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# Epipolar Geometry

COMP4102A

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Winter 2013

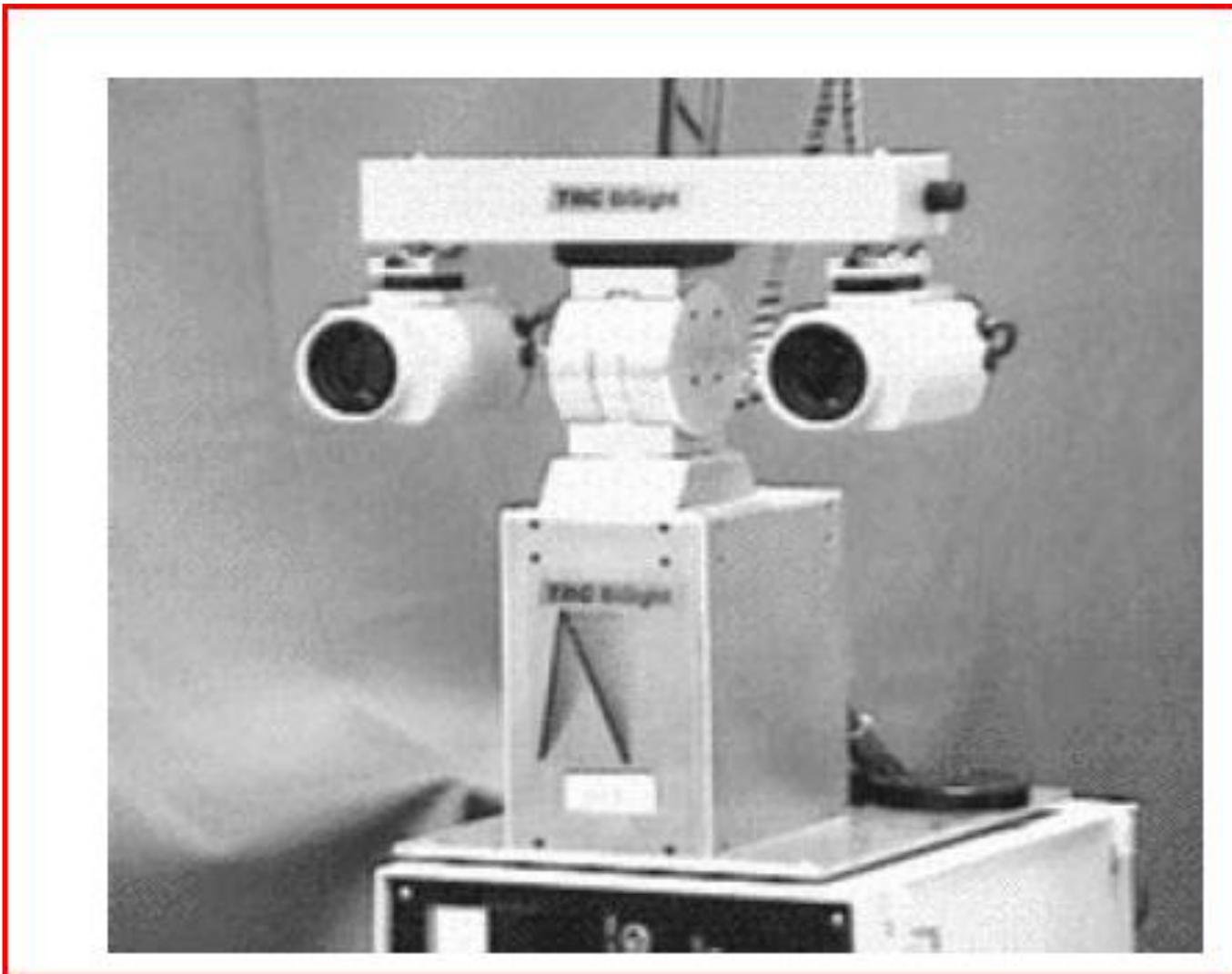
# Problem Definition

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- Simple stereo configuration
  - Corresponding points are on same horizontal line
  - This makes correspondence search a 1D search
  - Need only look for matches on same horizontal line
- if two cameras are in an arbitrary location is there a similar constraint to make search 1D?
  - Yes, called epipolar constraint
  - Based on epipolar geometry
  - We will derive this constraint
  - Consider two cameras that can see a single point P
  - They are in an arbitrary positions and orientation
    - One camera is rotated and translated relative to the other camera
    - Must be some overlap for correspondence and reconstruction!

# Controllable Stereo Head

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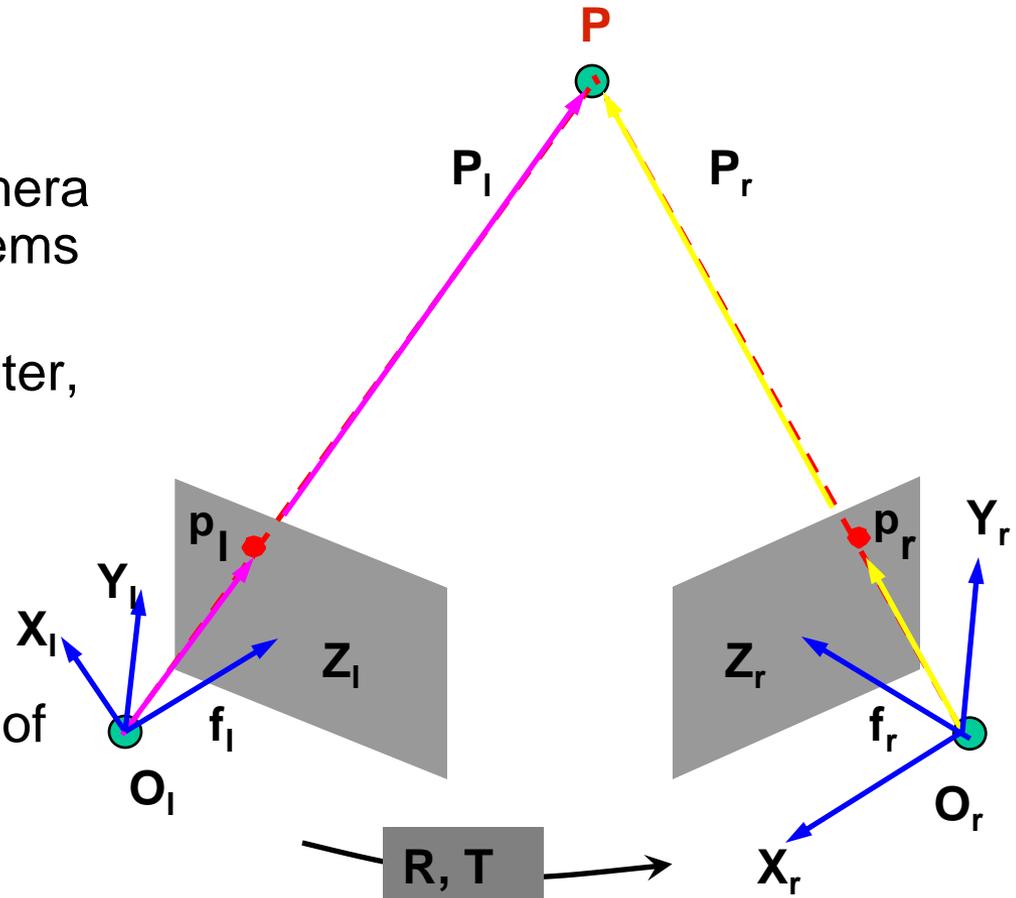
# Parameters of a Stereo System

## Intrinsic Parameters

- Characterize the transformation from camera to pixel coordinate systems of each camera
- Focal length, image center, aspect ratio

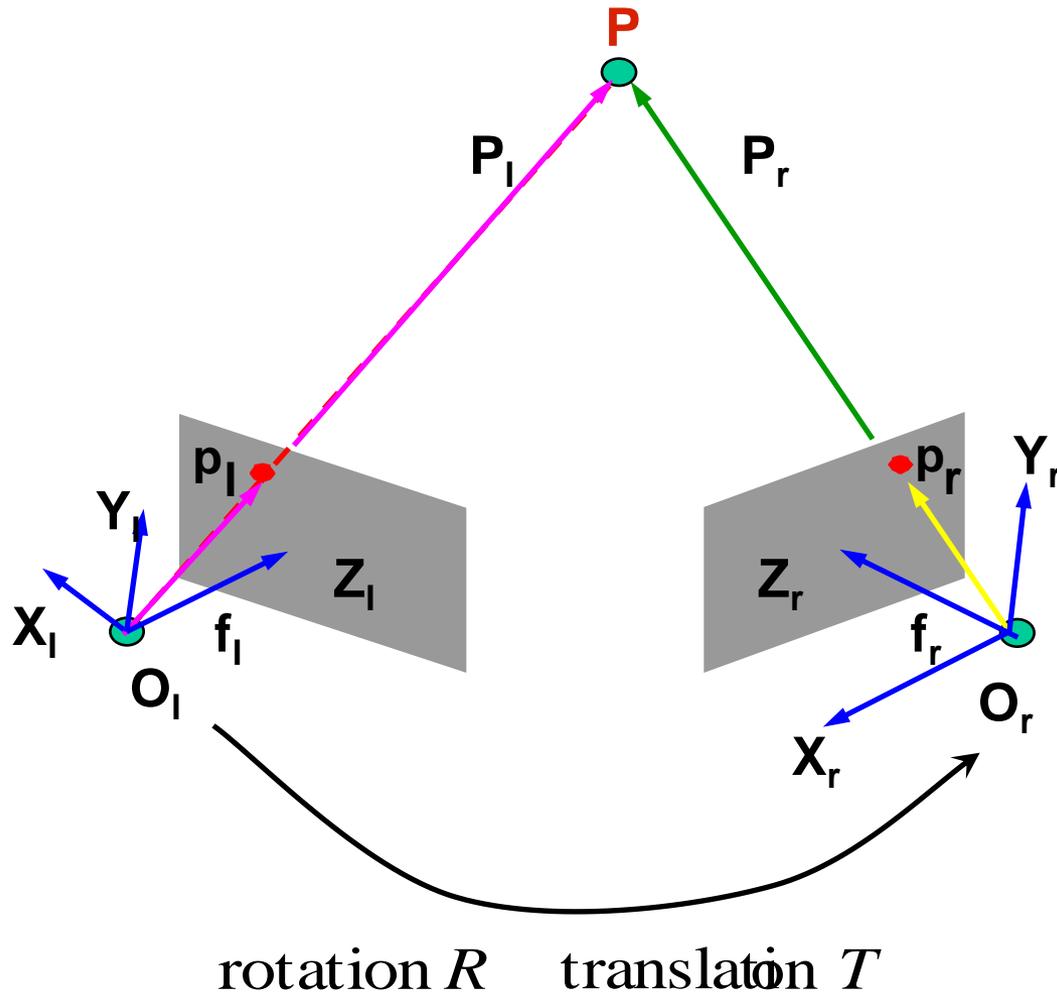
## Extrinsic parameters

- Describe the relative position and orientation of the two cameras
- Rotation matrix  $R$  and translation vector  $T$



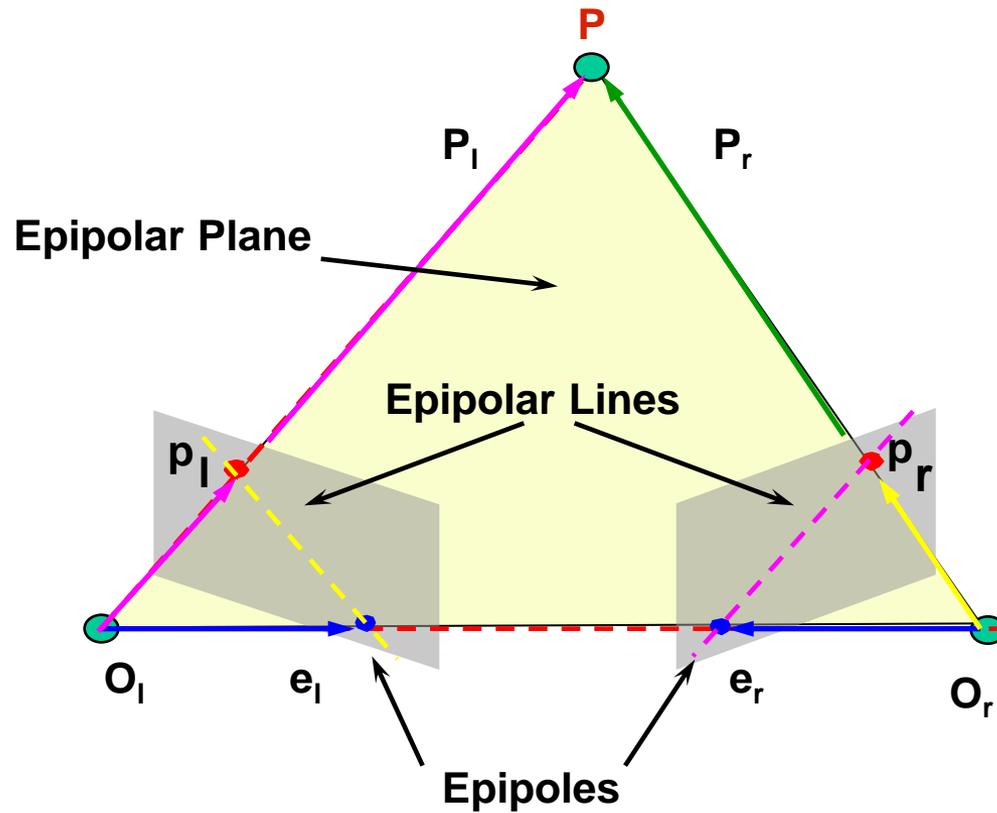
# Epipolar Geometry

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# Epipolar Geometry

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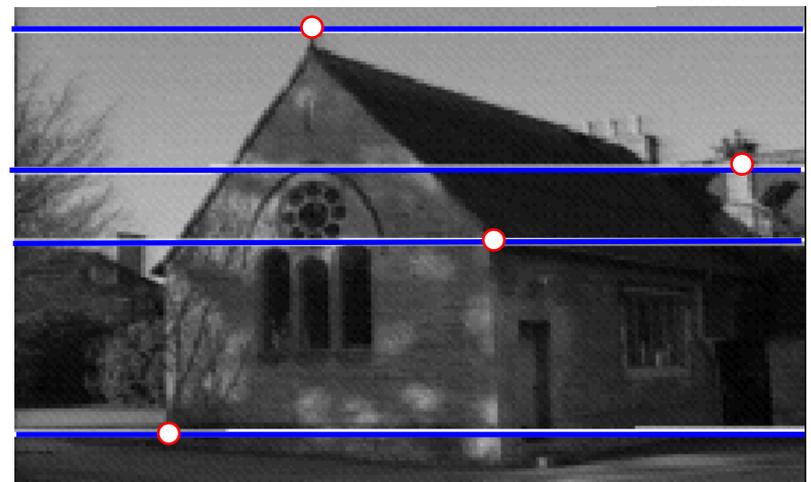
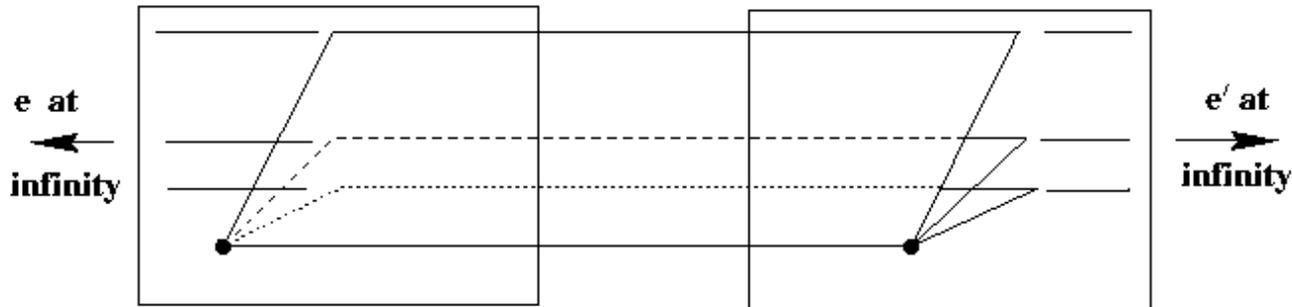
$$P_r = R(P_l - T)$$

# Shape of epipolar lines

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- Translating the cameras in the  $x, y$  plane without any camera rotation then the epipolar lines are parallel
- Translating the cameras in the direction of the camera  $y$  axis (horizontal) you get the simple stereo configuration of horizontal epipolar lines
- Translating the cameras with forward motion in the  $z$  axis produces epipolar lines that emanate from the epipole (sometimes called focus of projection)

# Epipolar geometry : parallel cameras



Epipolar geometry depends **only** on the relative pose (position and orientation) and internal parameters of the two cameras, i.e. the position of the camera centres and image planes. It does **not** depend on the scene structure (3D points external to the camera).

# Epipolar geometry : forward motion

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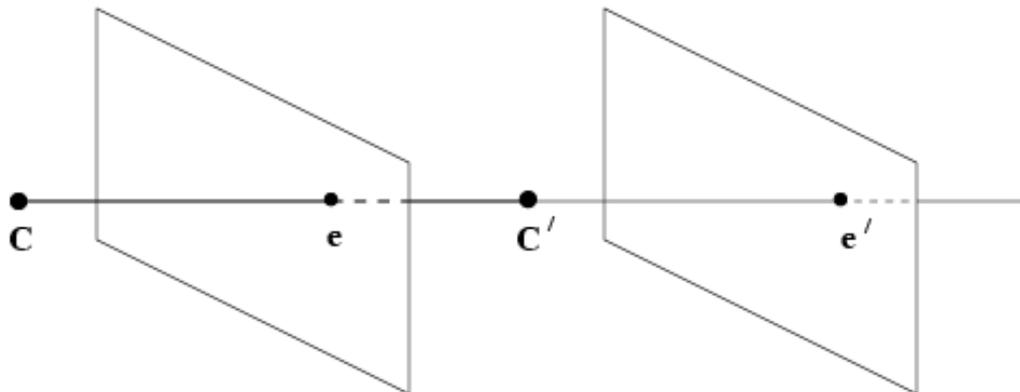
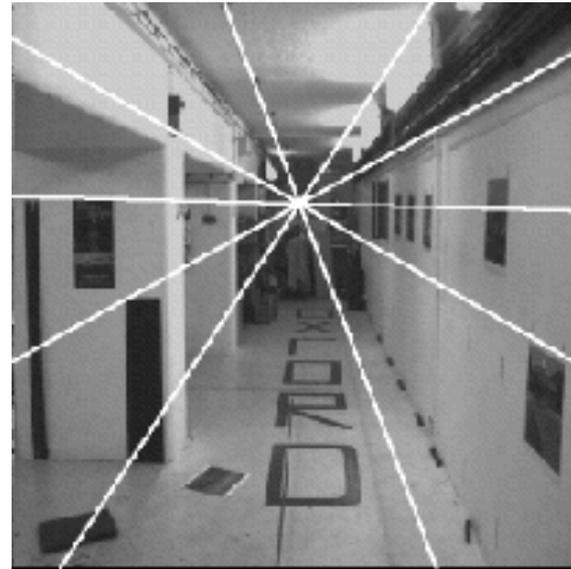
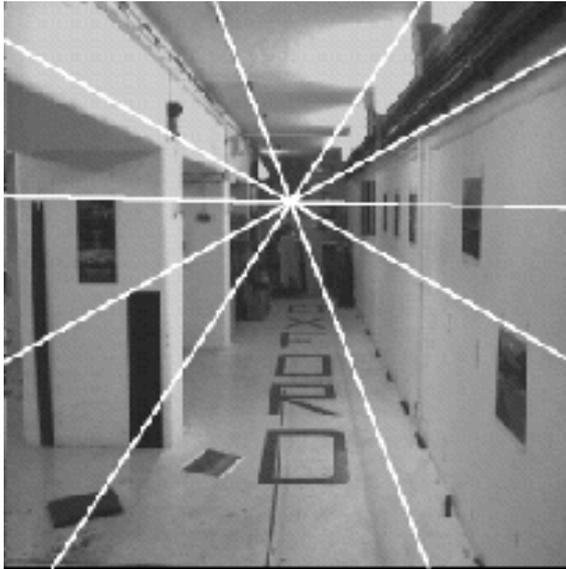


# Epipolar geometry : forward motion

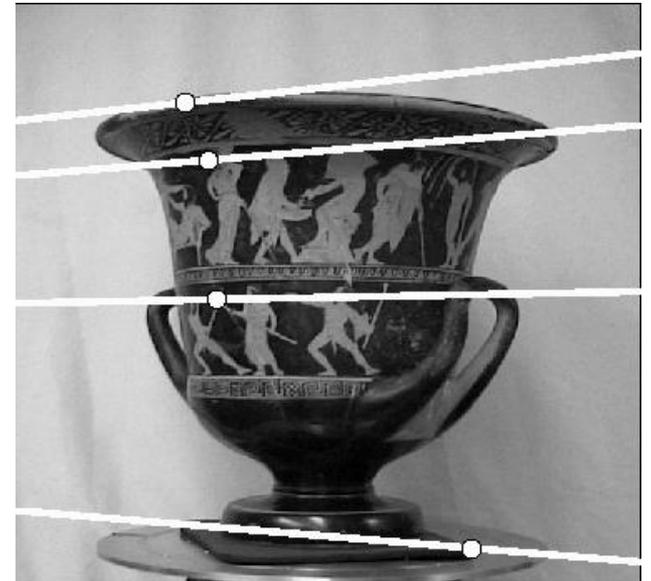
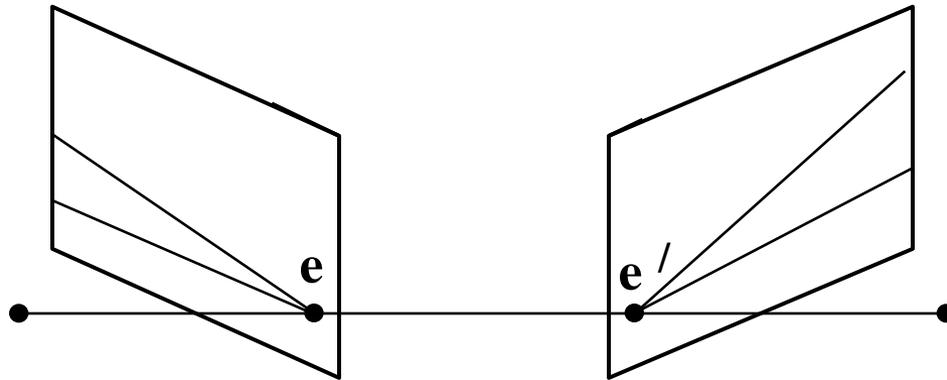


# Epipolar geometry : forward motion

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# Epipolar geometry : converging cameras



So this means that epipolar lines are in general **not** parallel (only for certain cases)

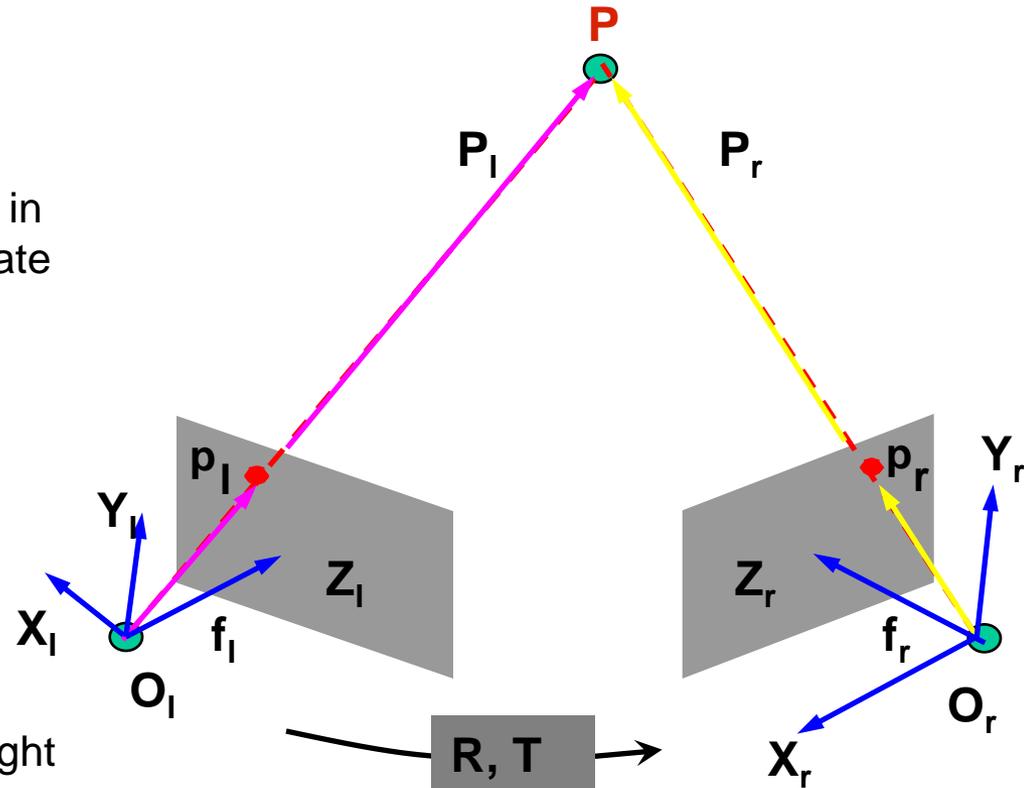
# Epipolar Geometry

## Notations

- $P_l = (X_l, Y_l, Z_l)$ ,  $P_r = (X_r, Y_r, Z_r)$ 
  - Vectors of the same 3-D point  $P$ , in the left and right camera coordinate systems respectively
- Extrinsic Parameters
  - Translation Vector  $T = (O_r - O_l)$
  - Rotation Matrix  $R$

$$P_r = R(P_l - T)$$

- $p_l = (x_l, y_l, z_l)$ ,  $p_r = (x_r, y_r, z_r)$ 
  - Projections of  $P$  on the left and right image plane respectively
  - For all image points, we have  $z_l = f_l$ ,  $z_r = f_r$



$$p_l = \frac{f_l}{Z_l} P_l$$

$$p_r = \frac{f_r}{Z_r} P_r$$

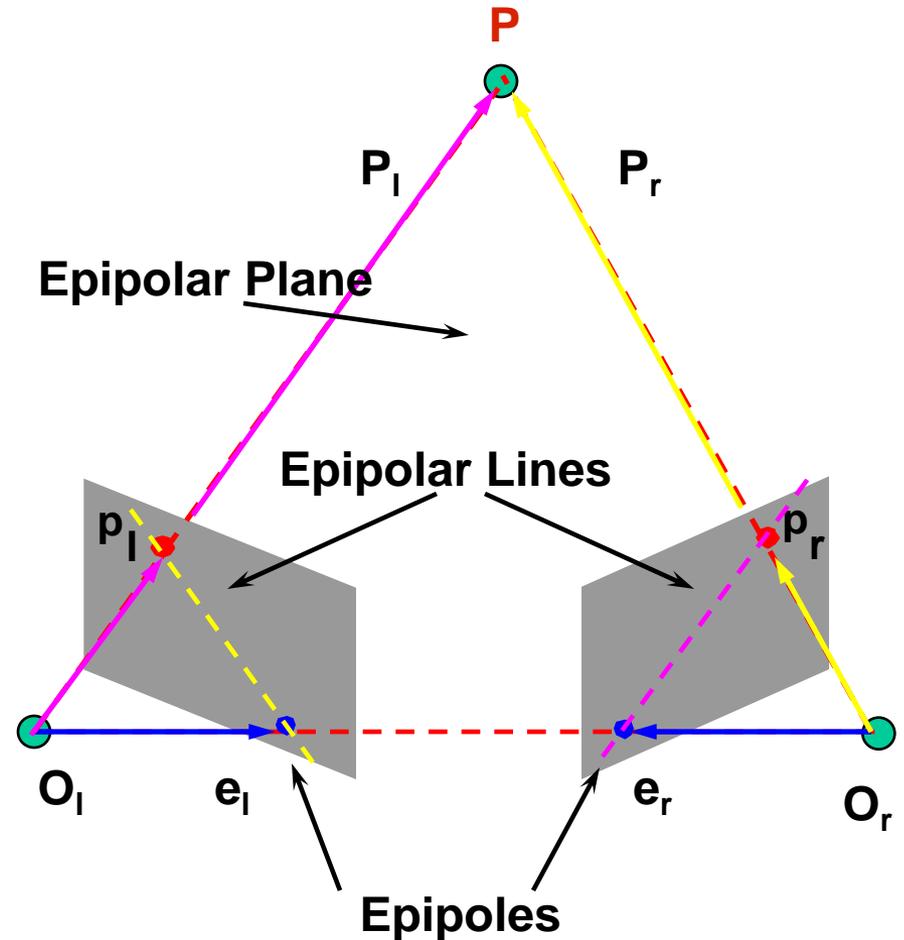
# Epipolar Geometry

Motivation: **where to search correspondences?**

- Epipolar Plane
  - A plane going through point  $P$  and the centers of projection (COPs) of the two cameras
- Epipolar Lines
  - Lines where epipolar plane intersects the image planes
- Epipoles
  - The image in one camera of the COP of the other

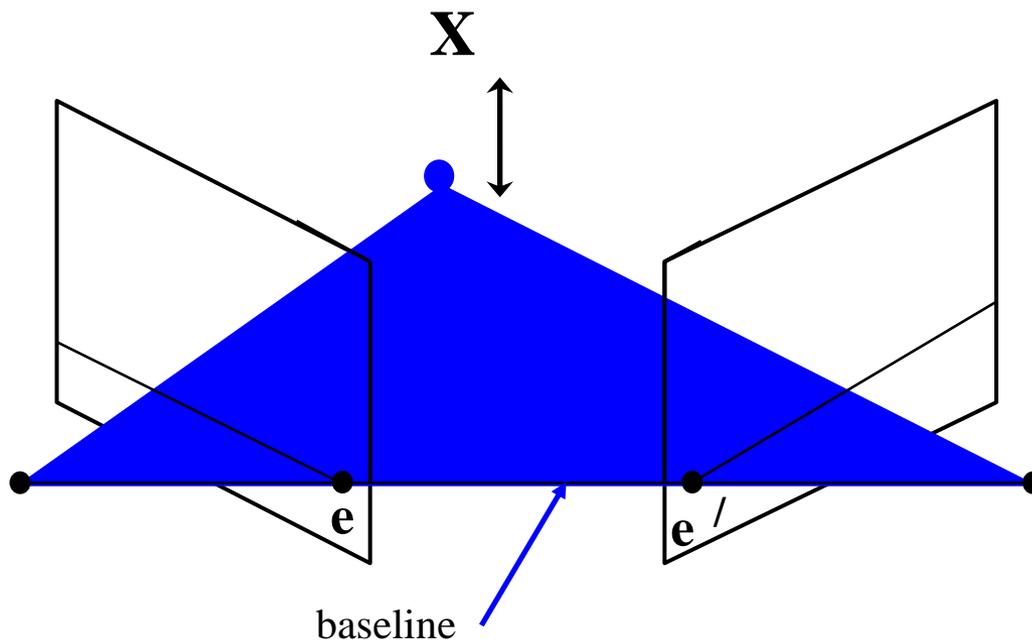
## Epipolar Constraint

- Corresponding points must lie on epipolar lines
- True for EVERY camera configuration, not depend on the geometry of the scene!



# The epipolar pencil

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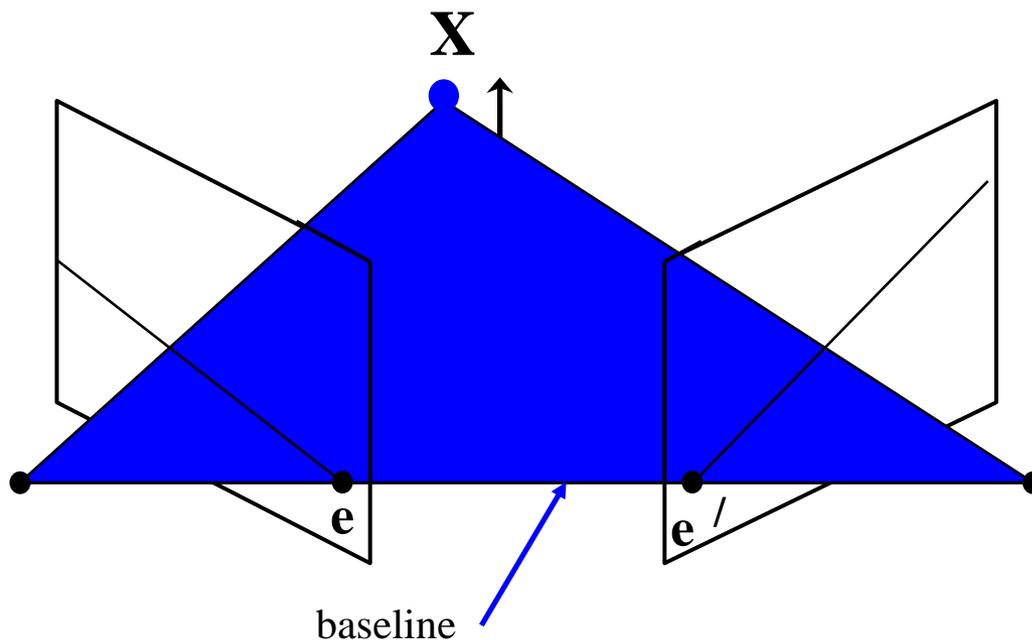


As the position of the 3D point  $X$  varies, the epipolar planes “rotate” about the baseline. This family of planes is known as an epipolar pencil. All epipolar lines intersect at the epipole.

(a pencil is a one parameter family)

# The epipolar pencil

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As the position of the 3D point  $X$  varies, the epipolar planes “rotate” about the baseline. This family of planes is known as an epipolar pencil. All epipolar lines intersect at the epipole.

(a pencil is a one parameter family)

# Epipolar Geometry

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Epipolar plane: plane going through point  $P$  and the centers of projection (COPs) of the two cameras

Epipolar lines: where this epipolar plane intersects the two image planes

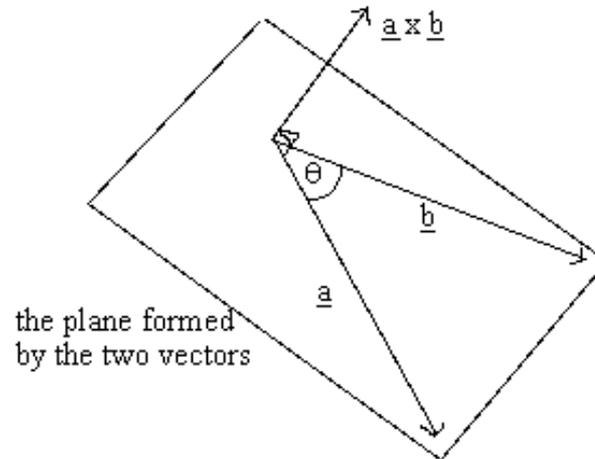
Epipoles: The image in one camera of the COP of the other

Epipolar Constraint: Corresponding points between the two images must lie on epipolar lines

# Cross product

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- Consider two vectors in 3D space
  - $(a_1, a_2, a_3)$  and  $(b_1, b_2, b_3)$
- $\underline{\mathbf{a}} \times \underline{\mathbf{b}} = \underline{\mathbf{n}} a b \sin \theta$
- Cross product is at 90 degrees to both vectors
  - Normal to the plane defined by the two vectors



# Cross product

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- Two possible normal directions
  - We use the right hand rule to compute direction
- $\underline{\mathbf{a}} \times \underline{\mathbf{b}} =$   
 $(a_1 \underline{\mathbf{i}} + a_2 \underline{\mathbf{j}} + a_3 \underline{\mathbf{k}}) \times (b_1 \underline{\mathbf{i}} + b_2 \underline{\mathbf{j}} + b_3 \underline{\mathbf{k}})$
- $\underline{\mathbf{a}} \times \underline{\mathbf{b}} =$   
 $(a_2 b_3 - a_3 b_2) \underline{\mathbf{i}} + (a_3 b_1 - a_1 b_3) \underline{\mathbf{j}} + (a_1 b_2 - a_2 b_1) \underline{\mathbf{k}}$
- Cross product can also be written as multiplication by a matrix
- $\underline{\mathbf{a}} \times \underline{\mathbf{b}} = \mathbf{S} \underline{\mathbf{b}}$

# Cross product as matrix multiplication

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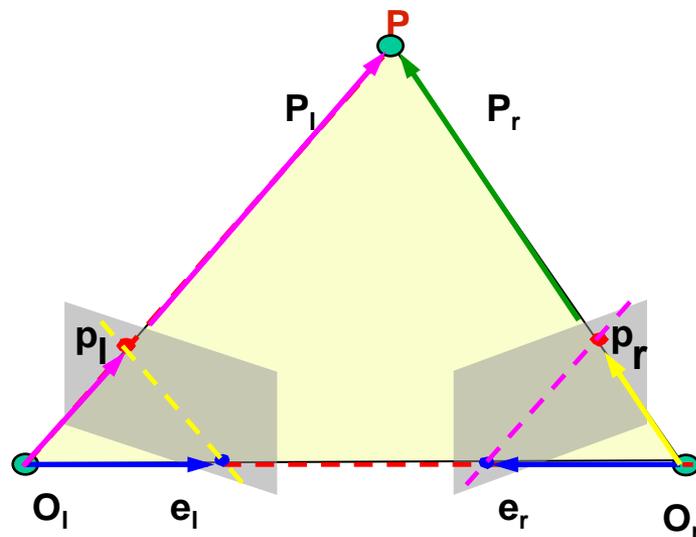
- Define matrix  $S$  as

$$\begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$$

- $\underline{\mathbf{a}} \times \underline{\mathbf{b}} = S \underline{\mathbf{b}}$

- Try the program `cross1.ch` on the course web site

# Essential Matrix



$$T \times P_l = SP_l$$

$$S = \begin{bmatrix} 0 & -T_z & T_y \\ T_z & 0 & -T_x \\ -T_y & T_x & 0 \end{bmatrix}$$

Coordinate Transformation:

$T, P_l, P_l - T$  are coplanar

Resolves to

Essential Matrix  $E = RS$

$$P_r = R(P_l - T)$$

$$(P_l - T)^T T \times P_l = 0$$

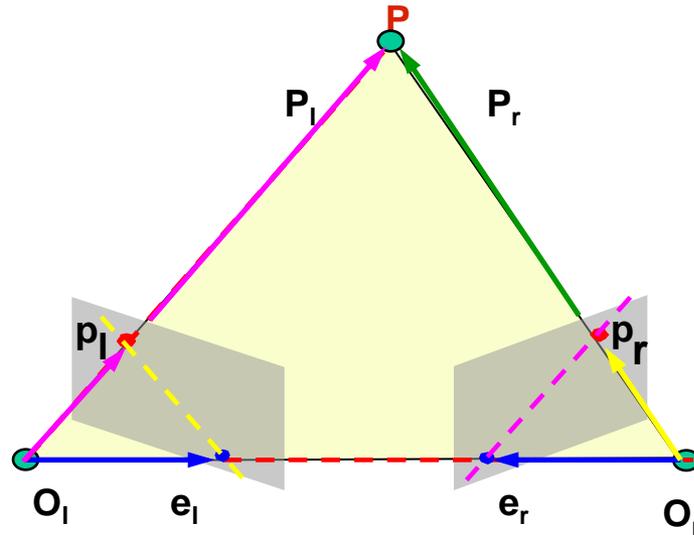
$$(R^T P_r)^T T \times P_l = 0$$

$$(R^T P_r)^T SP_l = 0$$

$$P_r^T RSP_l = 0$$

$$P_r^T EP_l = 0$$

# Essential Matrix



$$P_r^T E P_l = 0 \quad \Rightarrow \quad p_r^T E p_l = 0$$

$$P_r^T E P_l = 0$$

Big P are points  
In 3d space

$$p_l = \frac{f_l}{Z_l} P_l$$

$$p_r = \frac{f_r}{Z_r} P_r$$

Little p are points  
In image plane  
(camera co-ords)

$$p_r^T E p_l = 0$$

# Essential Matrix

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Essential Matrix  $E = RS$

$$\mathbf{p}_r^T \mathbf{E} \mathbf{p}_l = 0$$

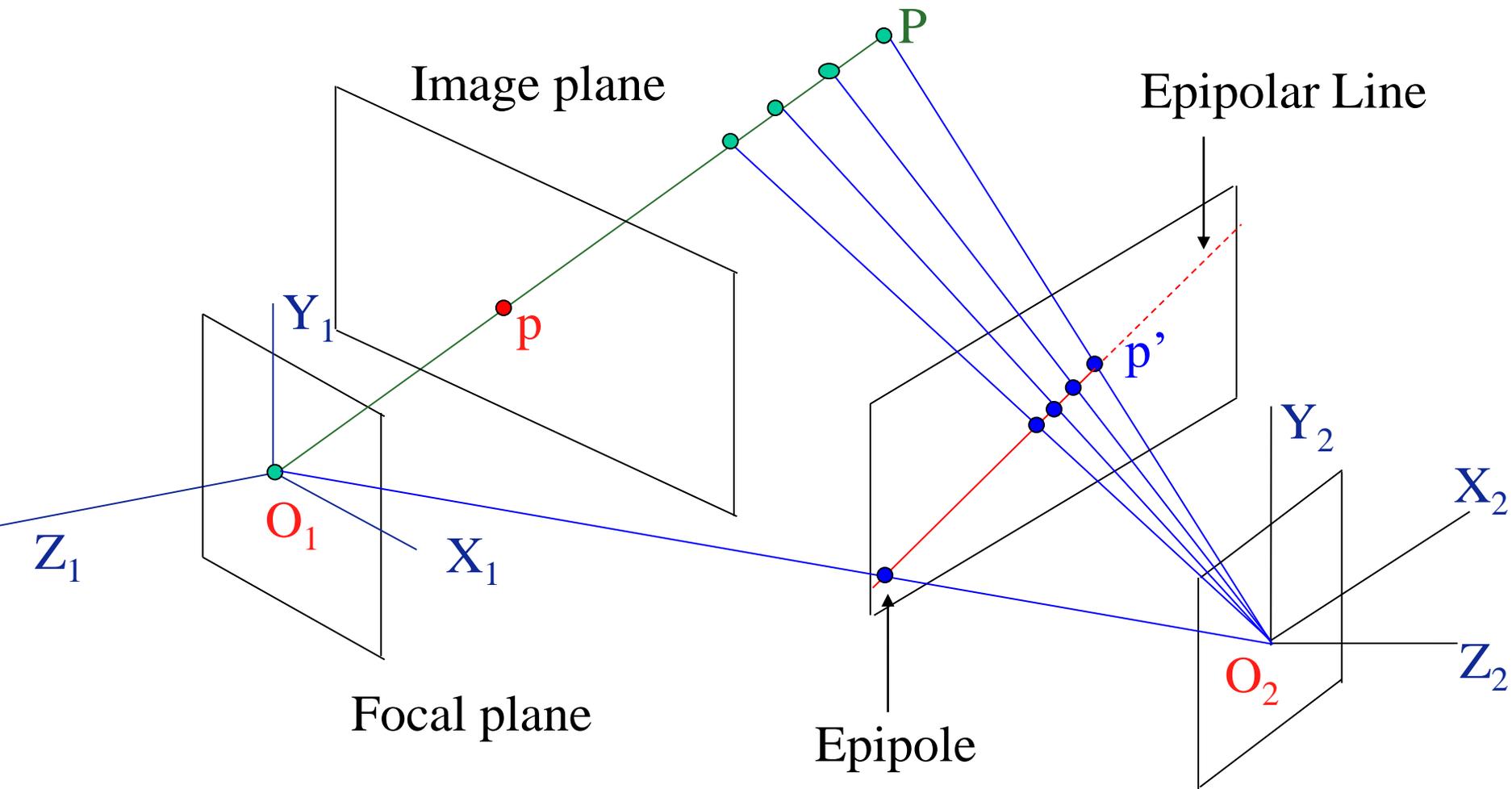
- A natural link between the stereo point pair and the extrinsic parameters of the stereo system
  - One correspondence  $\rightarrow$  a linear equation of 9 entries
  - Given 8 pairs of  $(p_l, p_r) \rightarrow E$
- Mapping between points and epipolar lines we are looking for
  - Given  $p_l, E \rightarrow p_r$  on the projective line in the right plane
  - Equation represents the epipolar line of either  $p_r$  (or  $p_l$ ) in the right (or left) image

## Note:

- $p_l, p_r$  are in the camera coordinate system, not pixel coordinates that we can more easily measure
- Notation in next slide:  $p, p'$  is equivalent to  $p_r$ , and  $p_l$

# What does Essential Matrix Mean?

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# Essential Matrix

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## Essential Matrix $E = RS$

- 3x3 matrix constructed from R and T (extrinsic only)
  - Rank (E) = 2, two equal nonzero singular values

$$\mathbf{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \quad \mathbf{S} = \begin{bmatrix} 0 & -T_z & T_y \\ T_z & 0 & -T_x \\ -T_y & T_x & 0 \end{bmatrix}$$

- **Rank (R) = 3**      **Rank (S) = 2**
- E has five degrees of freedom (3 rotation, 2 translation)
- If we know R and T it is easy to compute E
  - use the camera calibration method of Ch. 6 for two cameras
- But if we just have correspondences between the two stereo cameras then computing E is not as simple to compute

# Projective Geometry

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## •Projective Plane - $P^2$

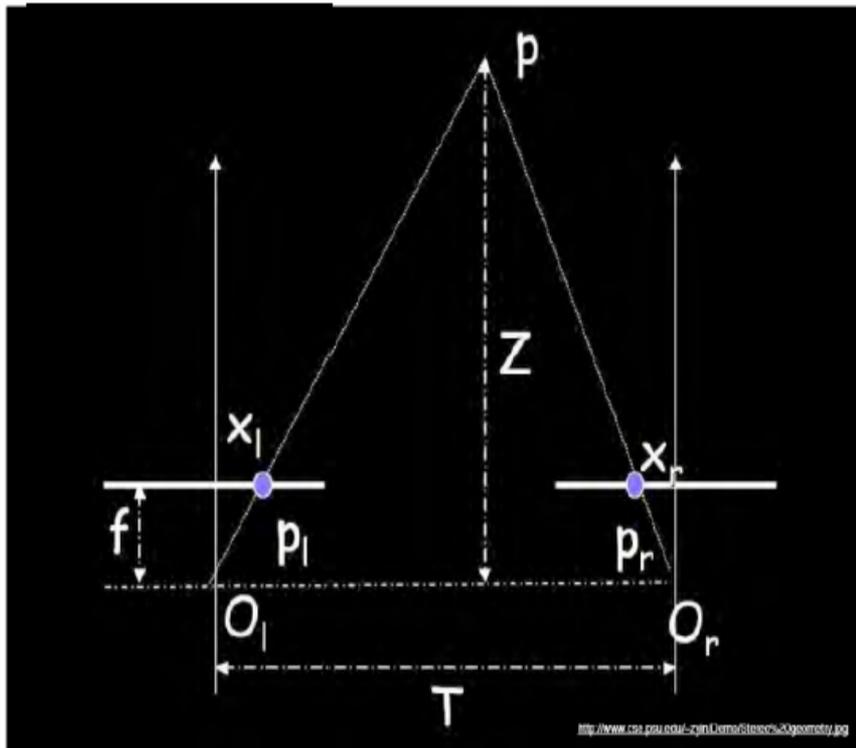
- Set of equivalence classes of triplets of real numbers
- $p = [x,y,z]^T$  and  $p' = [x',y',z']^T$  are equivalent if and only if there is a real number  $k$  such that  $[x,y,z]^T = k [x',y',z']^T$
- Each projective point  $p$  in  $P^2$  corresponds to a line through the origin in  $P^3$
- So points in  $P^2$ , the projective plane, and lines in  $P^3$ , ordinary space, are in a one to one correspondence
- A line in the projective plane is called a projective line represented by  $u = [ux, uy, uz]^T$
- Set of points  $p$  that satisfy the relation  $u^T \bullet p = 0$
- A projective line  $u$  can be represented by a 3d plane through the origin, that is the line defined by the equation  $u^T \bullet p = 0$
- $p$  is either a point lying on the line  $u$ , or a line going through the point  $u$

# Projective Line

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- If we have a point in one image then this means the 3D point  $P$  is on the line from the origin through that point in the image plane
  - So in the other image the corresponding point must be on the epipolar line
- What is the meaning of  $Ep_l$  ?
  - the line in the right plane that goes through  $p_r$  and the epipole  $e_r$
- Therefore the essential matrix is a mapping between points and epipolar lines
- $Ep_l$  defines the equation of the epipolar line in the right image for point  $p_r$  in the right image
- $E^T p_r$  defines the equation of the epipolar line in the left image for point  $p_l$  in the left image
- The relationship  $\mathbf{p}_r^T \mathbf{E} \mathbf{p}_l = 0$  maps a point to a line

# Essential Matrix for Parallel Cameras

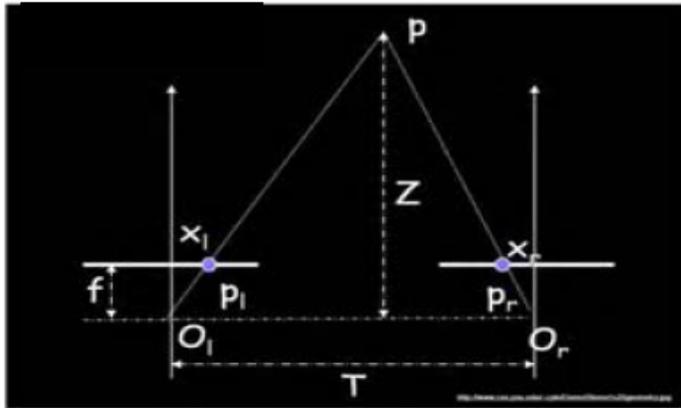


$$\mathbf{R} = \mathbf{I}$$

$$\mathbf{T} = [-T, 0, 0]^T$$

$$\mathbf{E} = [\mathbf{T}_x] \mathbf{R} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & T \\ 0 & -T & 0 \end{pmatrix}$$

# Essential Matrix for Parallel Cameras



$$\mathbf{R} = \mathbf{I}$$

$$\mathbf{T} = [-T, 0, 0]^T$$

$$\mathbf{E} = [\mathbf{T}_x] \mathbf{R} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & T \\ 0 & -T & 0 \end{pmatrix}$$

$$\mathbf{p}^T \mathbf{E} \mathbf{p}' = 0$$

$$\begin{bmatrix} x & y & f \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & T \\ 0 & -T & 0 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ f \end{bmatrix} = 0$$

$$\Leftrightarrow \begin{bmatrix} x & y & f \end{bmatrix} \begin{bmatrix} 0 \\ Tf \\ -Ty' \end{bmatrix} = 0$$

Image of any point must lie on same horizontal line in each image plane!

$$\Leftrightarrow y = y'$$

# Fundamental Matrix

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Same as Essential Matrix but points are in pixel coordinates and not camera coordinates

$$p_r^T E p_l = 0$$



$$\bar{p}_r^T F \bar{p}_l = 0$$

Pixel coordinates

$$F = K_r^{-T} E K_l^{-1}$$

Intrinsic parameters  
Book uses M and not K

# Fundamental Matrix

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Mapping between points and epipolar lines in the pixel coordinate systems

- With no prior knowledge of the stereo system parameters

From Camera to Pixels: Matrices of intrinsic parameters (M matrix in book)

$$\mathbf{K}_{\text{int}} = \begin{bmatrix} -f_x & 0 & o_x \\ 0 & -f_y & o_y \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{Rank}(\mathbf{K}_{\text{int}}) = 3$$

$$\mathbf{p}_r^T \mathbf{E} \mathbf{p}_l = 0$$

$$\mathbf{p}_l = \mathbf{K}_l^{-1} \bar{\mathbf{p}}_l$$

$$\mathbf{p}_r = \mathbf{K}_r^{-1} \bar{\mathbf{p}}_r$$

$$\bar{\mathbf{p}}_r^T \mathbf{F} \bar{\mathbf{p}}_l = 0$$

$$\mathbf{F} = \mathbf{K}_r^{-T} \mathbf{E} \mathbf{K}_l^{-1}$$

# Fundamental Matrix

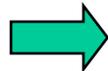
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## Fundamental Matrix

$$\mathbf{F} = \mathbf{K}_r^{-T} \mathbf{E} \mathbf{K}_l^{-1}$$

- Rank (F) = 2
- Encodes info on both intrinsic and extrinsic parameters
- Enables full reconstruction of the epipolar geometry
- In pixel coordinate systems without any knowledge of the intrinsic and extrinsic parameters
- Linear equation of the 9 entries of F but only 8 degrees of freedom because of homogeneous nature of equations

$$\bar{\mathbf{p}}_r^T \mathbf{F} \bar{\mathbf{p}}_l = 0$$



$$\begin{pmatrix} x_{im}^{(l)} & y_{im}^{(l)} & 1 \end{pmatrix} \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix} \begin{pmatrix} x_{im}^{(r)} \\ y_{im}^{(r)} \\ 1 \end{pmatrix} = 0$$

# Fundamental Matrix

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$$[x_r \ y_r \ 1] \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{bmatrix} \begin{bmatrix} x_l \\ y_l \\ 1 \end{bmatrix} = 0$$

- Now expand this equation as below
- One of  $x_l, y_l$  or  $x_r, y_r$  are known, others variables, and all the  $F$ 's are known

$$\begin{aligned} & F_{11}x_lx_r + F_{21}x_ly_r + F_{31}x_l \\ & + F_{12}x_ry_l + F_{22}y_ly_r + F_{32}y_l \\ & + F_{13}x_r + F_{23}y_r + F_{33} = 0 \end{aligned}$$

- This defines equation of a line in the form of  $ax + by + c = 0$  ( $x, y$  variables,  $a, b, c$  constant)!

# Computing E and F

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- If we know extrinsic parameters of stereo system we can compute  $E = R S$
- Using  $E$  and intrinsic parameters we can compute epipolar lines
- If we do not know intrinsic and extrinsic parameters can we compute  $F$ ?
- Yes, for at least 8 correspondences
  - Choose 8 correct correspondences manually
- Then compute  $F$  matrix
  - Then use  $F$  to guide search by computing the epipolar lines
  - This will simplify the process of finding correspondences

# Computing F: The Eight-point Algorithm

Input:  $n$  point correspondences ( $n \geq 8$ )

- Construct homogeneous system  $Ax=0$  from  $\bar{\mathbf{p}}_r^T \mathbf{F} \bar{\mathbf{p}}_l = 0$ 
  - $x = (f_{11}, f_{12}, f_{13}, f_{21}, f_{22}, f_{23}, f_{31}, f_{32}, f_{33})$ : entries in  $F$
  - Each correspondence give one equation
  - $A$  is a  $n \times 9$  matrix
- Obtain estimate  $F^\wedge$  by Eigenvector with smallest eigenvalue
  - $x$  (up to a scale) is column of  $V$  corresponding to the least singular value
- Enforce singularity constraint: since  $\text{Rank}(F) = 2$ 
  - Compute SVD of  $F^\wedge$   $\hat{\mathbf{F}} = \mathbf{U}\mathbf{D}\mathbf{V}^T$
  - Set the smallest singular value to 0:  $D \rightarrow D'$
  - Correct estimate of  $F$ :  $\mathbf{F}' = \mathbf{U}\mathbf{D}'\mathbf{V}^T$

Output: the fundamental matrix,  $F'$

can then compute  $E$  given intrinsic parameters

# Estimating Fundamental Matrix

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## The 8-point algorithm

$$u^T F u' = 0$$

Each point correspondence can be expressed as a linear equation

$$\begin{bmatrix} u & v & 1 \end{bmatrix} \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{bmatrix} \begin{bmatrix} u' \\ v' \\ 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} uu' & uv' & u & u'v & vv' & v & u' & v' & 1 \end{bmatrix} \begin{bmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \\ F_{33} \end{bmatrix} = 0$$

# Homogeneous System

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- M linear equations of form  $A\mathbf{x} = 0$
- If we have a given solution  $\mathbf{x}_1$ , s.t.  $A\mathbf{x}_1 = 0$  then  $c * \mathbf{x}_1$  is also a solution  $A(c * \mathbf{x}_1) = 0$
- Need to add a constraint on  $\mathbf{x}$ ,
  - Basically make  $\mathbf{x}$  a unit vector  $\mathbf{x}^T \mathbf{x} = 1$
- Can prove that the solution is the eigenvector corresponding to the single zero eigenvalue of that matrix  $A^T A$ 
  - This can be computed using eigenvector routine
  - Then finding the zero eigenvalue
  - Returning the associated eigenvector

# Singular Value Decomposition

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- Any  $m$  by  $n$  matrix  $A$  can be written as product of three matrices  $A = UDV^T$
- The columns of the  $m$  by  $m$  matrix  $U$  are mutually orthogonal unit vectors, as are the columns of the  $n$  by  $n$  matrix  $V$
- The  $m$  by  $n$  matrix  $D$  is diagonal, and the diagonal elements,  $\sigma_i$  are called the singular values
- It is the case that  $\sigma_1 \geq \sigma_2 \geq \dots \sigma_n \geq 0$
- Singular values squared are eigenvalues (square mat.)
- The rank of a square matrix is the number of linearly independent rows or columns
- For a square matrix ( $m = n$ ) then the number of non-zero singular values equals the rank of the matrix

# Essential/Fundamental Matrix

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- Essential and fundamental matrix differ
- Relate different quantities
  - Essential matrix is defined in terms of camera co-ordinates
  - Fundamental matrix defined in terms of pixel co-ordinates
- Need different things to calculate them
  - Essential matrix requires camera calibration and knowledge of correspondences
    - known intrinsic parameters, unknown extrinsic parameters
  - Fundamental matrix does not require any camera calibration, just knowledge of correspondences
    - Unknown intrinsic and unknown extrinsic
- Essential and fundamental matrix are related by the camera calibration parameters

# Essential/Fundamental matrix

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- Essential matrix  $E = RS$ 
  - Encodes information on the extrinsic parameters only
  - Has rank 2, since  $S$  has rank 2, and  $R$  is full rank
  - This means one singular value (eigenvalue) is zero
  - Its two non-zero singular values (eigenvalues) are equal
- Fundamental matrix  $F = K_r^{-T} E K_l^{-1}$ 
  - Encodes information on both intrinsic and extrinsic params
  - Has rank 2, since  $K_r$  and  $K_l$  have full rank
  - This means one singular value (eigenvalue) is zero
  - Its two non-zero singular values (eigenvalues) are not necessarily equal

# Essential/Fundamental Matrix

---

- We compute the fundamental matrix from the 2d pixel co-ordinates of correspondences between the left and right image
- If we have the fundamental matrix it is possible to compute the essential matrix if we know the camera calibration
- But we can still compute the epipolar lines using the fundamental matrix
- Therefore if we have the fundamental matrix this limits correspondence search to 1D search for general stereo camera positions in same way as for simple stereo

# Locating the Epipoles from F

$$\bar{\mathbf{p}}_r^T \mathbf{F} \bar{\mathbf{p}}_l = 0$$



$$\bar{\mathbf{p}}_r^T \mathbf{F} \bar{\mathbf{e}}_l = 0$$

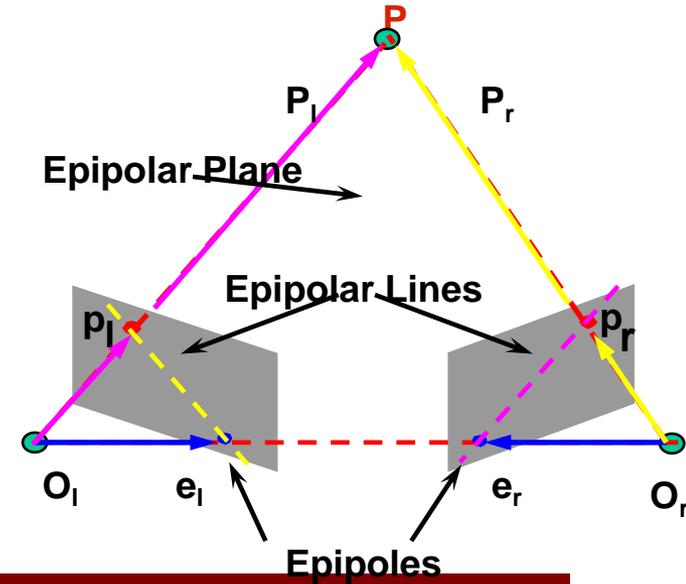


$$\mathbf{F} \bar{\mathbf{e}}_l = 0$$

$\bar{\mathbf{e}}_l$  lies on all the epipolar lines of the left image

For every  $\mathbf{p}_r$

F is not identically zero



Input: Fundamental Matrix F

$$\mathbf{F} = \mathbf{U} \mathbf{D} \mathbf{V}^T$$

- Find the SVD of F
- The epipole  $\mathbf{e}_l$  is the column of V corresponding to the null singular value (as shown above)
- The epipole  $\mathbf{e}_r$  is the column of U corresponding to the null singular value

Output: Epipole  $\mathbf{e}_l$  and  $\mathbf{e}_r$

# Fundamental and Essential matrix

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Both  $F$  and  $E$  are  $3 \times 3$  matrices, facts below apply to both

**Transpose :** If  $F$  is a fundamental matrix of cameras  $(P, P')$  then  $F^T$  is a fundamental matrix of camera  $(P', P)$

**Epipolar lines:** Think of  $p$  and  $p'$  as points in the projective plane then  $F p$  is projective line in the right image.

$$\text{That is } l' = F p \quad l = F^T p'$$

**Epipole:** Since for any  $p$  the epipolar line  $l' = F p$  contains the epipole  $e'$ . Thus  $(e'^T F) p = 0$  for a all  $p$ .

$$\text{Thus } e'^T F = 0 \quad \text{and } F e = 0$$

# Epipolar Geometry

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- Basic constraint used to help correspondence
- Makes search for matching points into a 1D search along epipolar lines
- If you have intrinsic and extrinsic parameters
  - Then compute essential matrix and find epipolar lines
- If you do not have intrinsic or extrinsic parameters but have at least 8 correct correspondences then
  - Compute fundamental matrix and find epipolar lines
- Can also compute the epipoles using SVD