COMP 3803 - Midterm Solutions

Solutions written in LaTeX, diagrams drawn in `ipe`

February 26, 2015

1. (a) **Q:** Show by induction that $n^3 + 2n$ is divisible by 3 for all $n \geq 0$.

   A: For the base case, $0^3 + 2 \cdot 0 = 0$ is clearly divisible by 3.
   Suppose that $n^3 + 2n$ is divisible by 3 for some n. Then:

   $$\begin{aligned}(n+1)^3 + 2(n+1) &= n^3 + 3n^2 + 3n + 1 + 2n + 2 \\ &= (n^3 + 2n) + 3(n^2 + n + 1)\end{aligned}$$

   By the induction hypothesis, $n^3 + 2n$ is divisible by 3. So by induction, we are done. □

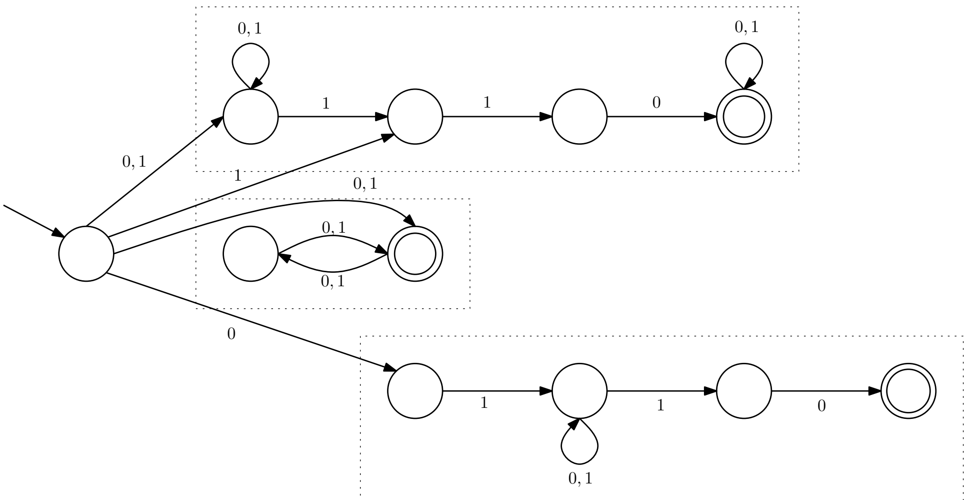
   (b) **Q:** Prove or disprove, using non-inductive arguments, a slightly more general claim that $n^6 + 4n$ is divisible by 6 for all $n \geq 0$.

   A: The claim is not true. As a counterexample, let $n = 1$, so $1^6 + 4 \cdot 1 = 5$ which is not divisible by 6. □

2. **Q:** Give the state diagram of a non-deterministic finite automaton (NFA) without ε-transitions that recongnizes the set L of all binary strings that have the following properties:

   - contains 110 as a substring, or
   - whose length is odd, or
   - start with 01 and end with 10.

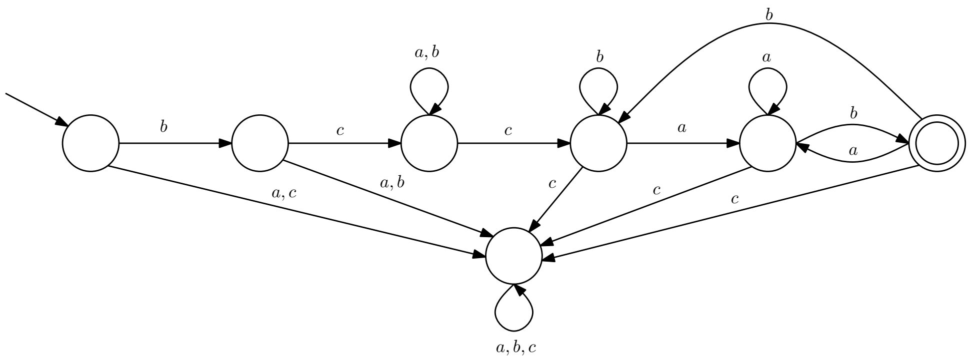
   A: *Note:* dotted lines delimit each part of the union of machines, for clarity.



3. (a) **Q:** Construct a deterministic finite automaton (DFA) accepting the following language:

$$\{W : W \in \{a,b,c\}^*, W \text{ starts with } bc, \text{ has a single } c \text{ after that, ends with } ab\}$$

A:

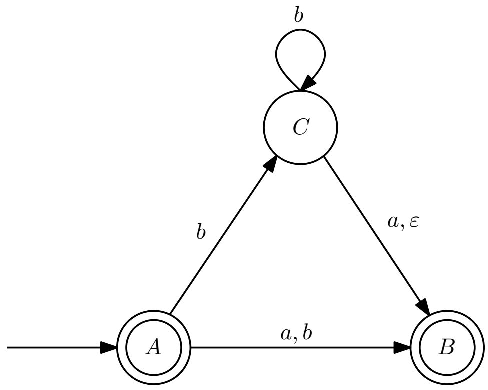


(b) **Q:** What is the regular expression for the language?

A:

$$bc(a \cup b)^* c(a \cup b)^* ab$$

4. **Q:** Give an equivalent deterministic finite automaton (DFA) for the following NFA:

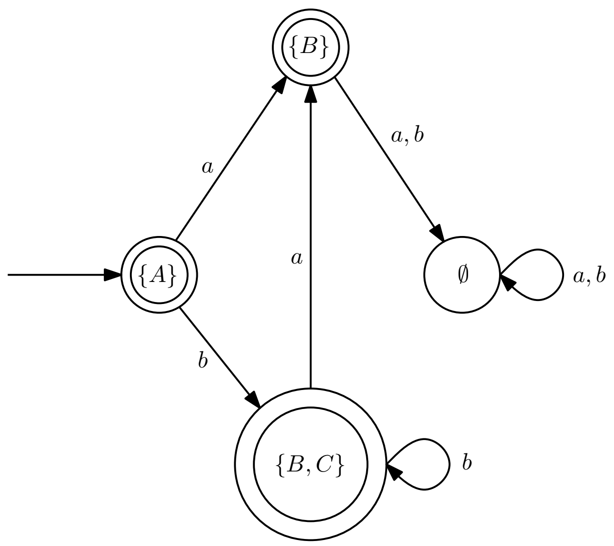


A:

The start state is $\{A\}$, and every state other than $\emptyset$ and $\{C\}$ are accepting.

	a	b
$\emptyset$	$\emptyset$	$\emptyset$
$\{A\}$	$\{B\}$	$\{B,C\}$
$\{B\}$	$\emptyset$	$\emptyset$
$\{C\}$	$\{B\}$	$\{B,C\}$
$\{A,B\}$	$\{B\}$	$\{B,C\}$
$\{A,C\}$	$\{B\}$	$\{B,C\}$
$\{B,C\}$	$\{B\}$	$\{B,C\}$
$\{A,B,C\}$	$\{B\}$	$\{B,C\}$

From the above table, we can see that $\{C\}$, $\{A,B\}$, $\{A,C\}$, and $\{A,B,C\}$ are unreachable, so we omit them from the following state diagram:



5. (a) **Q:** Let $A = \{3n : n \in \mathbb{N}$, $B \in \{3n+1 : n \in \mathbb{N}\}$, and $C = \{3n-1 : n \in \mathbb{N}\}$, where $\mathbb{N}$ is the set of natural numbers. Let D be the set of all non-negative odd numbers. Let $P = A \cup B \cup C$. What is the set P? Prove that the set $P \times D$ is countable.

   A: P is the set $\mathbb{N} \setminus \{1\}$, *i.e.* the set of all natural numbers strictly greater than 1. Each of the sets A, B, and C are subsets of $\mathbb{N} \setminus \{1\}$, so:

$$P = A \cup B \cup C \subseteq \mathbb{N} \setminus \{1\}$$

We prove by induction that $\mathbb{N} \setminus \{1\} \subseteq P$. First, $2 = 3 \cdot 1 - 1 \in C \subseteq P$, so $2 \in P$ and the base case is completed. Suppose that some integer $n \geq 2$ is in P. Then, $n = 3k + i$ for some $i \in \{-1, 0, 1\}$, so:

$$n + 1 = 3k + (i + 1)$$

If $i + 1 \in \{0, 1\}$, we are done, since then $n + 1 \in P$. Otherwise, $i + 1 = 2$, but then:

$$n + 1 = 3k + 2 = 3(k + 1) - 1 \in C$$

and once again, $n + 1 \in P$. By induction, $\mathbb{N} \setminus \{1\} \subseteq P$, so $\mathbb{N} \setminus \{1\} = P$.
Since D is a subset of $\mathbb{N}$, then D is countable. Clearly $\mathbb{N} \setminus \{1\}$ is countable. Since the cartesian product of two countable sets is countable, then $P \times D$ is countable. $\square$

(b) **Q:** Present brief arguments to demonstrate that the set of all subsets of $\mathbb{N}$ is uncountable. Your answer should be no more than 5 sentences.

   A: We can represent every set $S \subseteq \mathbb{N}$ uniquely by an infinite bitstring whose i^{th} bit is 0 if $i \notin S$, and 1 if $i \in S$. For example:

$$\begin{aligned} \emptyset \rightarrow& 000000\ldots \\ \{1,2\} \rightarrow& 110000\ldots \\ \{2,4,6,\ldots\} \rightarrow& 010101\ldots \end{aligned}$$

In this way, we can use diagonalization to produce an unlisted subset of the natural numbers in any such enumeration, so the set of all subsets of the natural numbers must be uncountable. $\square$

Remark. More is true! In fact, as you might notice from the above solution's eerie similarity to the usual diagonalization of $[0, 1]$, the set of subsets (also known as the power set) of $\mathbb{N}$ has the same cardinality as $\mathbb{R}$. Indeed, each subset of $\mathbb{N}$ corresponds to an infinite bitstring, which in turns corresponds to a real number in $[0, 1]$. Moreover:

$$\mathbb{R} = \bigcup_{i \in \mathbb{Z}} [i, i+1]$$

So $\mathbb{R}$ has the same cardinality as $[0, 1]$, and thus the same cardinality as the power set of $\mathbb{N}$.

The above might sound obvious, but it really isn't – not every uncountable set has the same cardinality as $\mathbb{R}$. In fact, the above question asks to prove a simple application of *Cantor's theorem*, which states that for any set S, the power set of S is strictly larger than S. Among other things, this immediately provides an explicit construction of (countably) infinitely many different uncountable cardinalities (called *beth numbers*). So, what is the cardinality of the set of all cardinalities?... Fun!

6. **Q:** Give regular expressions describing the following languages in which the alphabet Σ is $\{0, 1\}$:

   A:

   (a) $\{W : W$ has length at least 3, and its third symbol is $1\}$:

   $$\Sigma\Sigma 1 \Sigma^*$$

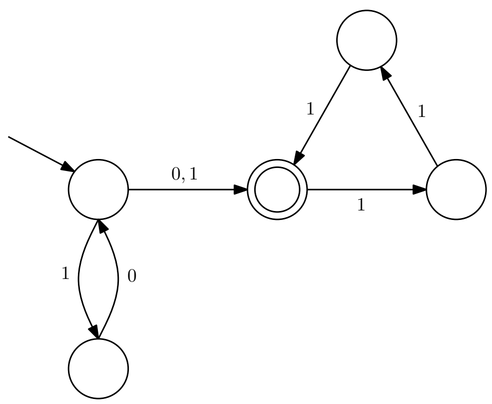
   (b) $\{W :$ Every odd position of W is a $1\}$:

   $$(1\Sigma)^*(1 \cup \varepsilon)$$

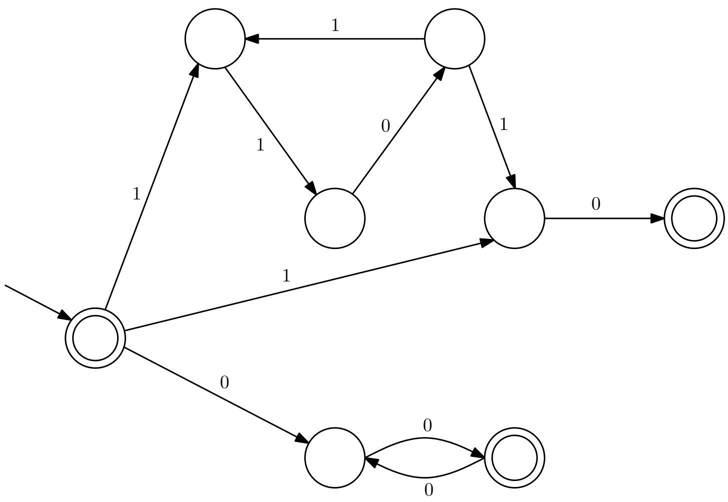
7. **Q:** Develop the NFA (recognizer) without ε-transitions for each of the following regular languages:

   A:

   (a) $(10)^*(0 \cup 1)(111)^*$:



(b) $(110)^*10 \cup (00)^*$:



8. **Q:** Let L_1 and L_2 be regular languages accepted by DFA M_1 and M_2 respectively. Let L_{1R} and L_{2R} be the languages containing the reversed strings from L_1 and L_2. Do the following:

A: (a) Formally describe the NFA that accepts L_{1R}:

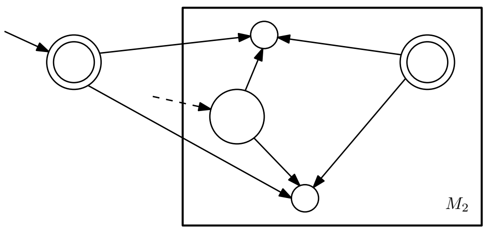
If $M_1 = (Q, \Sigma, \delta, q, F)$, then let $M_{1R} = (Q \cup \{q'\}, \Sigma, \delta', q', F')$, where:

- $F' = \{q\}$, the original starting state of M_1,
- q' is a new starting state, and
- $$\delta'(r, a) = \begin{cases} F & \text{if } r = q',\ a = \varepsilon; \\ \{s : s \in Q, \delta(s, a) = r\} & \text{if } r \neq q'. \end{cases}$$

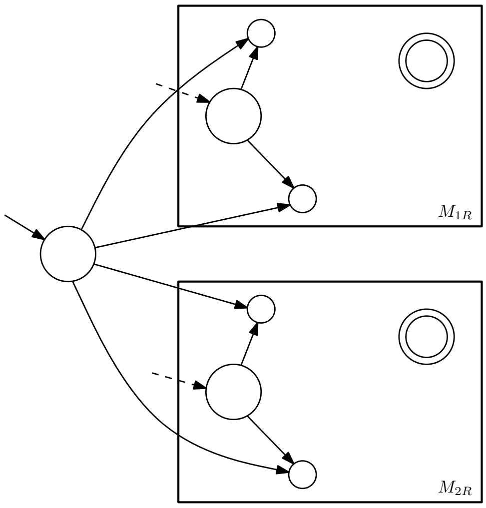
The NFA M_{1R} is the machine M_1 with all transitions reversed, and with a new starting state pointing to all potential final states of M_1. M_{1R} accepts L_{1R}.

(b) Show, without ε-transitions, the NFA that accepts L_2^*:

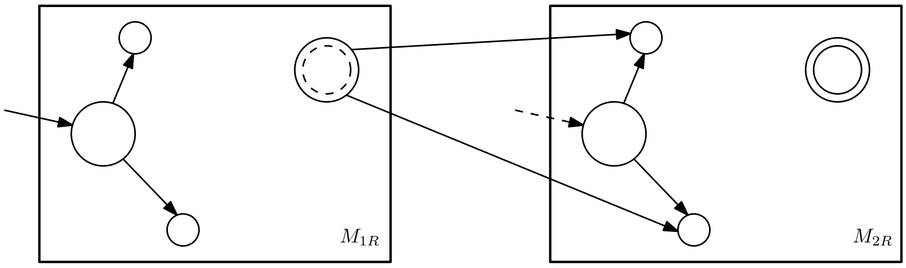
Note: dashed lines denote discarded transitions.



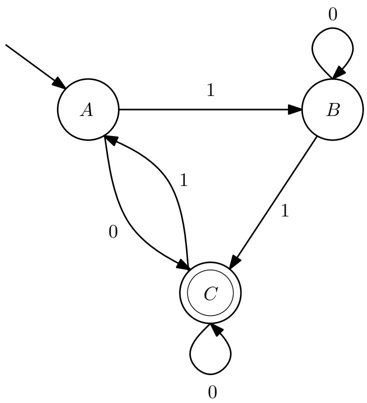
(c) Show, without ε-transitions, the NFA that accepts $L_{1R} \cup L_{2R}$:



(d) Show, without ε-transitions, the NFA that accepts $L_{1R} \cdot L_{2R}$:



9. **Q:** Set up the equations and solve for the regular expression of the language accepted by the DFA given below. You will be given partial credit for just setting up the equations!



A: We obtain, by inspection, the following system of equations:

$$\begin{aligned} L_A &= 1L_B \cup 0L_C \\ L_B &= 0L_B \cup 1L_C \\ L_C &= 0L_C \cup 1L_A \cup \varepsilon \end{aligned}$$

From this, we obtain:

$$L_C = 0^*(1L_A \cup \varepsilon) = 0^*1L_A \cup 0^*$$

and:

$$L_B = 0^*1L_C = 0^*1(0^*1L_A \cup 0^*) = 0^*10^*1L_A \cup 0^*10^*$$

Substituting these expressions into L_A, we get:

$$\begin{aligned} L_A &= 1(0^*10^*1L_A \cup 0^*10^*) \cup 0(0^*1L_A \cup 0^*) \\ &= (10^*10^*1 \cup 00^*1)L_A \cup (10^*10^* \cup 00^*) \\ &= (10^*10^*1 \cup 00^*1)^*(10^*10^* \cup 00^*) \end{aligned}$$

Since A is the starting state, then L_A is the language recognized by the given machine. $\square$