of edges on any cycle in G. The probabilistic method has been used to prove the existence of a graph that has "many" edges and whose girth is "high"; see for example Theorem 6.6 in Mitzenmacher and Upfal [3]. Chandran [1] has given a deterministic construction of such a graph.

In this note, we show that a trivial modification of the proof in [3] shows the existence of a dense *bipartite* graph whose girth is high. Gudmundsson and Smid [2] have shown that the construction in [1] can be modified to obtain a deterministic construction of such a bipartite graph.

We say that a graph G is an $n \times n$ bipartite graph, if its vertex set can be partitioned into two sets L and R, each having size n, such that every edge of G is between a vertex in L and a vertex in R.

We will prove the following result:

Theorem 1 *Let n and g be positive integers, such that g is even and $6 \leq g \leq 2\log n$. There exists a connected $n \times n$ bipartite graph with at least $\frac{1}{2}n^{1+1/g}$ edges, whose girth is at least g.*

We denote the complete $n \times n$ bipartite graph by K_{nn}. Thus, the edge set of K_{nn} is the set $\{\{u,v\} : u \in L, v \in R\}$.

Let $p := n^{-1+1/g}$. We construct a subgraph G of K_{nn}, by choosing each edge of K_{nn} independently and uniformly at random with probability p. Let

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X denote the number of edges of G, and let Y denote the number of cycles in G having length at most $g-2$. It is clear that

$$E(X) = pn^2 = n^{1+1/g}.$$

Below, we derive an upper bound on the expected value of the random variable Y.

First consider a fixed cycle C of length $2i$ in K_{nn}. (This cycle is fixed in the sense that we choose it *before* the graph G is constructed.) Obviously, the probability that C is contained in G is equal to p^{2i}.

Next, we analyze the number of cycles of length $2i$ in K_{nn}. Any such cycle is specified by

- choosing a subset A of L that consists of i vertices,
- choosing a subset B of R that consists of i vertices,
- taking one vertex in A to be the start vertex of the cycle and putting the remaining $i-1$ vertices of A in some order, and
- putting the vertices of B in some order.

In order to make the start vertex in A unique, we consider some fixed ordering of the vertices in L, and choose the start vertex to be the smallest element of A. In this way, we obtain every cycle of length $2i$ exactly twice, because any such cycle can be traversed in two directions. It follows that the number of cycles of length $2i$ is equal to

$$\frac{1}{2}\binom{n}{i}\binom{n}{i}(i-1)!\,i!,$$

and thus, by the linearity of expectation,

$$E(Y) = \frac{1}{2}\sum_{i=2}^{(g-2)/2}\binom{n}{i}\binom{n}{i}(i-1)!\,i!\,p^{2i}.$$

Since

$$\binom{n}{i}(i-1)! = \frac{n!}{i\,(n-i)!} \leq \frac{n^i}{i} \leq n^i$$

and

$$\binom{n}{i}i! = \frac{n!}{(n-i)!} \leq n^i,$$

we have

$$\begin{aligned}
E(Y) &\leq \frac{1}{2} \sum_{i=2}^{(g-2)/2} (pn)^{2i} \\
&= \frac{1}{2} \sum_{i=2}^{(g-2)/2} n^{2i/g} \\
&\leq \frac{1}{2} \sum_{i=0}^{(g-2)/2} \left(n^{2/g}\right)^{i} \\
&= \frac{1}{2} \frac{\left(n^{2/g}\right)^{g/2} - 1}{n^{2/g} - 1} \\
&\leq \frac{1}{2} \frac{n}{n^{2/g} - 1}.
\end{aligned}$$

Since $g \leq 2\log n$, we have $n^{2/g} \geq 2$ and, thus, $n^{2/g} - 1 \geq \frac{1}{2}n^{2/g}$. It follows that

$$E(Y) \leq \frac{1}{2} \frac{n}{\frac{1}{2}n^{2/g}} = n^{1-2/g}.$$

Since $g \leq 2\log n \leq 3\log n$, we have $n^{3/g} \geq 2$, which is equivalent to $n^{1-2/g} \leq \frac{1}{2}n^{1+1/g}$. Thus,

$$E(Y) \leq \frac{1}{2} n^{1+1/g}.$$

Let G' be the graph obtained by deleting, from G, one edge from each cycle whose length is at most $g-2$. Then, the girth of G' is at least g, and the expected number of edges of G' is at least

$$E(X - Y) = E(X) - E(Y) \geq n^{1+1/g} - \frac{1}{2} n^{1+1/g} = \frac{1}{2} n^{1+1/g}.$$

In conclusion, the above process constructs a random $n \times n$ bipartite graph G' with girth at least g and an expected number of edges that is at least $\frac{1}{2}n^{1+1/g}$. It follows that there *exists* a bipartite graph with girth at least g and that contains at least $\frac{1}{2}n^{1+1/g}$ edges. Let c be the number of connected components of this graph. If $c = 1$, then this graph satisfies the conditions in Theorem 1. Otherwise, we have $c \geq 2$, in which case we add $c - 1$ edges to the graph, in such a way that no new cycles are introduced, the graph is still bipartite, and the graph is connected. The resulting graph satisfies the conditions in Theorem 1.

References

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[3] M. Mitzenmacher and E. Upfal. *Probability and Computing*. Cambridge University Press, Cambridge, UK, 2005.