# Review of Linear Algebra

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# Linear algebra

- Is an important area of mathematics

- It is the basis of computer vision

- Is very widely taught, and there are many resources and books available

- We only focus on the basics

- Need to understand matrices, vectors and their basic operations

# Linear Equations

A system of linear equations, e.g.

$$2x_1 + 4x_2 = 2$$
$$4x_1 + 11x_2 = 1$$

can be written in matrix form, m rows and n columns:

$$\begin{bmatrix} 2 & 4 \\ 4 & 11 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

or in general:

$$Ax = b$$

# Matrix

A matrix is an $m \times n$ array of numbers.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} a_{ij} \end{bmatrix}$$

Example:

$$A = \begin{bmatrix} 2 & 3 & 5 & 4 \\ -4 & 1 & 3 & 9 \\ 0 & 7 & 10 & 11 \end{bmatrix}$$

# Matrix Arithmetic

Matrix addition

$$A_{m \times n} + B_{m \times n} = \left[ a_{ij} + b_{ij} \right]_{m \times n}$$

Matrix multiplication

$$A_{m \times n} B_{n \times p} = C_{m \times p} \qquad c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

Matrix transpose

$$A^T = \left[ a_{ji} \right]$$

$$(A + B)^T = A^T + B^T \qquad (AB)^T = B^T A^T$$

# Multiplication not commutative

Matrix multiplication is not commutative

$$AB \neq BA$$

Example:

$$\begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 17 & 29 \\ 19 & 11 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 18 & 32 \\ 17 & 10 \end{bmatrix}$$

# Symmetric Matrix

We say matrix A is symmetric if

$$A^T = A$$

Example:

$$A = \begin{bmatrix} 2 & 4 \\ 4 & 5 \end{bmatrix}$$

A symmetric matrix has to be a square matrix

# Inverse of matrix

If A is a square matrix, the inverse of A, written $A^{-1}$ satisfies:

$$AA^{-1} = I \qquad A^{-1}A = I$$

Where $I$, the identity matrix, is a diagonal matrix with all 1's on the diagonal.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

# Vectors – matrices with one column

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$
e.g. 
$$x = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$$

A vector is an n by 1 matrix, it is a column in our book

The **length** or the **norm** of a vector is

$$\|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

e.g. $\|x\| = \sqrt{2^2 + 3^2 + 5^2} = \sqrt{38}$

# Vector Arithmetic

**Vector addition**

$$u + v = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}$$

**Vector subtraction**

$$u - v = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} - \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} u_1 - v_1 \\ u_2 - v_2 \end{bmatrix}$$

**Multiplication by scalar**

$$\alpha u = \alpha \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \alpha u_1 \\ \alpha u_2 \end{bmatrix}$$

# Dot Product (inner product)

$$a = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} \qquad b = \begin{bmatrix} 4 \\ -3 \\ 2 \end{bmatrix}$$

$$a \cdot b = a^T b = \begin{bmatrix} 2 & 3 & 5 \end{bmatrix} \begin{bmatrix} 4 \\ -3 \\ 2 \end{bmatrix} = 2 \cdot 4 + 3 \cdot (-3) + 5 \cdot 2 = 9$$

$$a \cdot b = a^T b = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

# Vectors: Dot Product (Inner product)

$$A \cdot B = A^T B = \begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} d \\ e \\ f \end{bmatrix} = ad + be + cf$$

Think of the dot product as a matrix multiplication

$$\|A\|^2 = A^T A = aa + bb + cc$$

The magnitude is the dot product of a vector with itself

$$A \cdot B = \|A\| \|B\| \cos(\theta)$$

The dot product is also related to the angle between the two vectors

# Trace of Matrix

The trace of a matrix:

$$Tr(A) = \sum_{i=1}^{n} a_{ii}$$

# Orthogonal Matrix

A matrix A is orthogonal if

$$A^T A = I \quad \text{or} \quad A^T = A^{-1}$$

Example:

$$A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

# Matrix Transformation (and projections)
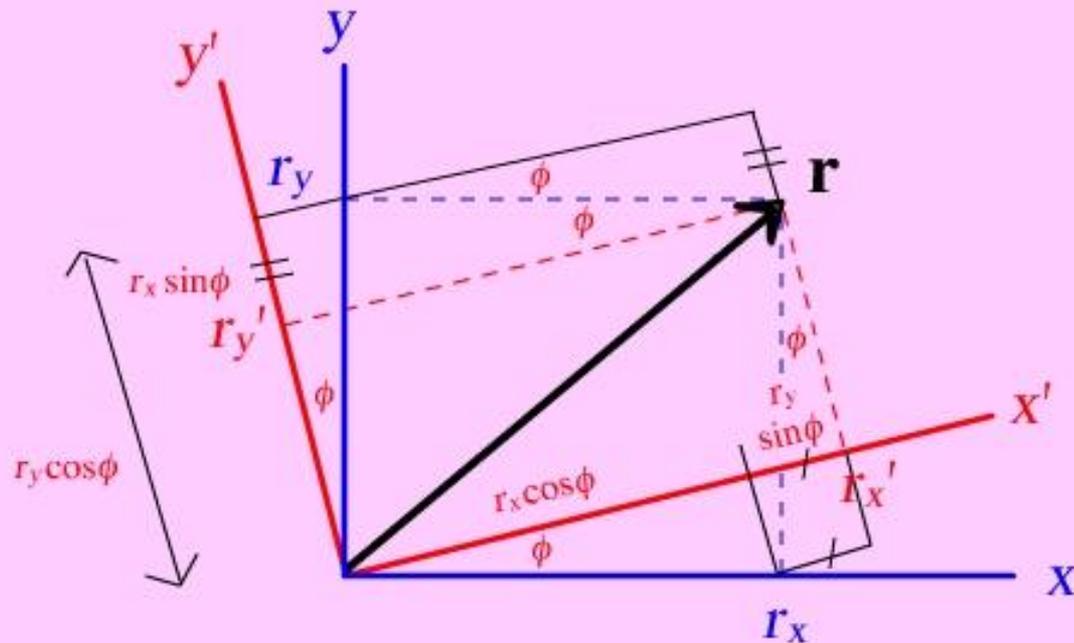
A matrix-vector multiplication transforms one vector to another

$$A_{m \times n} x_{n \times 1} = b_{m \times 1}$$

Example:

$$\begin{bmatrix} 2 & 5 \\ 3 & 1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 7 \end{bmatrix} = \begin{bmatrix} 41 \\ 16 \\ 26 \end{bmatrix}$$
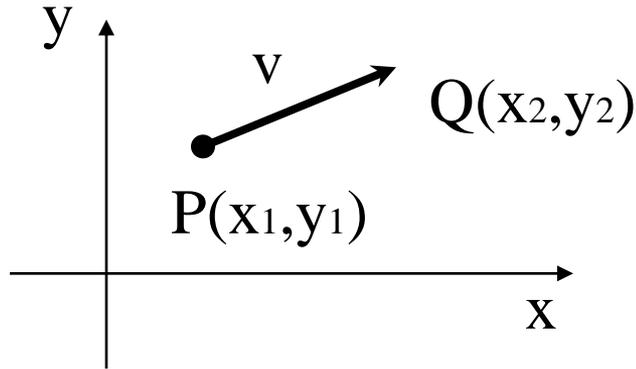
# Coordinate Rotation



$$r_x' = r_x \cos\phi + r_y \sin\phi$$
$$r_y' = -r_x \sin\phi + r_y \cos\phi$$

$$\begin{bmatrix} r_x' \\ r_y' \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} r_x \\ r_y \end{bmatrix}$$
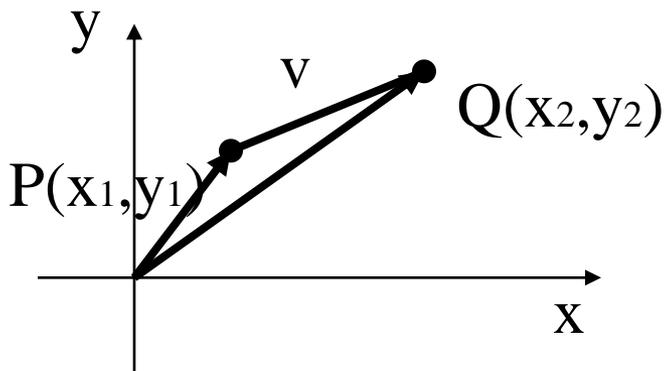
# Vectors and Points

Two points in a Cartesian coordinate system define a vector

$$v = \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix}$$

A point can also be represented as a vector, defined by the point and the origin (0,0).

$$P = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \qquad Q = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$$

$$v = Q - P \quad \text{or} \quad Q = P + v$$

Note: point and vector are different; vectors do not have positions

# Least Squares

When m>n for an m-by-n matrix A, $Ax = b$ has no solution.

In this case, we look for an approximate solution.
We look for vector $x$ such that

$$\|Ax - b\|^2$$

is as small as possible.

This is called the least squares solution.

# Least Squares

Least squares solution of linear system of equations
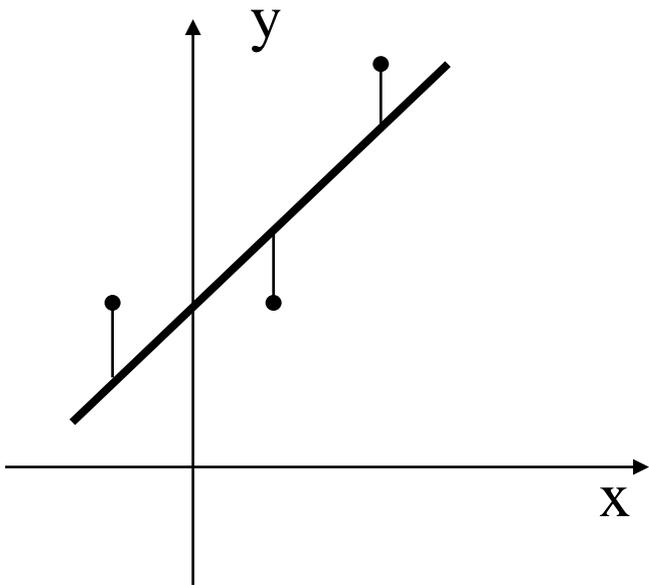
$$Ax = b$$

Normal equation: $A^T Ax = A^T b$

$A^T A$ is square and symmetric

The Least Square solution $\bar{x} = (A^T A)^{-1} A^T b$

makes $\|A\bar{x} - b\|^2$ minimal.

# Least Square Fitting of a Line

Line equations:

$$c + dx_1 = y_1$$
$$c + dx_2 = y_2$$
$$\vdots$$
$$c + dx_m = y_m$$

$$\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_m \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$$

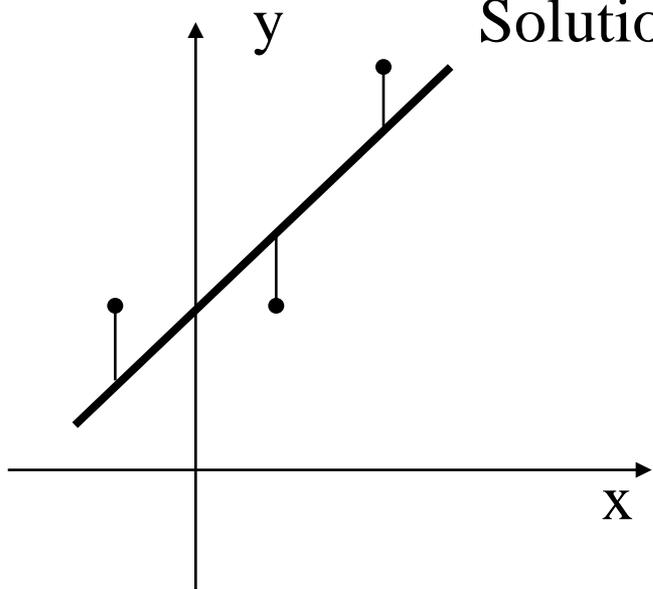$$Ax = y$$

The best solution c, d is the one that minimizes:

$$E^2 = \|y - Ax\|^2 = (y_1 - c - dx_1)^2 + \cdots + (y_m - c - dx_m)^2.$$

# Least Square Fitting - Example

Problem: find the line that best fit these three points:

$$P1=(-1,1), P2=(1,1), P3=(2,3)$$

Solution:

$$c - d = 1$$
$$c + d = 1$$
$$c + 2d = 3$$

or

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$$

$$A^T A x = A^T b \quad \text{is} \quad \begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

The solution is $c = \frac{9}{7}, d = \frac{4}{7}$ and best line is $\frac{9}{7} + \frac{4}{7}x = y$

# Homogeneous System

- m linear equations with n unknowns A$\mathbf{x}$ = 0
- Assume that m >= n-1 and rank(A) = n-1
- Trivial solution is $\mathbf{x}$ = 0 but there are more
- If we have a given solution $\mathbf{x}$, s.t. A$\mathbf{x}$ = 0 then c * $\mathbf{x}$ is also a solution since A(c* $\mathbf{x}$) = 0
- Need to add a constraint on $\mathbf{x}$,
  - Usually make $\mathbf{x}$ a unit vector $\mathrm{x}^{\mathrm{T}}\mathrm{x} = 1$
- Can prove that the solution of A$\mathbf{x}$ = 0 satisfying this constraint is the eigenvector corresponding to the only zero eigenvalue of that matrix $\mathrm{A}^{\mathrm{T}}\mathrm{A}$

# Homogeneous System

- This solution can be computed using the eigenvector or SVD routine

    - Find the zero eigenvalue (or the eigenvalue almost zero)

    - Then the associated eigenvector is the solution **x**

- And any scalar times **x** is also a solution

# Linear Independence

- A set of vectors is linear dependant if one of the vectors can be expressed as a linear combination of the other vectors.

$$v_k = \alpha_1 v_1 + \cdots + \alpha_{k-1} v_{k-1} + \alpha_{k+1} v_{k+1} + \cdots + \alpha_n v_n$$

- A set of vectors is linearly independent if none of the vectors can be expressed as a linear combination of the other vectors.

# Eigenvalue and Eigenvector

We say that x is an eigenvector of a square matrix A if

$$Ax = \lambda x$$

$\lambda$ is called eigenvalue and $x$ is called eigenvector.

The transformation defined by A changes only the magnitude of the vector $x$

Example:

$$\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

5 and 2 are eigenvalues, and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ are eigenvectors.

# Properties of Eigen Vectors

- If $\lambda_1, \lambda_2, \ldots, \lambda_q$ are distinct eigenvalues of a matrix, then the corresponding eigenvectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_q$ are linearly independent.

- A real, symmetric matrix has real eigenvalues with eigenvectors that can be chosen to be orthonormal.

# SVD: Singular Value Decomposition

An $m \times n$ matrix A can be decomposed into:

$$A = UDV^T$$

U is $m \times m$, V is $n \times n$, both of them have orthogonal columns:

$$U^T U = I \qquad V^T V = I$$

D is an $m \times n$ diagonal matrix.

Example:

$$\begin{bmatrix} 2 & 0 \\ 0 & -3 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

# Singular Value Decomposition

- Any m by n matrix A can be written as product of three matrices $A = UDV^T$

- The columns of the m by m matrix U are mutually orthogonal unit vectors, as are the columns of the n by n matrix V

- The m by n matrix D is diagonal, and the diagonal elements, $\sigma_i$ are called the singular values

- It is the case that $\sigma_1 \geq \sigma_2 \geq \ldots \sigma_n \geq 0$

- A matrix is non-singular if and only all of the singular values are not zero

- The condition number of the matrix is $\dfrac{\sigma_1}{\sigma_n}$

- If the condition number is large, then then matrix is almost singular and is called ill-conditioned

# Singular Value Decomposition

- The rank of a square matrix is the number of linearly independent rows or columns

- For a square matrix (m = n) the number of non-zero singular values equals the rank of the matrix

- If A is a square, non-singular matrix, it's inverse can be written as $A^{-1} = VD^{-1}U^T$ where $A = UDV^T$

- The squares of the non zero singular values are the non-zero eigenvalues of both the n by n matrix $A^T A$ and of the m by m matrix $AA^T$

- The columns of U are the eigenvectors of $AA^T$

- The columns of V are the eigenvectors of $A^T A$