
Epipolar Geometry

COMP4102A

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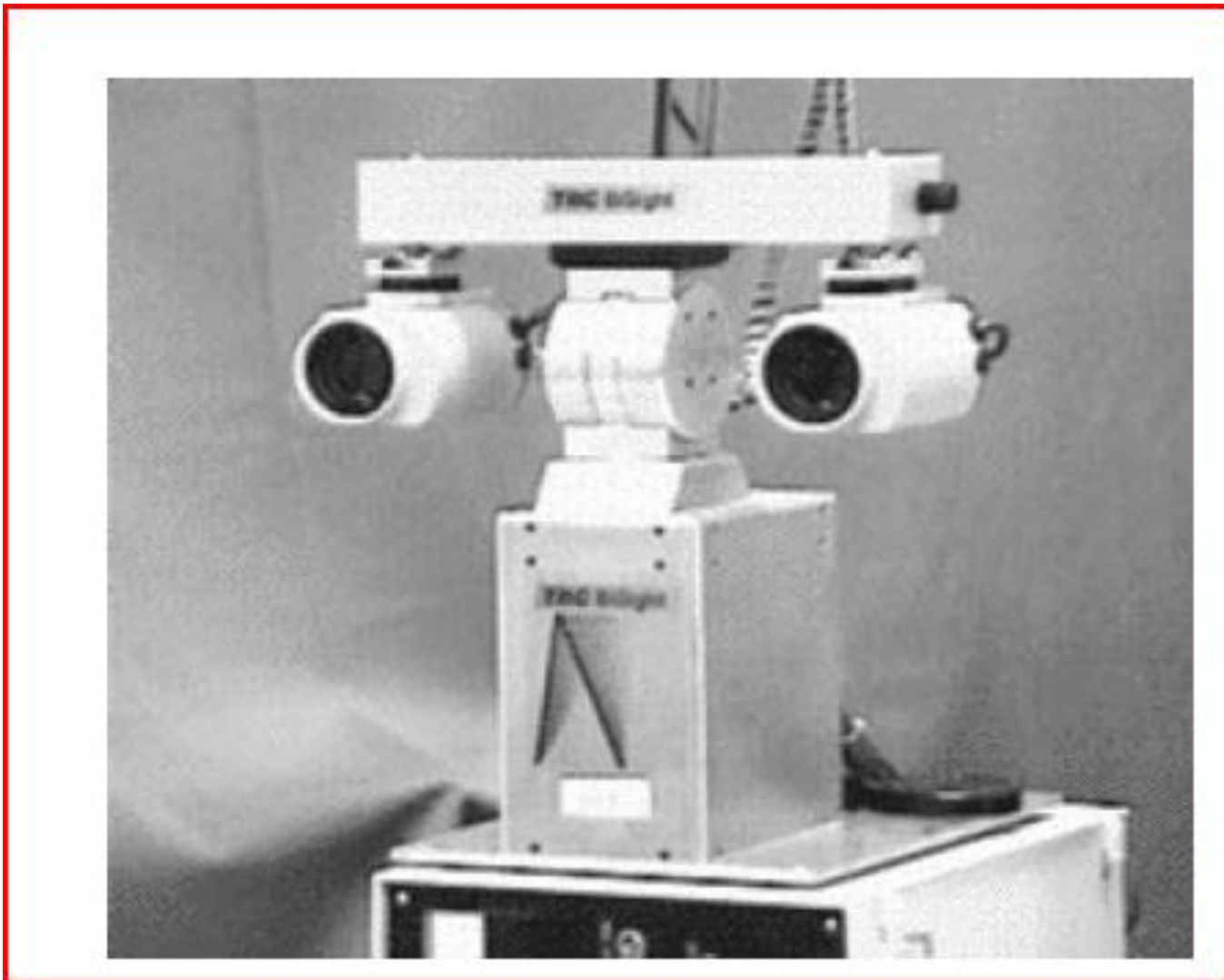
Winter 2014

Version 2

Problem Definition

- Simple stereo configuration
 - Corresponding points are on same horizontal line
 - This makes correspondence search a 1D search
 - Need only look for matches on same horizontal line
- if two cameras are in an arbitrary location is there a similar constraint to make search 1D?
 - Yes, called epipolar constraint
 - Based on epipolar geometry
 - We will derive this constraint
 - Consider two cameras that can see a single point P
 - They are in an arbitrary positions and orientation
 - One camera is rotated and translated relative to the other camera
 - Must be some overlap for correspondence and reconstruction!

Controllable Stereo Head



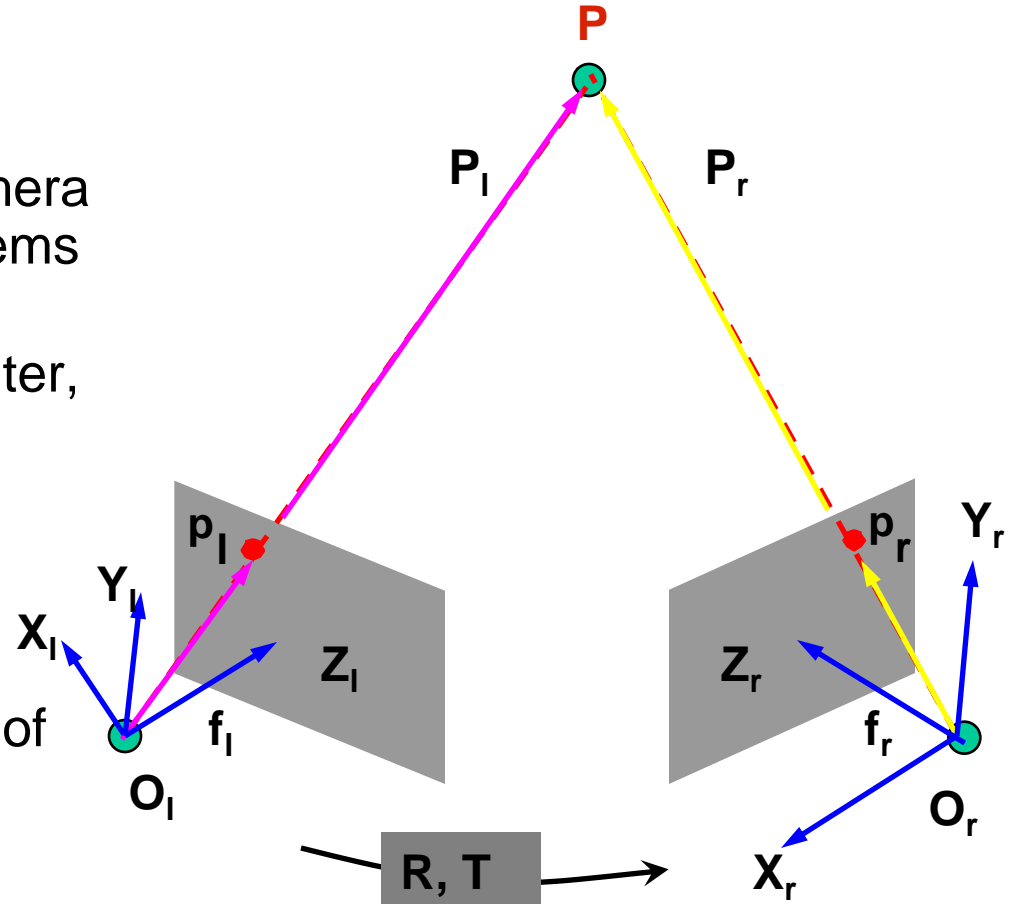
Parameters of a Stereo System

Intrinsic Parameters

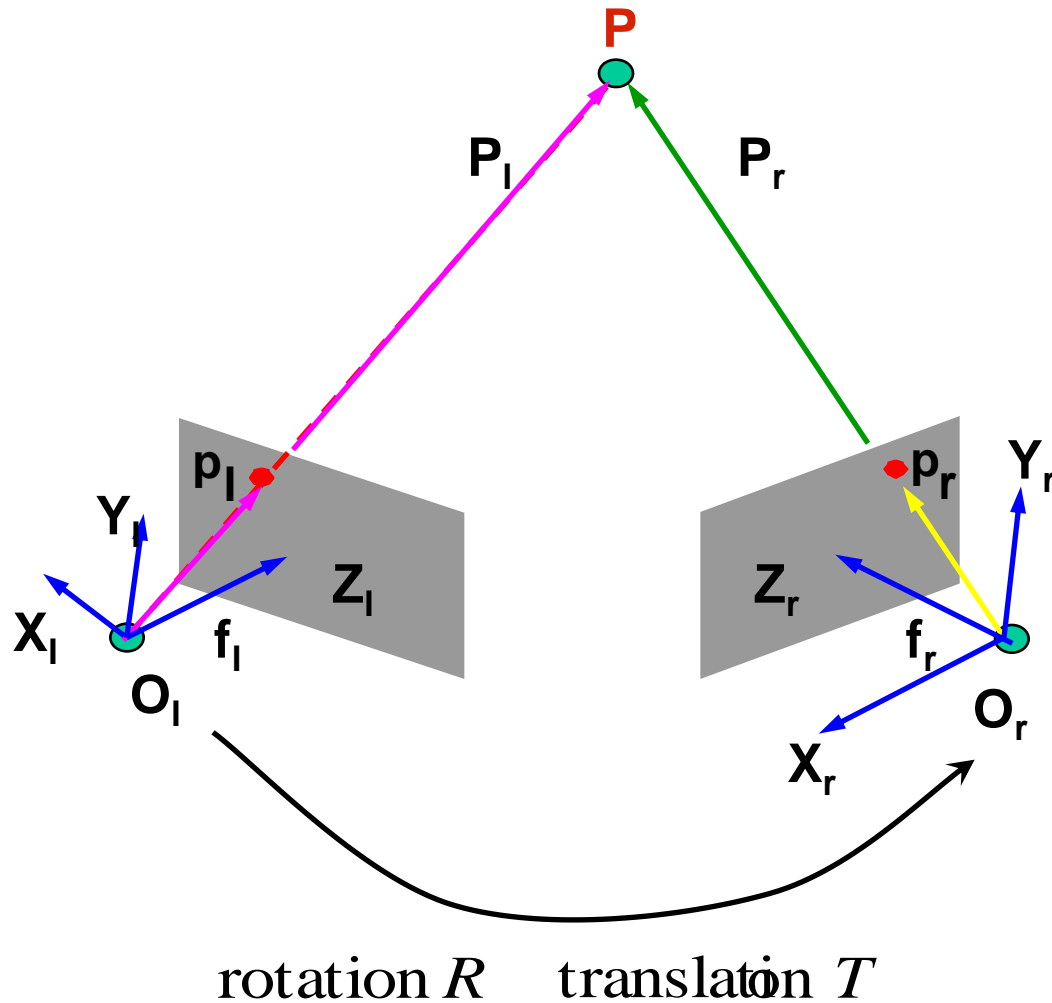
- Characterize the transformation from camera to pixel coordinate systems of each camera
- Focal length, image center, aspect ratio

Extrinsic parameters

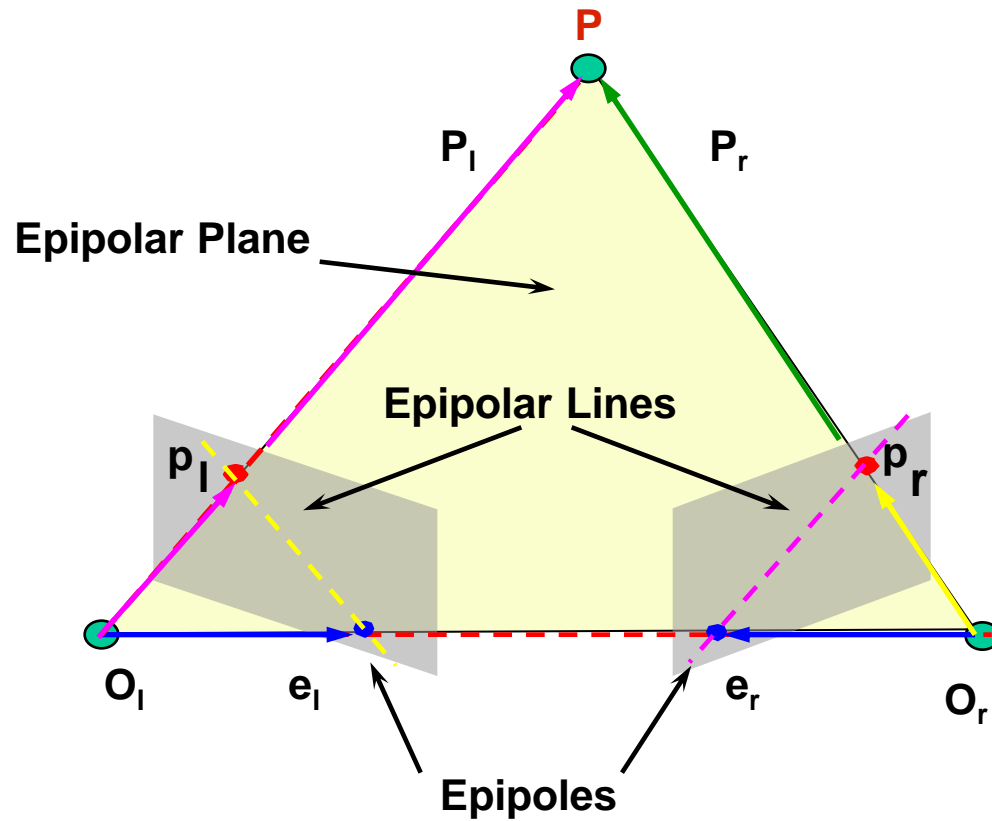
- Describe the relative position and orientation of the two cameras
- Rotation matrix R and translation vector T



Epipolar Geometry



Epipolar Geometry

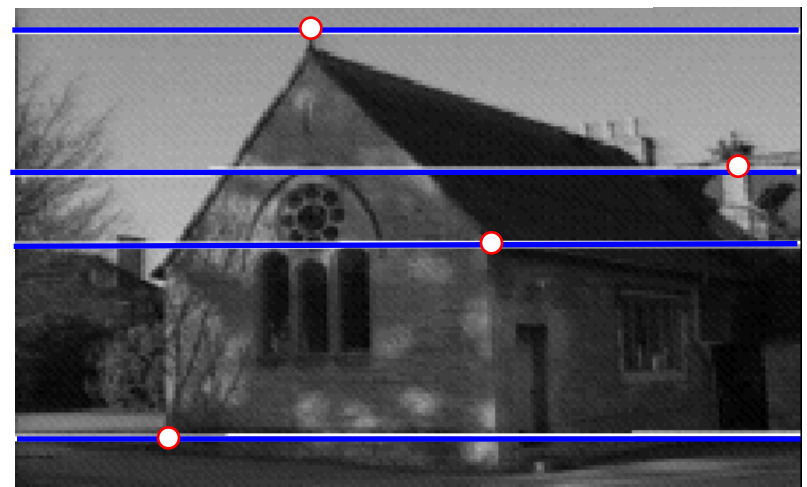
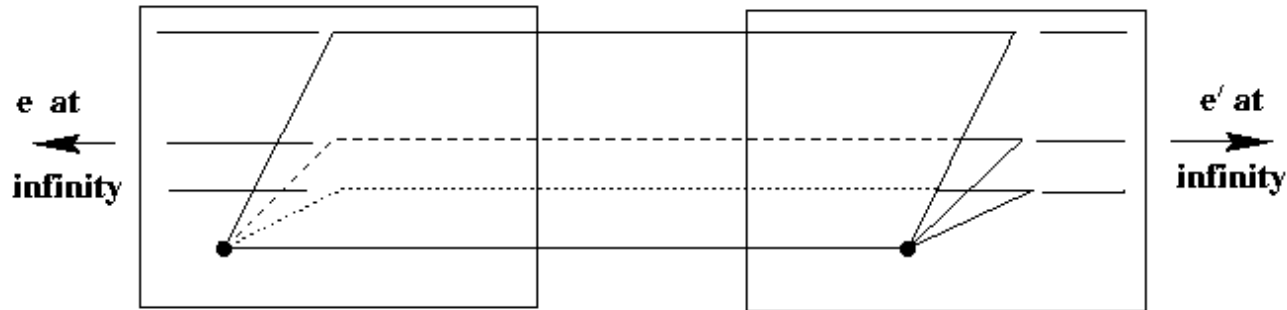


$$P_r = R(P_l - T)$$

Shape of epipolar lines

- Translating the cameras in the x, y plane without any camera rotation then the epipolar lines are parallel
- Translating the cameras in the direction of the camera y axis (horizontal) you get the simple stereo configuration of horizontal epipolar lines
- Translating the cameras with forward motion in the z axis produces epipolar lines that emanate from the epipole (sometimes called focus of projection)

Epipolar geometry : parallel cameras



Epipolar geometry depends **only** on the relative pose (position and orientation) and internal parameters of the two cameras, i.e. the position of the camera centres and image planes. It does **not** depend on the scene structure (3D points external to the camera).

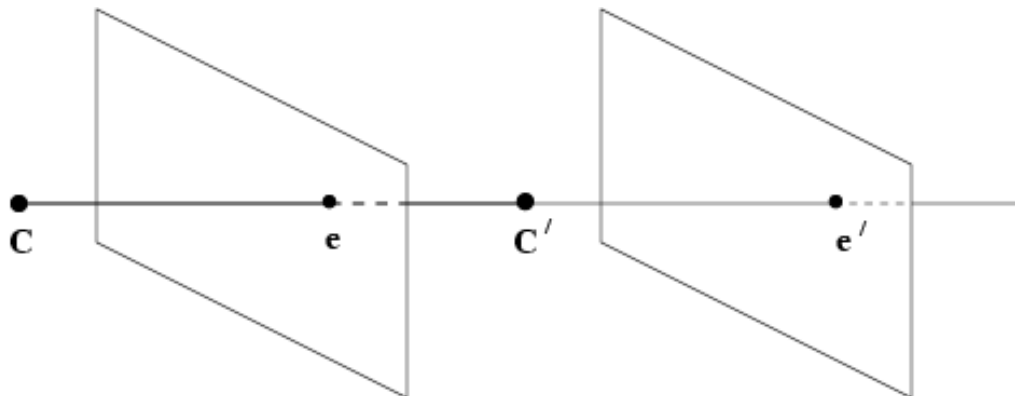
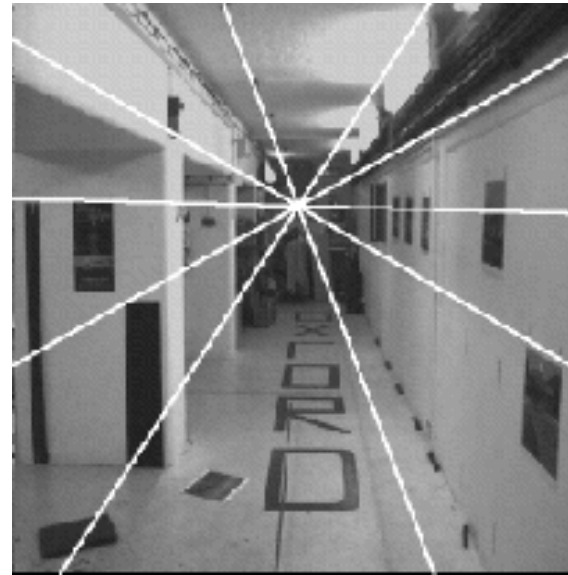
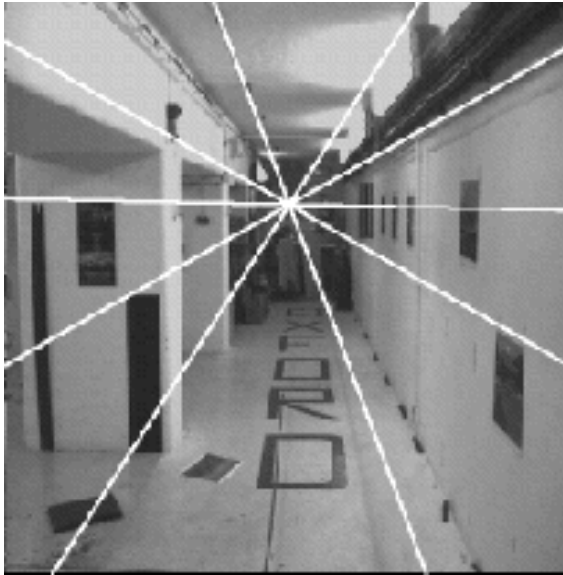
Epipolar geometry : forward motion



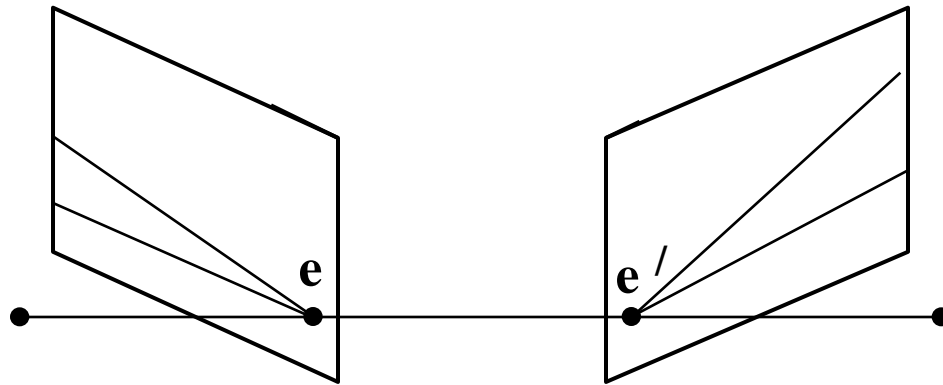
Epipolar geometry : forward motion



Epipolar geometry : forward motion



Epipolar geometry : converging cameras



So this means that epipolar lines are in general **not** parallel (only for certain cases)

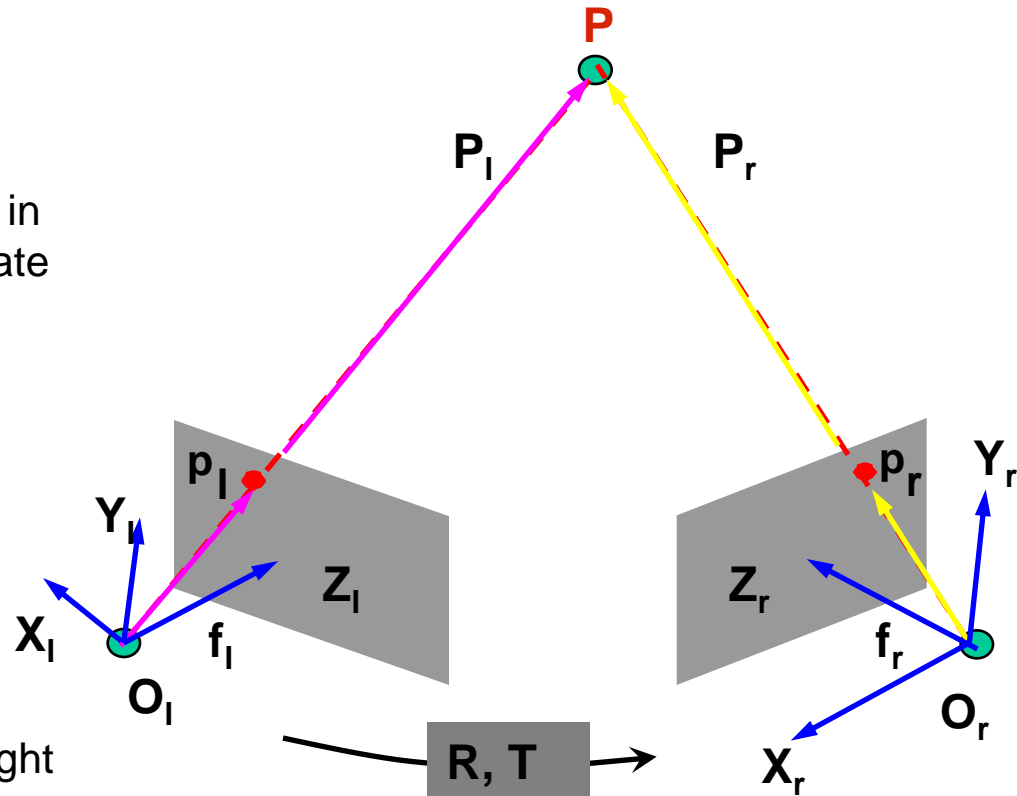
Epipolar Geometry

Notations

- $P_l = (X_l, Y_l, Z_l)$, $P_r = (X_r, Y_r, Z_r)$
 - Vectors of the same 3-D point P , in the left and right camera coordinate systems respectively
- Extrinsic Parameters
 - Translation Vector $T = (O_r - O_l)$
 - Rotation Matrix R

$$P_r = R(P_l - T)$$

- $p_l = (x_l, y_l, z_l)$, $p_r = (x_r, y_r, z_r)$
 - Projections of P on the left and right image plane respectively
 - For all image points, we have $z_l = f_l$, $z_r = f_r$



$$p_l = \frac{f_l}{Z_l} P_l$$

$$p_r = \frac{f_r}{Z_r} P_r$$

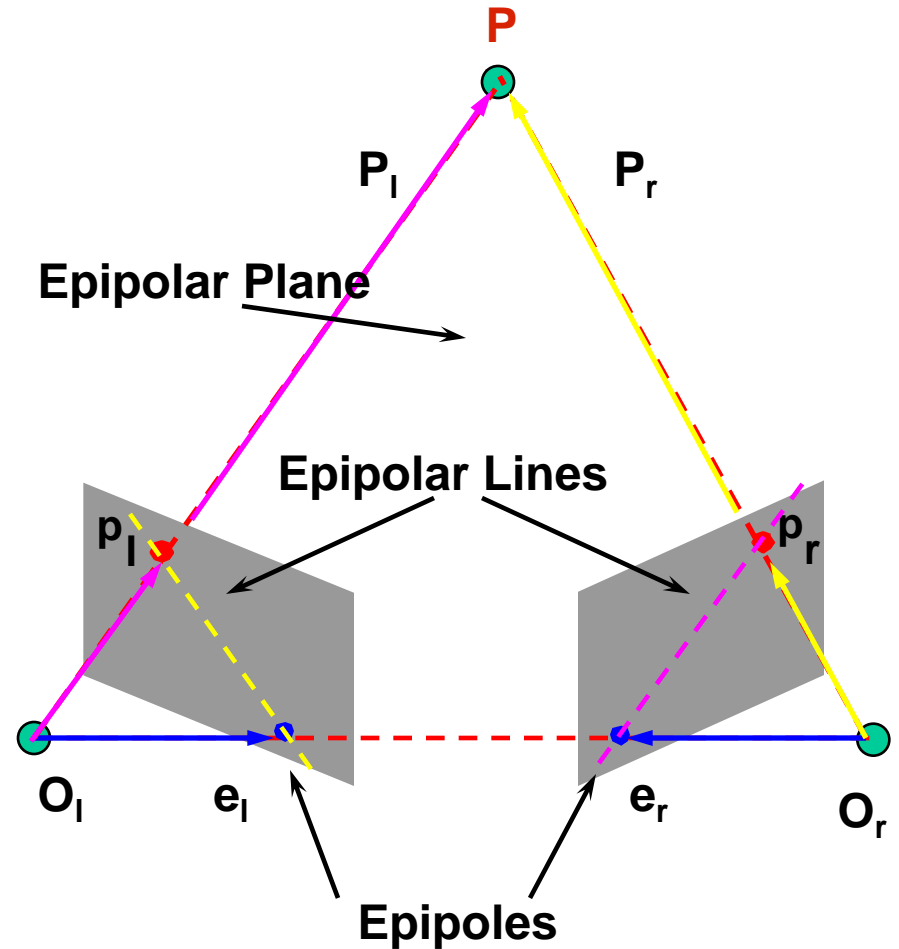
Epipolar Geometry

Motivation: **where to search correspondences?**

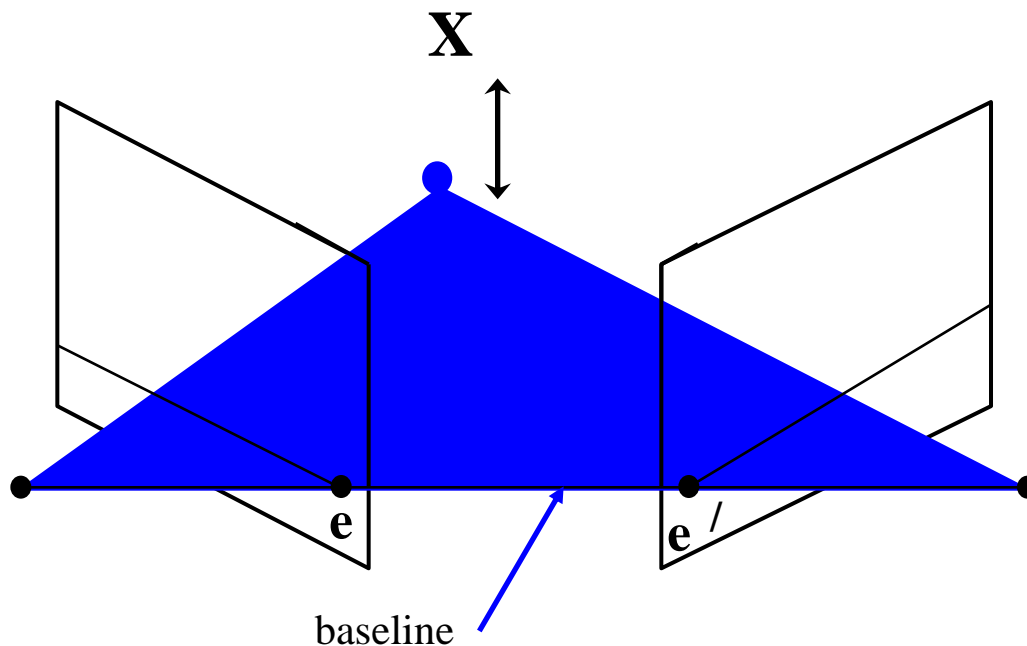
- Epipolar Plane
 - A plane going through point P and the centers of projection (COPs) of the two cameras
- Epipolar Lines
 - Lines where epipolar plane intersects the image planes
- Epipoles
 - The image in one camera of the COP of the other

Epipolar Constraint

- Corresponding points must lie on epipolar lines
- True for EVERY camera configuration, not depend on the geometry of the scene!



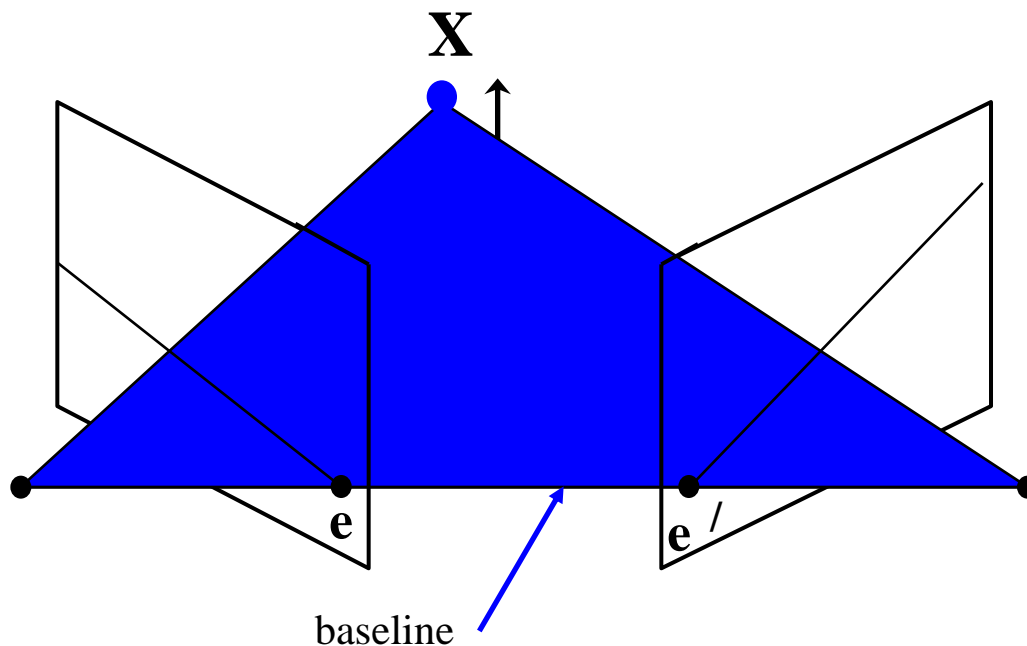
The epipolar pencil



As the position of the 3D point X varies, the epipolar planes “rotate” about the baseline. This family of planes is known as an epipolar pencil. All epipolar lines intersect at the epipole.

(a pencil is a one parameter family)

The epipolar pencil



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(a pencil is a one parameter family)

Epipolar Geometry

Epipolar plane: plane going through point P and the centers of projection (COPs) of the two cameras

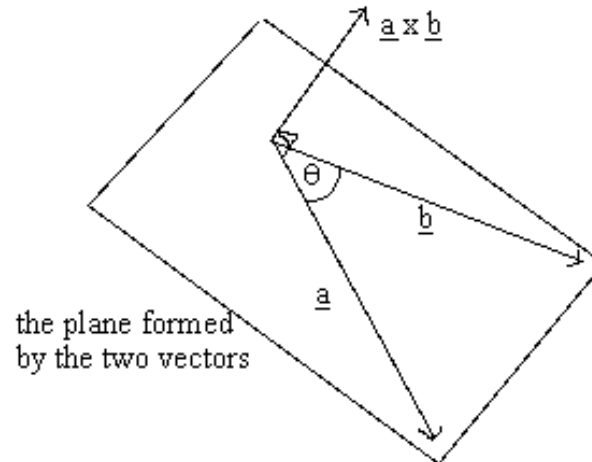
Epipolar lines: where this epipolar plane intersects the two image planes

Epipoles: The image in one camera of the COP of the other

Epipolar Constraint: Corresponding points between the two images must lie on epipolar lines

Cross product

- Consider two vectors in 3D space
 - (a_1, a_2, a_3) and (b_1, b_2, b_3)
- $\underline{\mathbf{a}} \times \underline{\mathbf{b}} = \underline{\mathbf{n}} a b \sin \theta$
- Cross product is at 90 degrees to both vectors
 - Normal to the plane defined by the two vectors



Cross product

- Two possible normal directions
 - We use the right hand rule to compute direction
- $\underline{\mathbf{a}} \times \underline{\mathbf{b}} =$
 $(a_1 \underline{\mathbf{i}} + a_2 \underline{\mathbf{j}} + a_3 \underline{\mathbf{k}}) \times (b_1 \underline{\mathbf{i}} + b_2 \underline{\mathbf{j}} + b_3 \underline{\mathbf{k}})$
- $\underline{\mathbf{a}} \times \underline{\mathbf{b}} =$
 $(a_2 b_3 - a_3 b_2) \underline{\mathbf{i}} + (a_3 b_1 - a_1 b_3) \underline{\mathbf{j}} + (a_1 b_2 - a_2 b_1) \underline{\mathbf{k}}$
- Cross product can also be written as multiplication by a matrix
- $\underline{\mathbf{a}} \times \underline{\mathbf{b}} = \mathbf{S} \underline{\mathbf{b}}$

Cross product as matrix multiplication

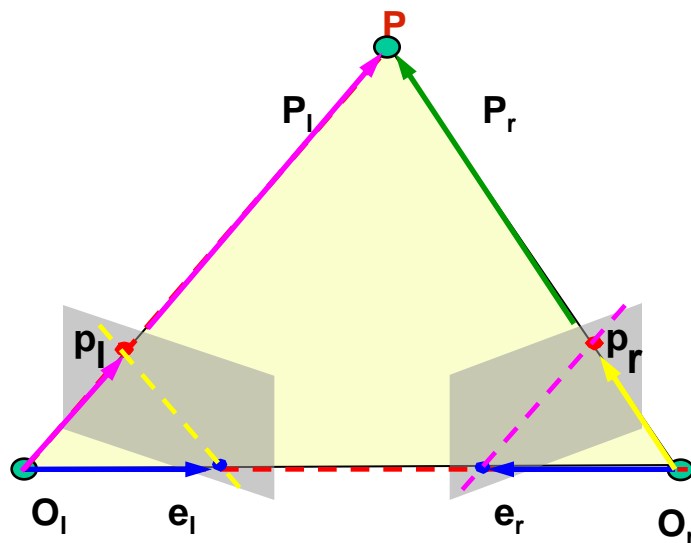
- Define matrix S as

$$\begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$$

- $\underline{\mathbf{a}} \times \underline{\mathbf{b}} = S \underline{\mathbf{b}}$

- Try the program `cross1.ch` on the course web site

Essential Matrix



$$T \times P_l = SP_l$$

$$S = \begin{bmatrix} 0 & -T_z & T_y \\ T_z & 0 & -T_x \\ -T_y & T_x & 0 \end{bmatrix}$$

Coordinate Transformation:

$T, P_l, P_l - T$ are coplanar

Resolves to

Essential Matrix $E = RS$

$$P_r = R(P_l - T)$$

$$(P_l - T)^T T \times P_l = 0$$

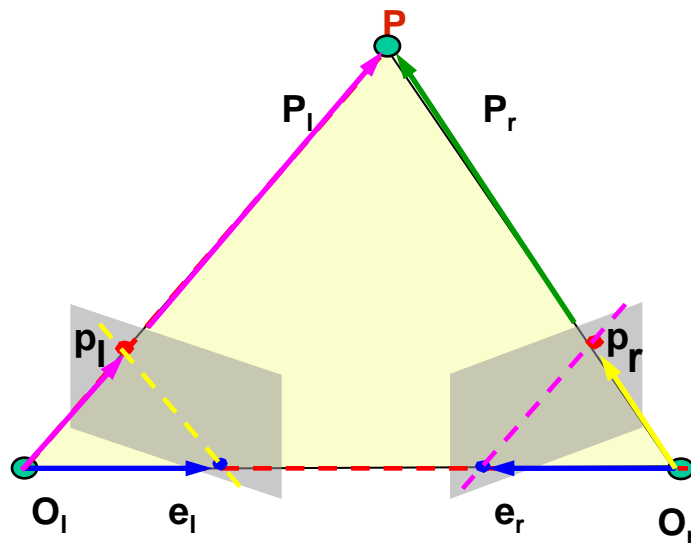
$$(R^T P_r)^T T \times P_l = 0$$

$$(R^T P_r)^T SP_l = 0$$

$$P_r^T RSP_l = 0$$

$$P_r^T EP_l = 0$$

Essential Matrix



$$P_r^T E P_l = 0 \quad \Rightarrow \quad p_r^T E p_l = 0$$

$$P_r^T E P_l = 0$$

Big P are points
In 3d space

$$p_l = \frac{f_l}{Z_l} P_l$$

$$p_r = \frac{f_r}{Z_r} P_r$$

Little p are points
In image plane
(camera co-ords)

$$p_r^T E p_l = 0$$

Essential Matrix

Essential Matrix $E = RS$

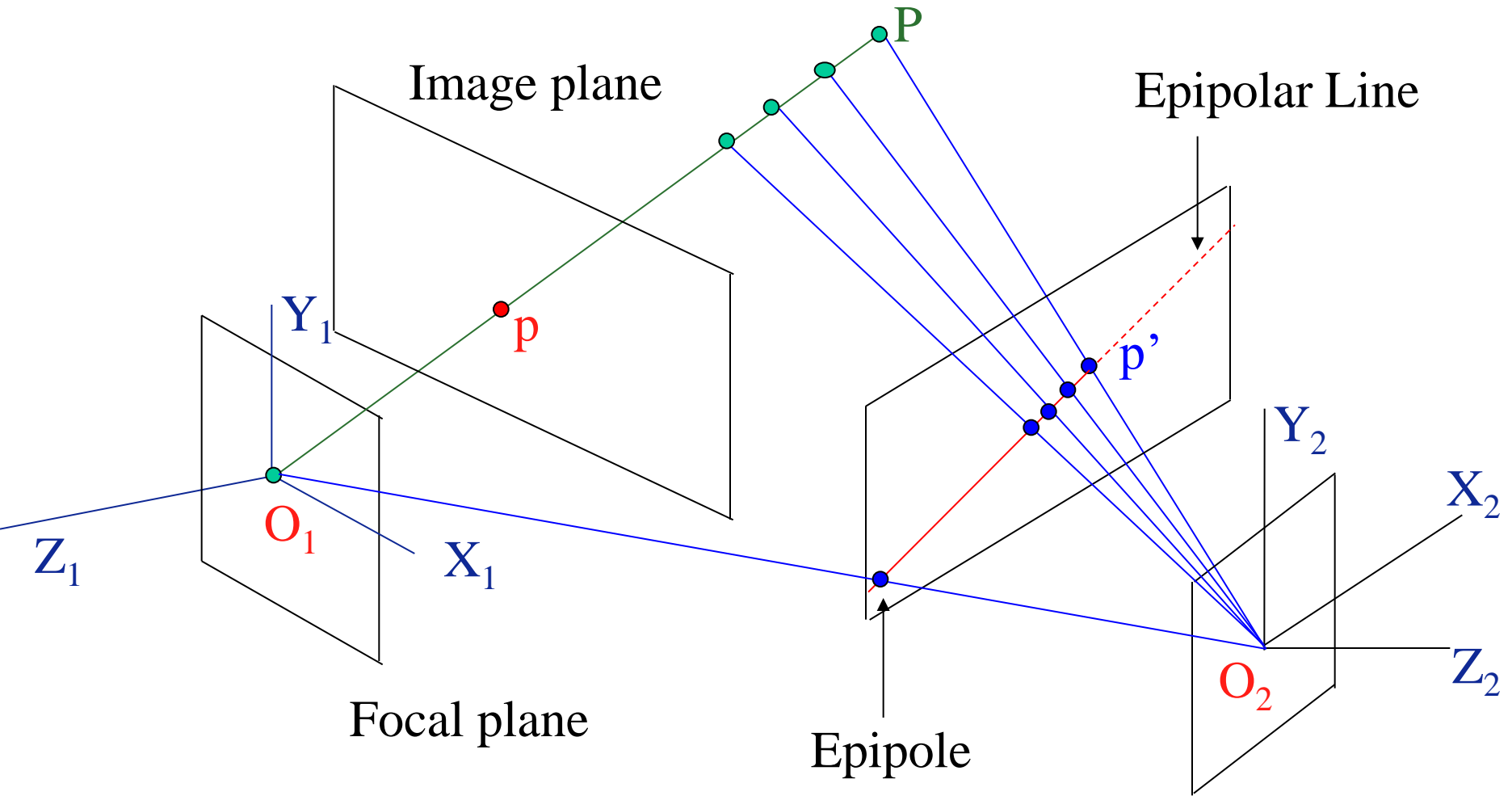
$$\mathbf{p}_r^T \mathbf{E} \mathbf{p}_l = 0$$

- A natural link between the stereo point pair and the extrinsic parameters of the stereo system
 - One correspondence \rightarrow a linear equation of 9 entries
 - Given 8 pairs of (p_l, p_r) \rightarrow E
- Mapping between points and epipolar lines we are looking for
 - Given $p_l, E \rightarrow p_r$ on the projective line in the right plane
 - Equation represents the epipolar line of either p_r (or p_l) in the right (or left) image

Note:

- p_l, p_r are in the camera coordinate system, not pixel coordinates that we can more easily measure
- Notation in next slide: p, p' is equivalent to p_r , and p_l

What does Essential Matrix Mean?



Essential Matrix

Essential Matrix $E = RS$

- 3x3 matrix constructed from R and T (extrinsic only)
 - Rank (E) = 2, two equal nonzero singular values

$$\mathbf{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \quad \mathbf{S} = \begin{bmatrix} 0 & -T_z & T_y \\ T_z & 0 & -T_x \\ -T_y & T_x & 0 \end{bmatrix}$$

- **Rank (R) = 3** **Rank (S) = 2**
- E has five degrees of freedom (3 rotation, 2 translation)
- If we know R and T it is easy to compute E
 - use the camera calibration method of Ch. 6 for two cameras
- But if we just have correspondences between the two stereo cameras then computing E is not as simple to compute

Projective Geometry

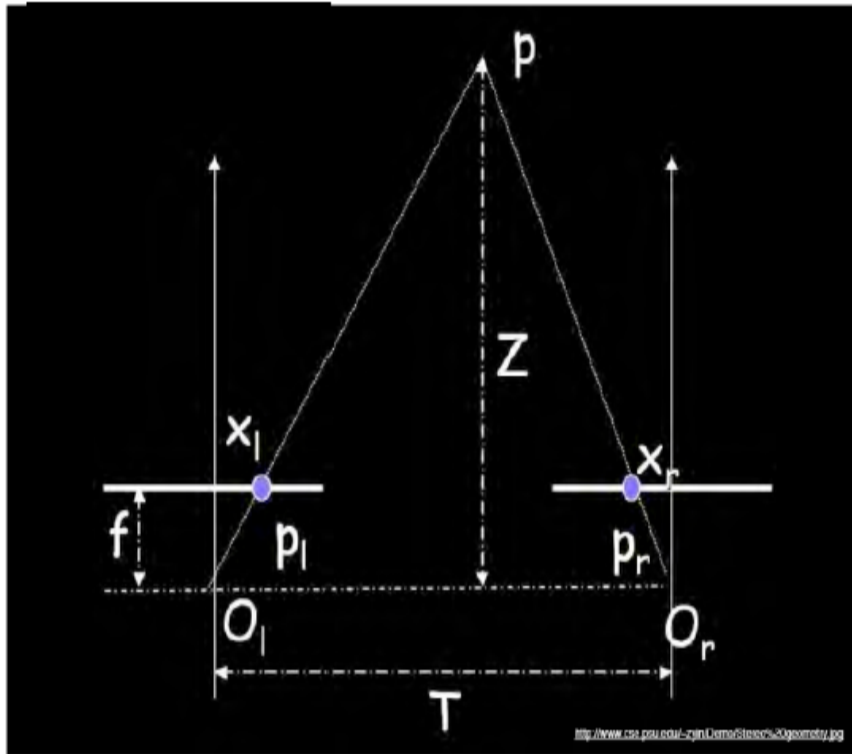
•Projective Plane - P^2

- Set of equivalence classes of triplets of real numbers
- $p = [x,y,z]^T$ and $p' = [x',y',z']^T$ are equivalent if and only if there is a real number k such that $[x,y,z]^T = k [x',y',z']^T$
- Each projective point p in P^2 corresponds to a line through the origin in P^3
- So points in P^2 , the projective plane, and lines in P^3 , ordinary space, are in a one to one correspondence
- A line in the projective plane is called a projective line represented by $u = [ux, uy, uz]^T$
- Set of points p that satisfy the relation $u^T \bullet p = 0$
- A projective line u can be represented by a 3d plane through the origin, that is the line defined by the equation $u^T \bullet p = 0$
- p is either a point lying on the line u , or a line going through the point u

Projective Line

- If we have a point in one image then this means the 3D point P is on the line from the origin through that point in the image plane
 - So in the other image the corresponding point must be on the epipolar line
- What is the meaning of Ep_l ?
 - the line in the right plane that goes through p_r and the epipole e_r
- Therefore the essential matrix is a mapping between points and epipolar lines
- Ep_l defines the equation of the epipolar line in the right image for point p_r in the right image
- $E^T p_r$ defines the equation of the epipolar line in the left image for point p_l in the left image
- The relationship $\mathbf{p}_r^T \mathbf{E} \mathbf{p}_l = 0$ maps a point to a line

Essential Matrix for Parallel Cameras

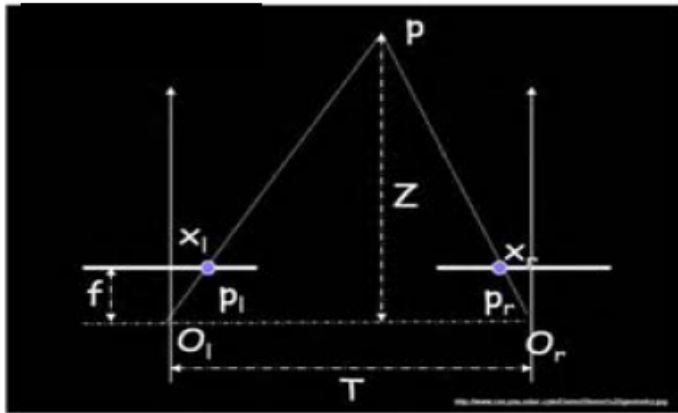


$$\mathbf{R} = \mathbf{I}$$

$$\mathbf{T} = [-T, 0, 0]^T$$

$$\mathbf{E} = [\mathbf{T}_x] \mathbf{R} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & T \\ 0 & -T & 0 \end{pmatrix}$$

Essential Matrix for Parallel Cameras



$$\mathbf{R} = \mathbf{I}$$

$$\mathbf{T} = [-T, 0, 0]^T$$

$$\mathbf{E} = [\mathbf{T}_x] \mathbf{R} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & T \\ 0 & -T & 0 \end{pmatrix}$$

$$\mathbf{p}^T \mathbf{E} \mathbf{p}' = 0$$

$$\begin{bmatrix} x & y & f \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & T \\ 0 & -T & 0 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ f \end{bmatrix} = 0$$

$$\Leftrightarrow \begin{bmatrix} x & y & f \end{bmatrix} \begin{bmatrix} 0 \\ Tf \\ -Ty' \end{bmatrix} = 0$$

Image of any point must lie on same horizontal line in each image plane!

$$\Leftrightarrow y = y'$$

Fundamental Matrix

Same as Essential Matrix but points are in pixel coordinates and not camera coordinates

$$p_r^T E p_l = 0$$



$$\bar{p}_r^T F \bar{p}_l = 0$$

Pixel coordinates

$$F = K_r^{-T} E K_l^{-1}$$

Intrinsic parameters
Book uses M and not K

Fundamental Matrix

Mapping between points and epipolar lines in the pixel coordinate systems

- With no prior knowledge of the stereo system parameters

From Camera to Pixels: Matrices of intrinsic parameters (M matrix in book)

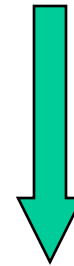
$$\mathbf{K}_{\text{int}} = \begin{bmatrix} -f_x & 0 & o_x \\ 0 & -f_y & o_y \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{Rank}(\mathbf{K}_{\text{int}}) = 3$$

$$\mathbf{p}_r^T \mathbf{E} \mathbf{p}_l = 0$$

$$\mathbf{p}_l = \mathbf{K}_l^{-1} \bar{\mathbf{p}}_l$$

$$\mathbf{p}_r = \mathbf{K}_r^{-1} \bar{\mathbf{p}}_r$$



$$\bar{\mathbf{p}}_r^T \mathbf{F} \bar{\mathbf{p}}_l = 0$$

$$\mathbf{F} = \mathbf{K}_r^{-T} \mathbf{E} \mathbf{K}_l^{-1}$$

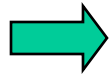
Fundamental Matrix

Fundamental Matrix

$$\mathbf{F} = \mathbf{K}_r^{-T} \mathbf{E} \mathbf{K}_l^{-1}$$

- Rank (F) = 2
- Encodes info on both intrinsic and extrinsic parameters
- Enables full reconstruction of the epipolar geometry
- In pixel coordinate systems without any knowledge of the intrinsic and extrinsic parameters
- Linear equation of the 9 entries of F but only 8 degrees of freedom because of homogeneous nature of equations

$$\bar{\mathbf{p}}_r^T \mathbf{F} \bar{\mathbf{p}}_l = 0$$



$$\begin{pmatrix} x_{im}^{(l)} & y_{im}^{(l)} & 1 \end{pmatrix} \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix} \begin{pmatrix} x_{im}^{(r)} \\ y_{im}^{(r)} \\ 1 \end{pmatrix} = 0$$

Fundamental Matrix

$$[x_r \ y_r \ 1] \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{bmatrix} \begin{bmatrix} x_l \\ y_l \\ 1 \end{bmatrix} = 0$$

- Now expand this equation as below
- One of x_l, y_l or x_r, y_r are known, others variables, and all the F 's are known

$$\begin{aligned} & F_{11}x_l x_r & + & F_{21}x_l y_r & + & F_{31}x_l \\ + & F_{12}x_r y_l & + & F_{22}y_l y_r & + & F_{32}y_l \\ + & F_{13}x_r & + & F_{23}y_r & + & F_{33} = 0 \end{aligned}$$

- This defines equation of a line in the form of $ax + by + c = 0$ (x, y variables, a, b, c constant)!

Computing E and F

- If we know extrinsic parameters of stereo system we can compute $E = R S$
- Using E and intrinsic parameters we can compute epipolar lines
- If we do not know intrinsic and extrinsic parameters can we compute F ?
- Yes, for at least 8 correspondences
 - Choose 8 correct correspondences manually
- Then compute F matrix
 - Then use F to guide search by computing the epipolar lines
 - This will simplify the process of finding correspondences

Computing F: The Eight-point Algorithm

Input: n point correspondences ($n \geq 8$)

- Construct homogeneous system $Ax = 0$ from $\bar{\mathbf{p}}_r^T \mathbf{F} \bar{\mathbf{p}}_l = 0$
 - $x = (f_{11}, f_{12}, f_{13}, f_{21}, f_{22}, f_{23}, f_{31}, f_{32}, f_{33})$: entries in F
 - Each correspondence give one equation
 - A is a $n \times 9$ matrix
- Obtain estimate F^\wedge by Eigenvector with smallest eigenvalue
 - x (up to a scale) is column of V corresponding to the least singular value
- Enforce singularity constraint: since $\text{Rank}(F) = 2$
 - Compute SVD of F^\wedge $\hat{\mathbf{F}} = \mathbf{U}\mathbf{D}\mathbf{V}^T$
 - Set the smallest singular value to 0: $D \rightarrow D'$
 - Correct estimate of F : $\mathbf{F}' = \mathbf{U}\mathbf{D}'\mathbf{V}^T$

Output: the fundamental matrix, F'

can then compute E given intrinsic parameters

Estimating Fundamental Matrix

The 8-point algorithm

$$u^T F u' = 0$$

Each point correspondence can be expressed as a linear equation

$$\begin{bmatrix} u & v & 1 \end{bmatrix} \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{bmatrix} \begin{bmatrix} u' \\ v' \\ 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} uu' & uv' & u & u'v & vv' & v & u' & v' & 1 \end{bmatrix} \begin{bmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \\ F_{33} \end{bmatrix} = 0$$

Homogeneous System

- M linear equations of form $A\mathbf{x} = 0$
- If we have a given solution \mathbf{x}_1 , s.t. $A\mathbf{x}_1 = 0$ then $c * \mathbf{x}_1$ is also a solution $A(c * \mathbf{x}_1) = 0$
- Need to add a constraint on \mathbf{x} ,
 - Basically make \mathbf{x} a unit vector $\mathbf{x}^T \mathbf{x} = 1$
- Can prove that the solution is the eigenvector corresponding to the single zero eigenvalue of that matrix $A^T A$
 - This can be computed using eigenvector routine
 - Then finding the zero eigenvalue
 - Returning the associated eigenvector

Singular Value Decomposition

- Any m by n matrix A can be written as product of three matrices $A = UDV^T$
- The columns of the m by m matrix U are mutually orthogonal unit vectors, as are the columns of the n by n matrix V
- The m by n matrix D is diagonal, and the diagonal elements, σ_i are called the singular values
- It is the case that $\sigma_1 \geq \sigma_2 \geq \dots \sigma_n \geq 0$
- Singular values squared are eigenvalues (square mat.)
- The rank of a square matrix is the number of linearly independent rows or columns
- For a square matrix ($m = n$) then the number of non-zero singular values equals the rank of the matrix

Essential/Fundamental Matrix

- Essential and fundamental matrix differ
- Relate different quantities
 - Essential matrix is defined in terms of camera co-ordinates
 - Fundamental matrix defined in terms of pixel co-ordinates
- Need different things to calculate them
 - Essential matrix requires camera calibration and knowledge of correspondences
 - known intrinsic parameters, unknown extrinsic parameters
 - Fundamental matrix does not require any camera calibration, just knowledge of correspondences
 - Unknown intrinsic and unknown extrinsic
- Essential and fundamental matrix are related by the camera calibration parameters

Essential/Fundamental matrix

- Essential matrix $E = RS$
 - Encodes information on the extrinsic parameters only
 - Has rank 2, since S has rank 2, and R is full rank
 - This means one singular value (eigenvalue) is zero
 - Its two non-zero singular values (eigenvalues) are equal
- Fundamental matrix $F = K_r^{-T} E K_l^{-1}$
 - Encodes information on both intrinsic and extrinsic params
 - Has rank 2, since K_r and K_l have full rank
 - This means one singular value (eigenvalue) is zero
 - Its two non-zero singular values (eigenvalues) are not necessarily equal

Essential/Fundamental Matrix

- We compute the fundamental matrix from the 2d pixel co-ordinates of correspondences between the left and right image
- If we have the fundamental matrix it is possible to compute the essential matrix if we know the camera calibration
- But we can still compute the epipolar lines using the fundamental matrix
- Therefore if we have the fundamental matrix this limits correspondence search to 1D search for general stereo camera positions in same way as for simple stereo

Locating the Epipoles from F

$$\bar{\mathbf{p}}_r^T \mathbf{F} \bar{\mathbf{p}}_l = 0$$



$$\bar{\mathbf{p}}_r^T \mathbf{F} \bar{\mathbf{e}}_l = 0$$

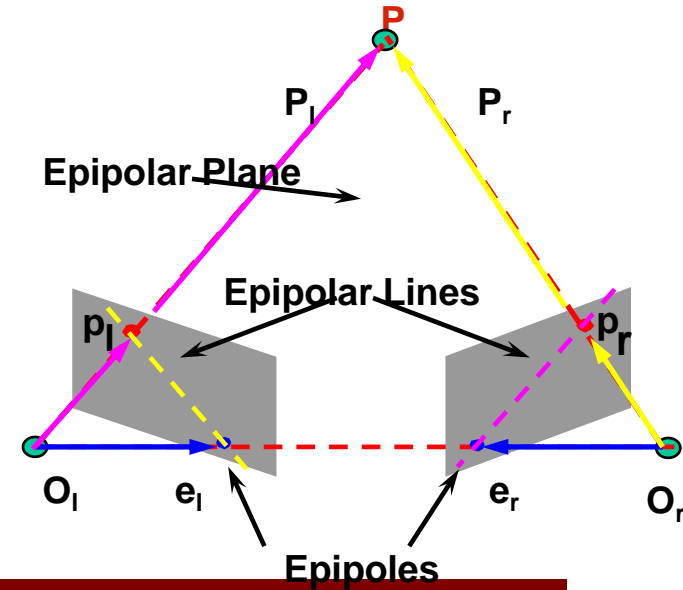


$$\mathbf{F} \bar{\mathbf{e}}_l = 0$$

$\bar{\mathbf{e}}_l$ lies on all the epipolar lines of the left image

For every $\bar{\mathbf{p}}_r$

F is not identically zero



Input: Fundamental Matrix F

$$\mathbf{F} = \mathbf{U} \mathbf{D} \mathbf{V}^T$$

- Find the SVD of F
- The epipole $\bar{\mathbf{e}}_l$ is the column of V corresponding to the null singular value (as shown above)
- The epipole $\bar{\mathbf{e}}_r$ is the column of U corresponding to the null singular value

Output: Epipole $\bar{\mathbf{e}}_l$ and $\bar{\mathbf{e}}_r$

Fundamental and Essential matrix

Both F and E are 3×3 matrices, facts below apply to both

Transpose : If F is a fundamental matrix of cameras (P, P') then F^T is a fundamental matrix of camera (P', P)

Epipolar lines: Think of p and p' as points in the projective plane then $F p$ is projective line in the right image.

$$\text{That is } l' = F p \quad l = F^T p'$$

Epipole: Since for any p the epipolar line $l' = F p$ contains the epipole e' . Thus $(e'^T F) p = 0$ for a all p .

$$\text{Thus } e'^T F = 0 \quad \text{and } F e = 0$$

Epipolar Geometry

- Basic constraint used to help correspondence
- Makes search for matching points into a 1D search along epipolar lines
- If you have intrinsic and extrinsic parameters
 - Then compute essential matrix and find epipolar lines
- If you do not have intrinsic or extrinsic parameters but have at least 8 correct correspondences then
 - Compute fundamental matrix and find epipolar lines
- Can also compute the epipoles using SVD

Similarities and Differences

- **Projection, homography and fund. matrix**
 - All three can be computed given the intrinsic and extrinsic camera parameters; K , R and T
 - Just simple substitution and algebraic manipulation
 - Projection and fundamental matrix is many to one transform
 - Homography is one to one transform
- **Given correspondences from 3d to 2d points**
 - Can compute the projection matrix
- **Given correspondences from 2d to 2d**
 - Can compute homography and the fundamental matrix
- **Approach is very similar in all cases**
 - A matrix from correspondences, eigenvector solution
 - Fund matrix requires some post processing, others do not