Geometric Model of Camera

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COMP 4102A
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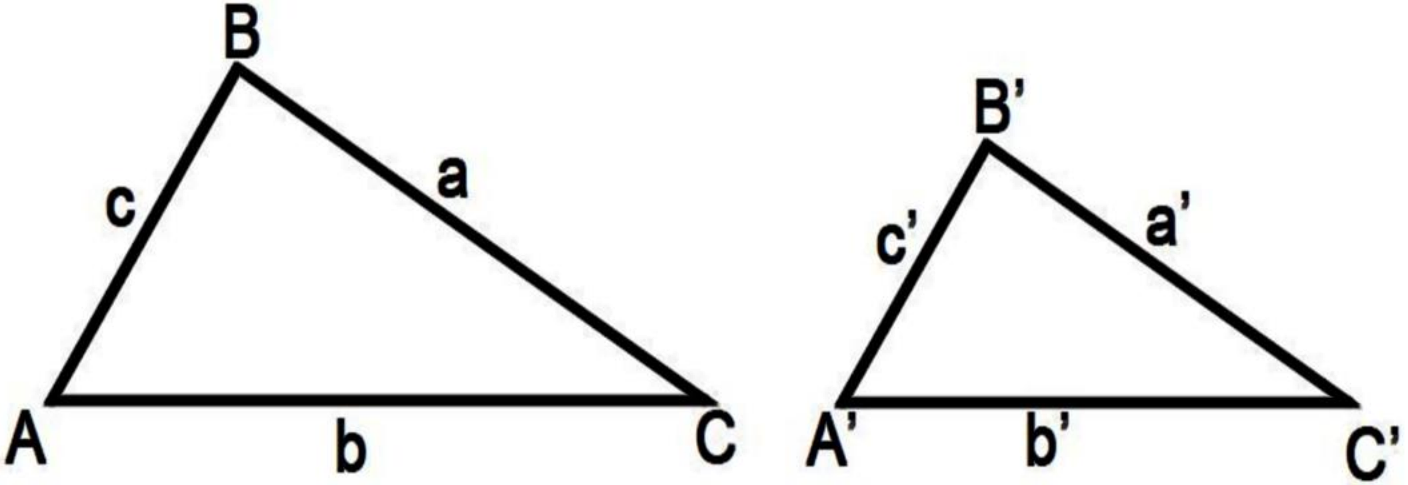
Version 2

Similar Triangles



property (i): corresponding angles are equal

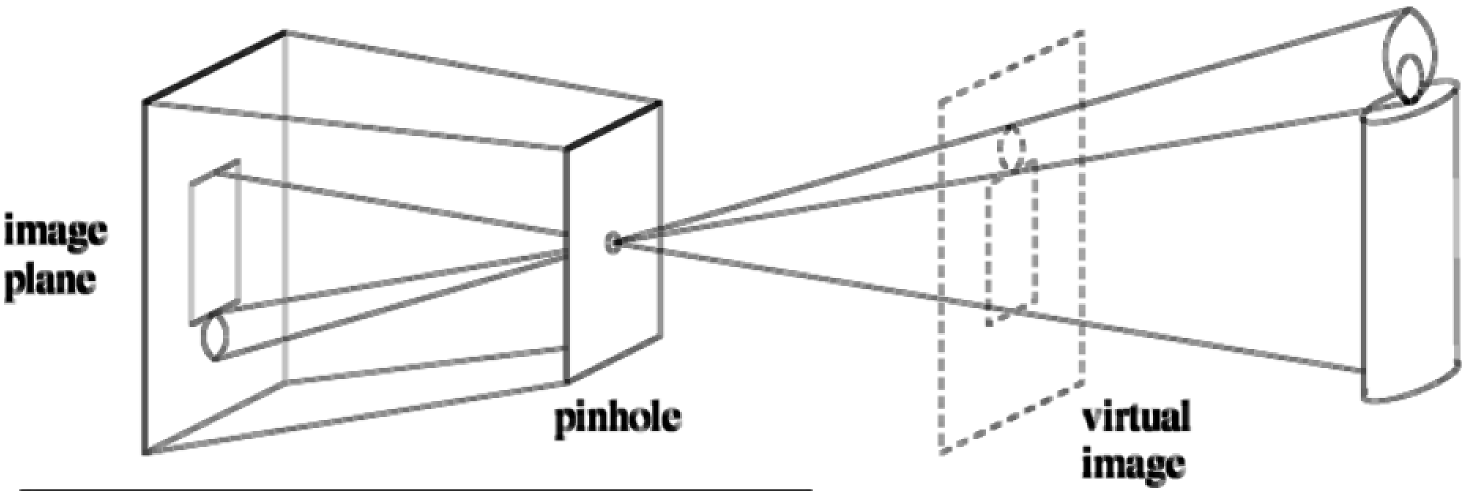
(A = A' and B = B' and C = C')

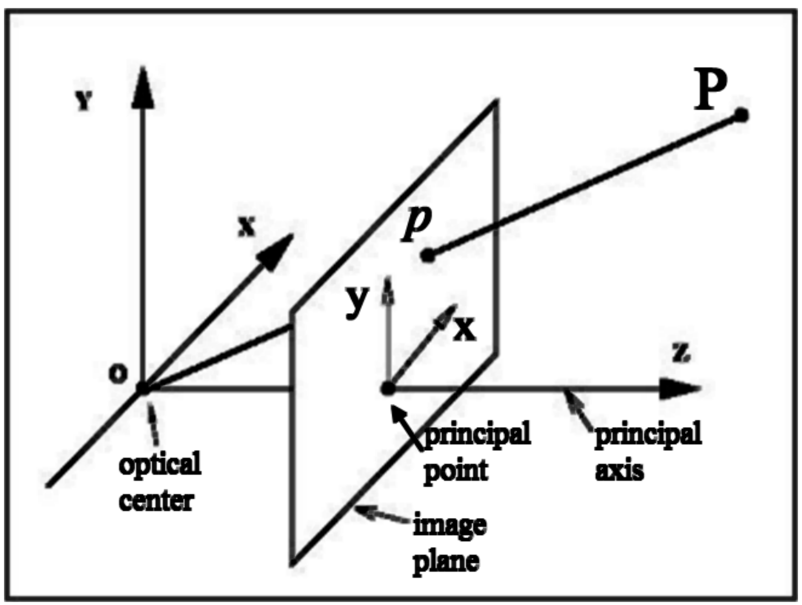
property (ii): corresponding sides have proportional lengths

$$\left(\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}\right)$$

Geometric Model of Camera

Perspective projection





P(X,Y,Z) $\longrightarrow$ p(x,y)

$$x = f\,\frac{X}{Z} \qquad y = f\,\frac{Y}{Z}$$

Parallel lines aren't…

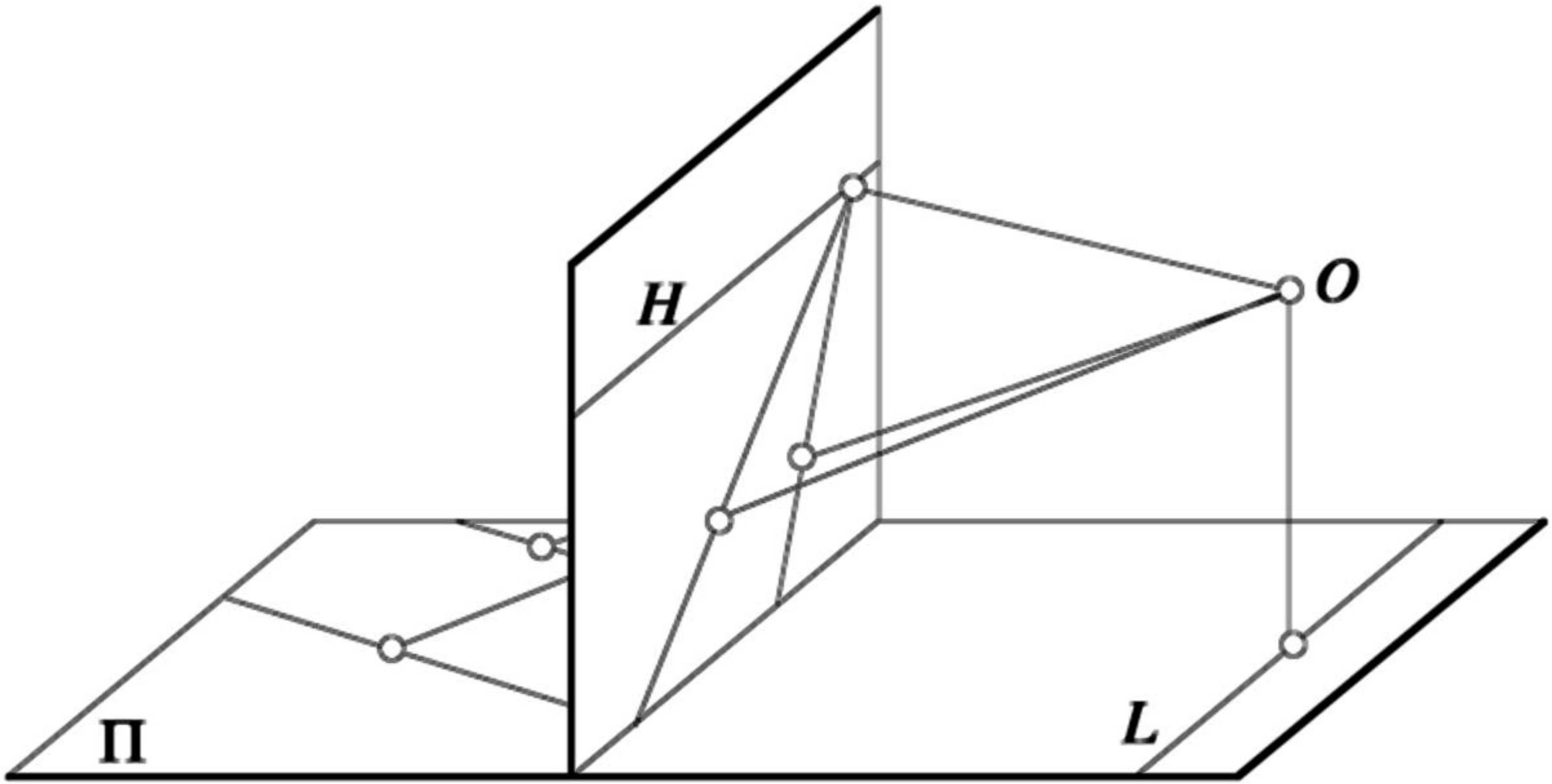
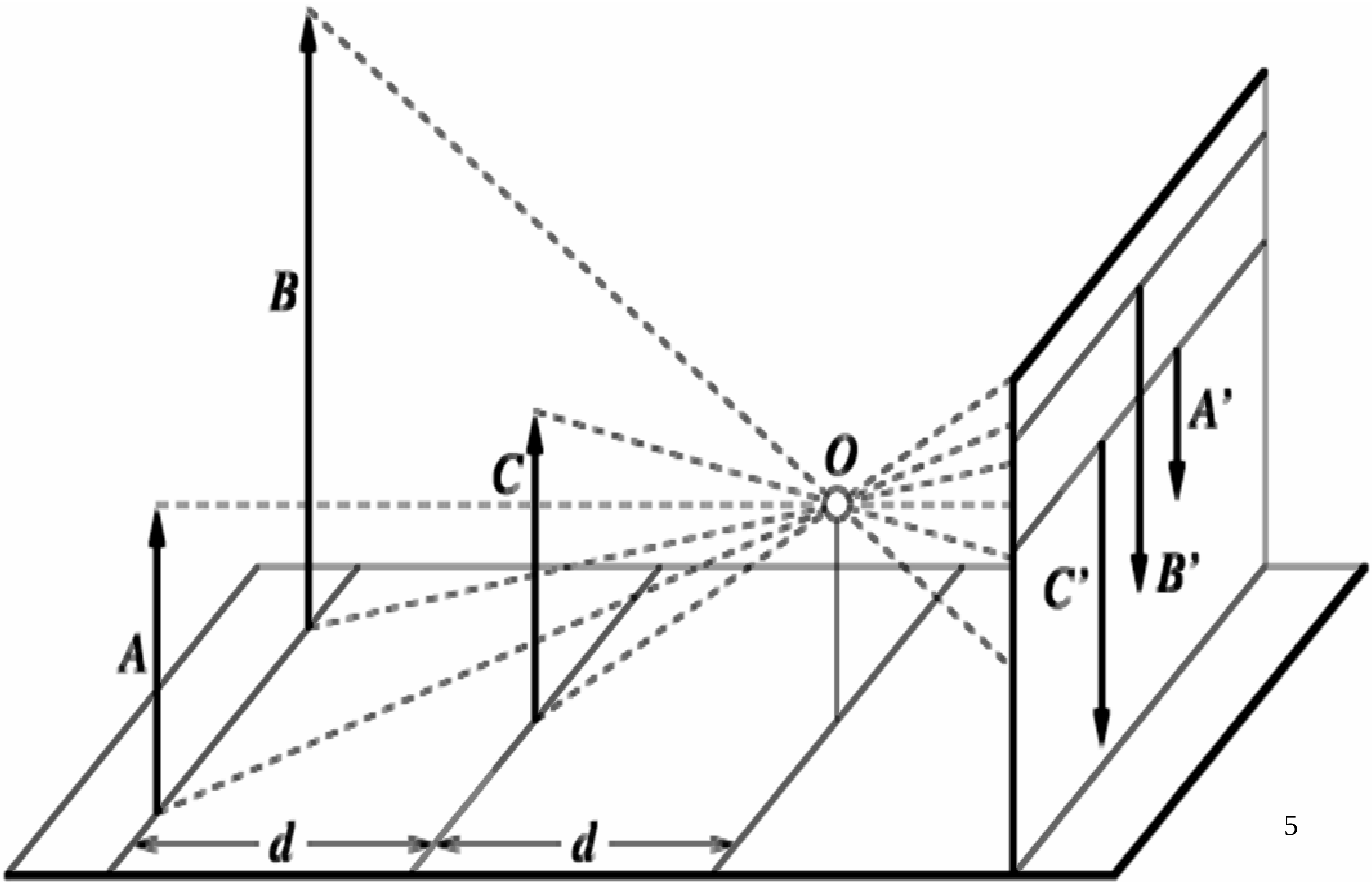
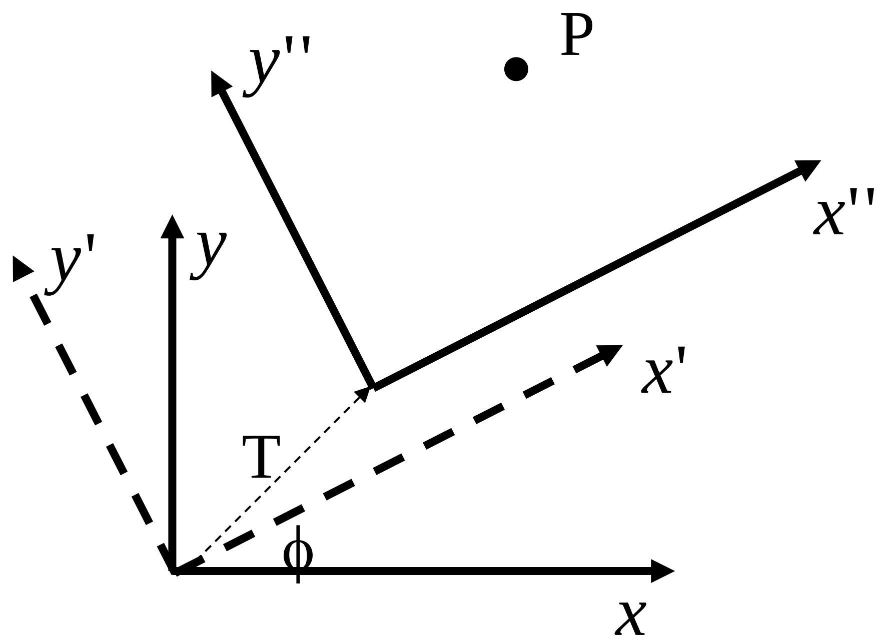


Figure by David Forsyth

Lengths can’t be trusted...



Coordinate Transformation – 2D



Rotation and Translation

$$\boldsymbol{p}' = \begin{bmatrix} p_x' \\ p_y' \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} p_x \\ p_y \end{bmatrix}$$

$$\boldsymbol{p}'' = \begin{bmatrix} p_x' \\ p_y' \end{bmatrix} + \begin{bmatrix} T_x \\ T_y \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} p_x \\ p_y \end{bmatrix} + \begin{bmatrix} T_x \\ T_y \end{bmatrix}$$

Homogeneous Coordinates

Go one dimensional higher:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \rightarrow \begin{bmatrix} wx \\ wy \\ wz \\ w \end{bmatrix}$$

w is an arbitrary non-zero scalar, usually we choose 1.

From homogeneous coordinates to Cartesian coordinates:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \rightarrow \begin{bmatrix} x_1 / x_3 \\ x_2 / x_3 \end{bmatrix} \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \rightarrow \begin{bmatrix} x_1 / x_4 \\ x_2 / x_4 \\ x_3 / x_4 \end{bmatrix}$$

2D Transformation with Homogeneous Coordinates

2D coordinate transformation:

$$\boldsymbol{p}'' = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} p_x \\ p_y \end{bmatrix} + \begin{bmatrix} T_x \\ T_y \end{bmatrix}$$

2D coordinate transformation using homogeneous coordinates:

$$\begin{bmatrix} p_x'' \\ p_y'' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & T_x \\ -\sin\phi & \cos\phi & T_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ 1 \end{bmatrix}$$

Homogeneous coordinates (In 2d)

Two points are equal if and only if:

$x'/w' = x/w$ and $y'/w' = y/w$

$w=0$: points at infinity

- useful for projections and curve drawing

Homogenize = divide by w.

Homogenized point example:

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} 64400 \\ 70800 \\ 170 \end{bmatrix}$$

$x=x'/w' = 64400/170 = 378.8$
$y=y'/w' = 70800/170 = 416.5$

Why use homogeneous co-ordinates

- We require a composition (sequence of) rotations, translations and projections
- Even the projection is a matrix multiplication
- Each of these can be described by matrix operations using homogeneous coordinates
- Composing them together, applying them one after the other just matrix multiplication
- The final operation, which takes a 3d point and produces a 2d image is one big matrix multiplication

Translations with homogeneous

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

$$\begin{cases} x' = x + wt_x \\ y' = y + wt_y \\ w' = w \end{cases}$$

$$\begin{cases} \dfrac{x'}{w'} = \dfrac{x}{w} + t_x \\[2ex] \dfrac{y'}{w'} = \dfrac{y}{w} + t_y \end{cases}$$

Scaling with homogeneous

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

$$\begin{cases} x' = s_x x \\ y' = s_y y \\ w' = w \end{cases}$$

$$\begin{cases} \dfrac{x'}{w'} = S_x \dfrac{x}{w} \\ \dfrac{y'}{w'} = S_y \dfrac{y}{w} \end{cases}$$

Rotation with homogeneous co-ord's

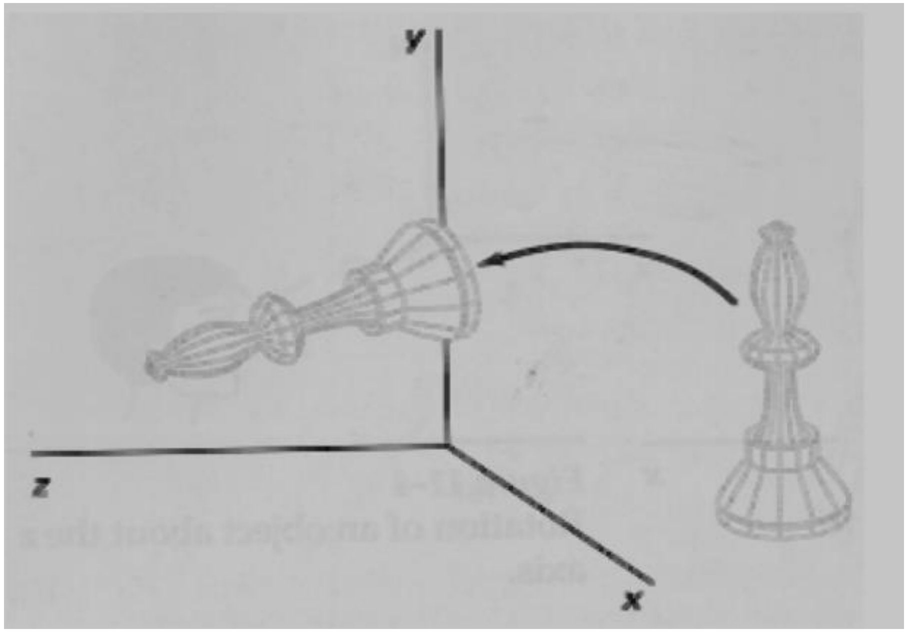
$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

$$\begin{cases} \frac{x'}{w'} = \cos\theta \frac{x}{w} - \sin\theta \frac{y}{w} \\ \frac{y'}{w'} = \sin\theta \frac{x}{w} + \cos\theta \frac{y}{w} \end{cases}$$

$$\begin{cases} x' = \cos\theta x - \sin\theta y \\ y' = \sin\theta x + \cos\theta y \\ w' = w \end{cases}$$

3D Rotation about X-Axis

Representation in homogeneous co-ordinates



$$x' = x$$

$$y' = y\cos(\theta) - z\sin(\theta)$$

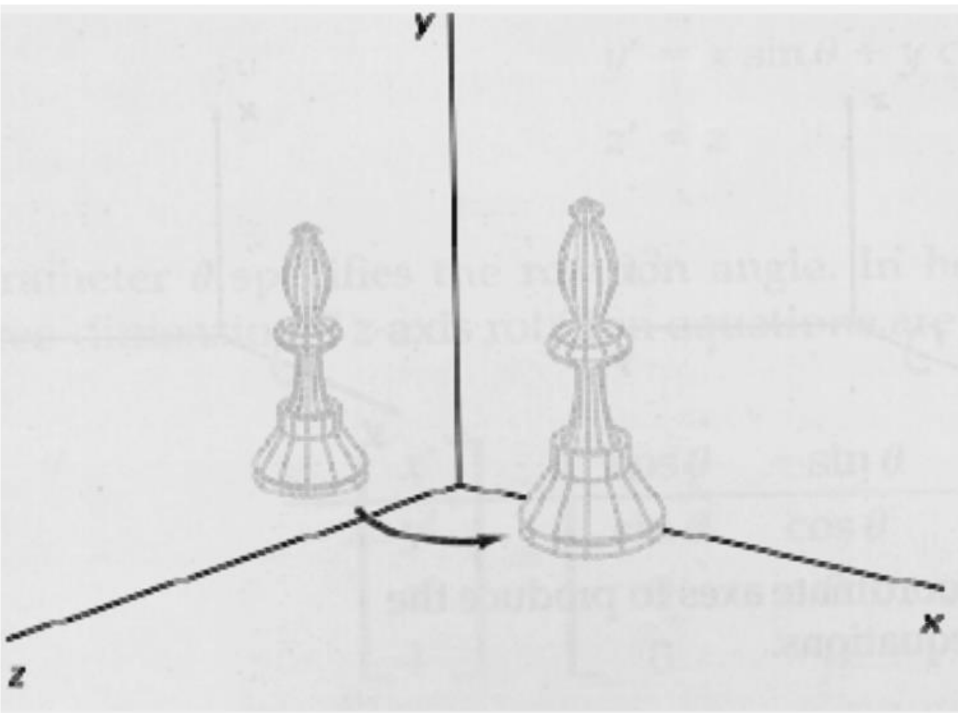
$$z' = y\sin(\theta) + z\cos(\theta)$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) & 0 \\ 0 & \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\underline{P' = R_x(\theta)\ P}$$

3D Rotation about Y-Axis

Representation in homogeneous co-ordinates



$$x' = z\sin(\theta) + x\cos(\theta)$$

$$y' = y$$

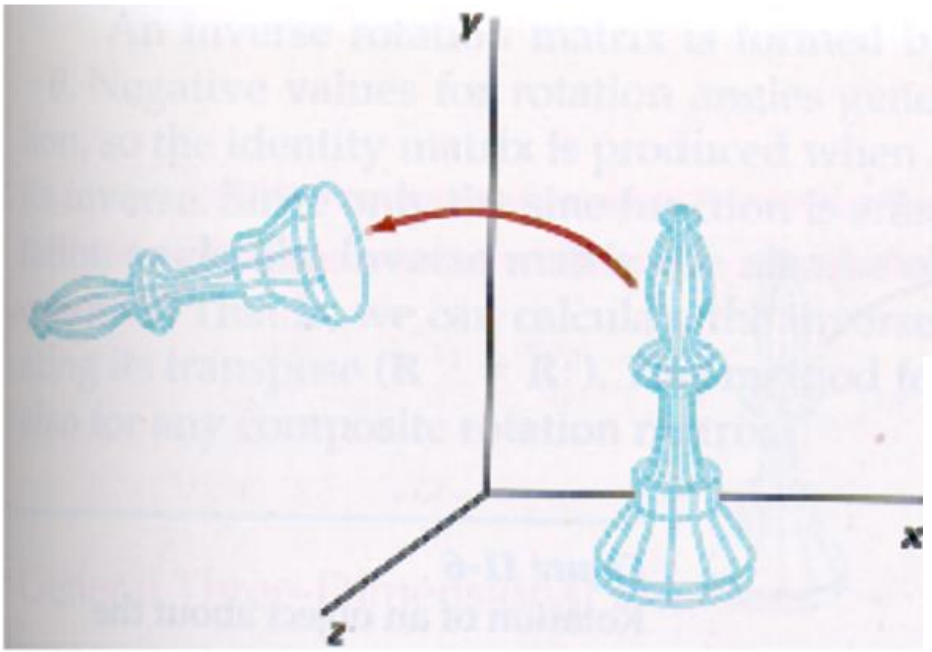
$$z' = z\cos(\theta) - x\sin(\theta)$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\underline{P' = R_y(\theta)\ P}$$

3D Rotation about Z-Axis

Representation in homogeneous co-ordinates



$$x' = x\cos(\theta) - y\sin(\theta)$$

$$y' = x\sin(\theta) + y\cos(\theta)$$

$$z' = z$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 & 0 \\ \sin(\theta) & \cos(\theta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\underline{P' = R_z(\theta)\ P}$$

3D Rotation Matrix – Euler Angles

Rotate around each coordinate axis:

Rx Ry Rz

$$\mathbf{R}_1(\alpha)=\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\alpha & -\sin\alpha\\ 0 & \sin\alpha & \cos\alpha\end{bmatrix}\quad \mathbf{R}_2(\beta)=\begin{bmatrix}\cos\beta & 0 & \sin\beta\\ 0 & 1 & 0\\ -\sin\beta & 0 & \cos\beta\end{bmatrix}\quad \mathbf{R}_3(\gamma)=\begin{bmatrix}\cos\gamma & -\sin\gamma & 0\\ \sin\gamma & \cos\gamma & 0\\ 0 & 0 & 1\end{bmatrix}$$

Combine the three rotations: $R = RxRyRz$

- Can describe any 3d rotation by a sequence of rotations about the three axis
- So R = RxRyRz or RxRzRy or RyRxRz or RyRzRx or RzRxRy or RzRyRx

3d Rotation Matrix

- When you specify the values for each axis you musts also specify order of operations
- Different orders have different angle values
- Succeeding rotations are about an already modified set of three axis
- Remember matrix multiplication is not commutative AB is not same as BA
- A rotation matrix has 9 elements but we need only three numbers to specify a 3d rotation uniquely!

3d Rotation Matrix - Examples

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

```
45 degrees ccw about X-axis
   1.00      0.00      0.00      0.00
   0.00      0.71     -0.71      0.00
   0.00      0.71      0.71      0.00
   0.00      0.00      0.00      1.00
```

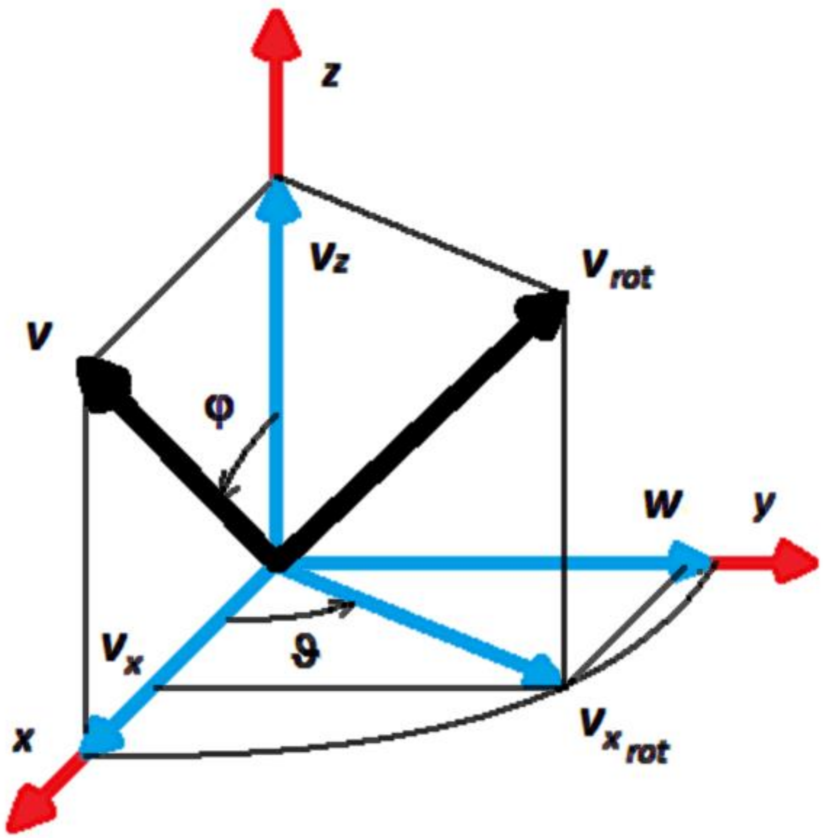
```
30 degrees ccw about Y-axis
   0.87      0.00      0.50      0.00
   0.00      1.00      0.00      0.00
  -0.50      0.00      0.87      0.00
   0.00      0.00      0.00      1.00
```

```
30 degrees ccw about Z-axis
   0.87     -0.50      0.00      0.00
   0.50      0.87      0.00      0.00
   0.00      0.00      1.00      0.00
   0.00      0.00      0.00      1.00

-30 degrees ccw about Z-axis
   0.87      0.50      0.00      0.00
  -0.50      0.87      0.00      0.00
   0.00      0.00      1.00      0.00
   0.00      0.00      0.00      1.00
```

3D Rotation – Rodriguez

- Rotations also described by axis and angle
- Rotation axis (2 parameters) and amount of rotation (angle – 1 parameter)



Can convert between Euler Angles and Rodriguez representation

Degrees of freedom

- This is the number of parameters necessary to generate all possible instances of a given geometric object
- A 2d rotation has one degree of freedom
  - Varying one parameter (angle) will generate every possible 2d rotation
- What are degrees of freedom of a 3d rotation
  - It is a 3 by 3 matrix with nine numbers so is it 9?
- No, the answer is three (always!)
  - By varying three parameters we can generate every possible 3d rotation matrix (for every representation!)
- Why is it 3 and not 9?

Rotation Matrices

- Both 2d and 3d rotation matrices have two characteristics
- They are orthogonal (also called orthonormal)

$$R^T R = I \qquad\qquad R^T = R^{-1}$$

- Their determinant is 1
- Matrix below is orthogonal but not a rotation matrix because the determinate is not 1

$$\begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{pmatrix} \longleftarrow \text{this is a reflection matrix}$$

Rotation Matrices

- Rows and columns of a rotation matrix are unit vectors
- Every row is orthogonal to every other rows
  - Their dot product is zero
- Every column is orthogonal to every other column
  - Their dot product is zero
- These extra constraints mean that the entries in the rotation matrix are not independent
- This reduces the degrees of freedom

3d Rotation Matrix - Inversions

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

Remember that cos(-t) = cos(t)
And that sin(-t) = -sin(t)

$$R_x(\theta)R_x(-\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} = \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} = I$$

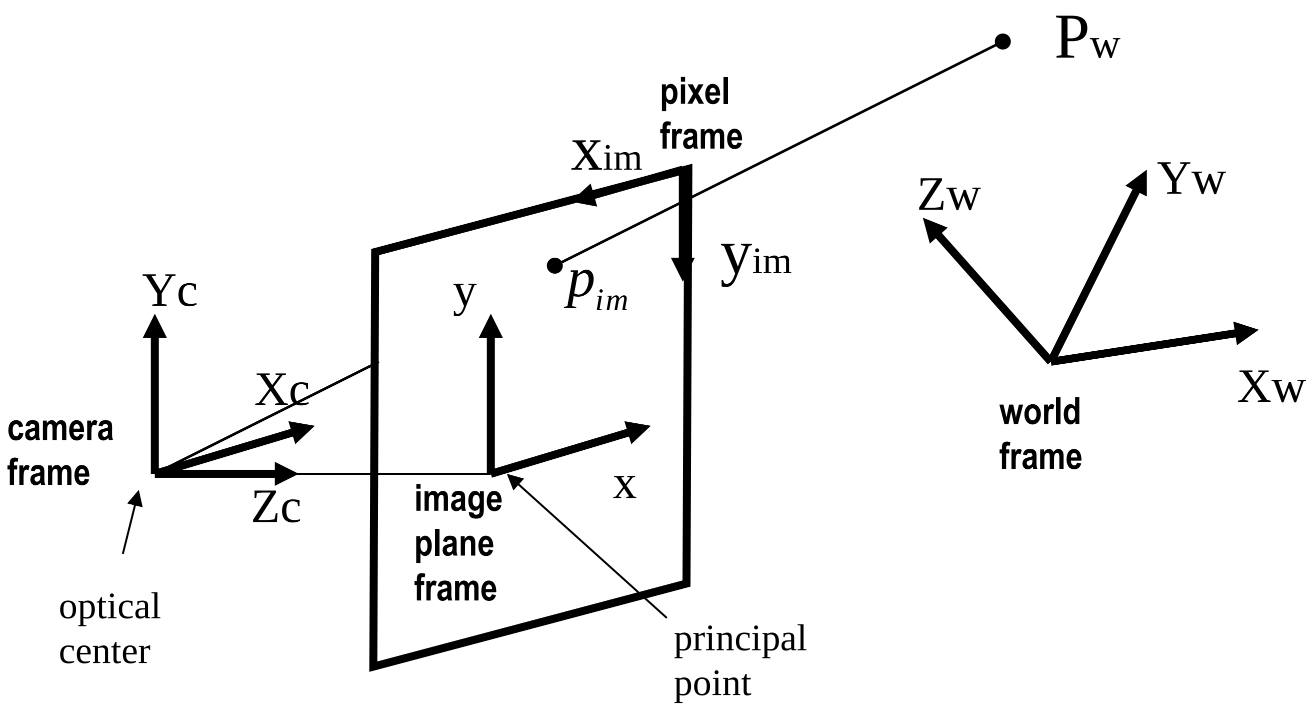
- Also for rotation matrices which we can see
-

$$R^T R = I \qquad R^T = R^{-1}$$

Homogeneous co-ordinates

- Transformation – transform a point in an n dimensional space to another n dim point
  - Transformations are scale, rotations, translations, etc.
  - You can represent all these by multiplication by one appropriate matrix using homogeneous co-ordinates
- Projection – transform a point in an n dimensional space to an m dim point
  - For projection m is normally less than n
  - Perspective projection is a projection (3d to 2d)
  - From the 3d world to a 2d point in the image
  - You can also represent a projection as matrix multiplication with one appropriate matrix and homogeneous co-ordinates

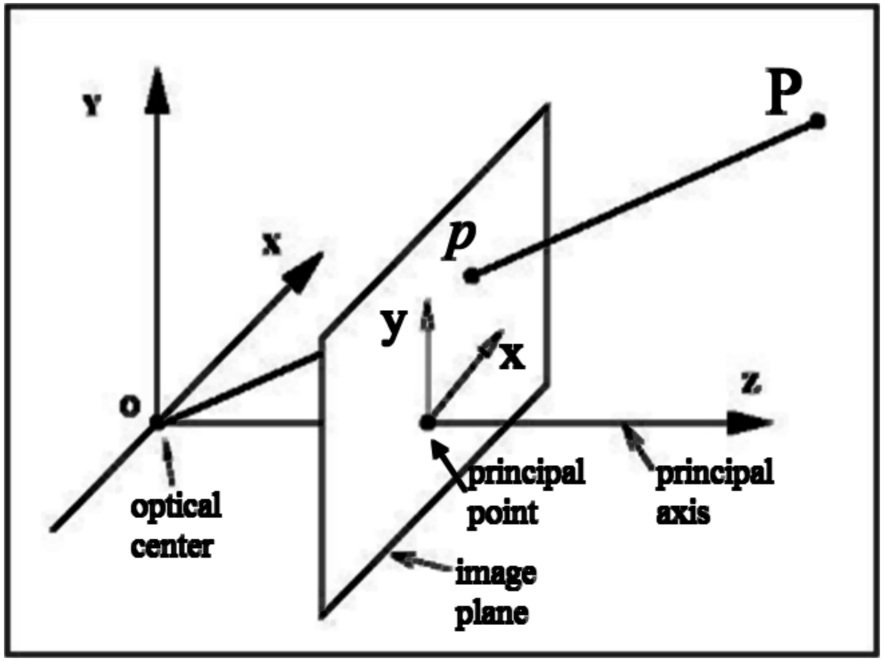
Four Coordinate Frames



Camera model:

$$\boldsymbol{p_{im}} = \begin{bmatrix} \textit{\textbf{transformation}} \\ \textit{\textbf{matrix}} \end{bmatrix} \boldsymbol{P_w}$$

Perspective Projection



$$x = f\frac{X}{Z} \qquad y = f\frac{Y}{Z}$$

These are *nonlinear*.

Using homogenous coordinate, we have a *linear* relation:

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

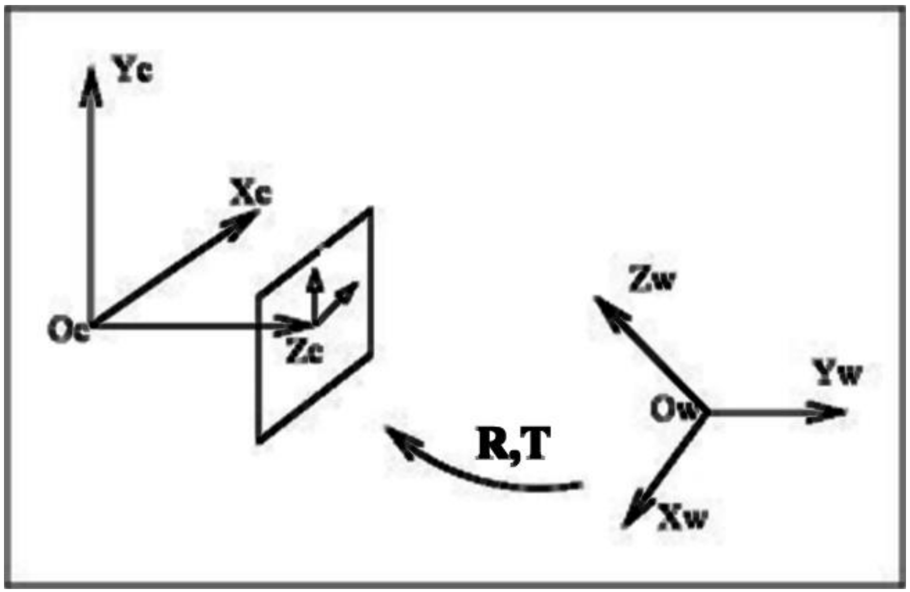
$$x = u / w$$

$$y = v / w$$

World to Camera Coordinate

Transformation between the camera and world coordinates. Here we rotate, then translate, to go from world to camera co-ordinates which is opposite of book, but is simpler and is the way in which OpenCV routines do it:

$$\mathbf{X}_c = \mathbf{R}\mathbf{X}_w + \mathbf{T}$$



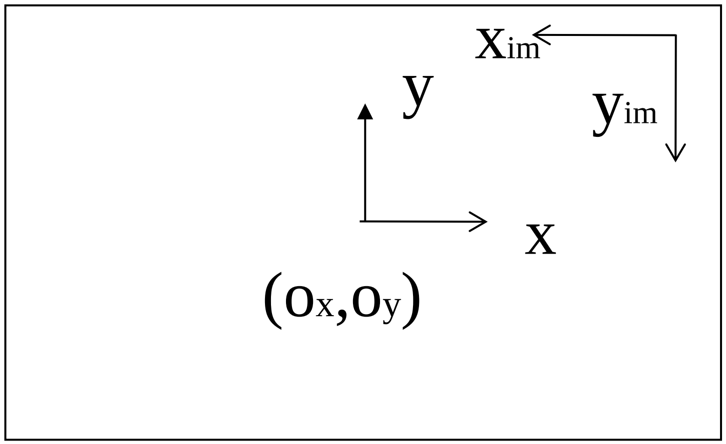
$$\begin{bmatrix} X_c \\ Y_c \\ Z_c \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{R} & \mathbf{T} \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix}$$

After R, and T we have converted from world to camera frame. In the camera frame the z axis is along the optical center.

Camera Coordinates to Image Coordinates

$$x = (o_x - x_{im})s_x \qquad y = (o_y - y_{im})s_y$$

S_x, S_y : pixel sizes in millimeters per pixel



$$\begin{bmatrix} x_{im} \\ y_{im} \\ 1 \end{bmatrix} = \begin{bmatrix} -1/s_x & 0 & o_x \\ 0 & -1/s_y & o_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Camera co-ordinates *x*, and *y* are in millimetres

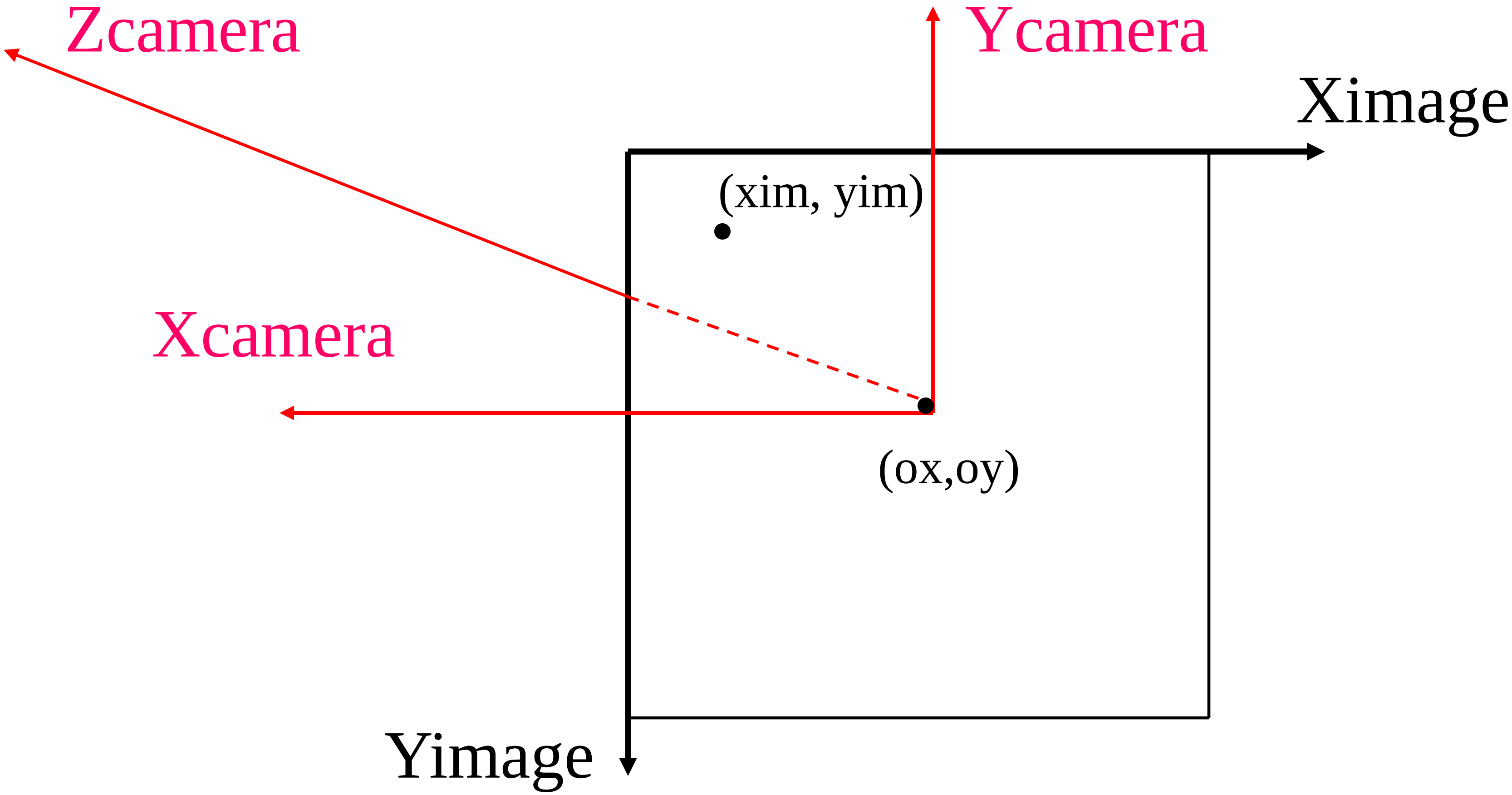
Image co-ordinates x_{im}, y_{im}, are in pixels

Center of projection o_x, o_y is in pixels

Sign change because horizontal and vertical axis of the image and camera frame have opposite directions.

Image and Camera frames

Now we look from the camera outward and image origin is the top left pixel (0,0)



Put All Together – World to Pixel

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1/s_x & 0 & o_x \\ 0 & -1/s_y & o_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

$$\boxed{x_{im} = x_1 / x_3 \quad y_{im} = x_2 / x_3}$$

From camera to pixel

$$= \begin{bmatrix} -1/s_x & 0 & o_x \\ 0 & -1/s_y & o_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} X_c \\ Y_c \\ Z_c \\ 1 \end{bmatrix}$$

Add projection

$$= \begin{bmatrix} -1/s_x & 0 & o_x \\ 0 & -1/s_y & o_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} R & T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix}$$

Add Rotation
And Translation

$$= \begin{bmatrix} -f/s_x & 0 & o_x \\ 0 & -f/s_y & o_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} R & T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix} = K\begin{bmatrix} R & T \end{bmatrix} \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix}$$

Perspective Projection Matrix

Projection is a matrix multiplication using homogeneous coordinates

$$\begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} fx \\ fy \\ z \end{bmatrix} \qquad \Rightarrow \left(f\frac{x}{z}, f\frac{y}{z} \right)$$

divide by the third coordinate

In practice: lots of coordinate transformations…

$$\begin{pmatrix} \text{2D} \\ \text{point} \\ \text{(3x1)} \end{pmatrix} = \begin{pmatrix} \text{Camera to} \\ \text{pixel coord.} \\ \text{trans. matrix} \\ \text{(3x3)} \end{pmatrix} \begin{pmatrix} \text{Perspective} \\ \text{projection matrix} \\ \text{(3x4)} \end{pmatrix} \begin{pmatrix} \text{World to} \\ \text{camera coord.} \\ \text{trans. matrix} \\ \text{(4x4)} \end{pmatrix} \begin{pmatrix} \text{3D} \\ \text{point} \\ \text{(4x1)} \end{pmatrix}$$

Camera Parameters

- Extrinsic parameters define the location and orientation of the camera reference frame with respect to a world reference frame
  - Depend on the external world, so they are extrinsic
- Intrinsic parameters link pixel co-ordinates in the image with the corresponding co-ordinates in the camera reference frame
  - An intrinsic characteristic of the camera
- Image co-ordinates are in pixels
- Camera co-ordinates are in millimetres
  - In formulas that do conversions the units must match!

Intrinsic Camera Parameters

$$K = \begin{bmatrix} -f/s_x & 0 & o_x \\ 0 & -f/s_y & o_y \\ 0 & 0 & 1 \end{bmatrix}$$

K is a 3x3 upper triangular matrix, called the **Camera Calibration Matrix**.

There are five intrinsic parameters:

(a) The pixel sizes in x and y directions s_x, s_y in millimeters/pixel

(b) The focal length f in millimeters

(c) The principal point (o_x,o_y) in pixels, which is the point where the optic axis intersects the image plane.

(d) The units of f/Sx and f/Sy are in pixels, why is this so?

Camera intrinsic parameters

- Can write three of these parameters differently by letting f/sx = fx and f/sy = fy
  - Then intrinsic parameters are ox,oy,fx,fy
  - The units of these parameters are pixels!
- In practice pixels are square (sx = sy) so that means fx should equal fy for most cameras
  - However, every explicit camera calibration process (using calibration objects) introduces some small errors
  - These calibration errors make fx not exactly equal to fy
- So in OpenCV the intrinsic camera parameters are the four following ox,oy,fx,fy
  - However fx is usually very close to fy and if this is not the case then there is a problem

Extrinsic Parameters and Proj. Matrix

$$\mathbf{p}_{im} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = K\begin{bmatrix} R & T \end{bmatrix} \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix} = M \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix}$$

[R|T] defines the **extrinsic parameters**.
The 3x4 matrix M = K[R|T] is called the **projection matrix**.
It takes 3d points in the world co-ordinate system and maps them to the appropriate image co-ordinates in pixels

Create a complete projection matrix

- Camera located at $T_{cw}=[T_x, T_y, T_z]^t = [10,20,30]^t$
- Camera aimed along z-axis, x,y axes parallel to world axes (R=I)

$$T_{cw} = \begin{bmatrix} T_x \\ T_y \\ T_z \end{bmatrix} = \begin{bmatrix} 10 \\ 20 \\ 30 \end{bmatrix} \qquad R = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad S = \begin{bmatrix} -10 \\ -20 \\ -30 \end{bmatrix}$$

- Camera has focal length $f_x = f_y = 1000$
- image=640x480, assume $u_0=320$, $v_0=240$
- Calculate complete 3x4 projection matrix

$$\begin{bmatrix} P_{11} & P_{12} & P_{13} & P_{14} \\ P_{21} & P_{22} & P_{23} & P_{24} \\ P_{31} & P_{32} & P_{33} & P_{34} \end{bmatrix} = \begin{bmatrix} 1000 & 0 & 320 \\ 0 & 1000 & 240 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -10 \\ 0 & 1 & 0 & -20 \\ 0 & 0 & 1 & -30 \end{bmatrix} = \begin{bmatrix} 1000 & 0 & 320 & -19600 \\ 0 & 1000 & 240 & -27200 \\ 0 & 0 & 1 & -30 \end{bmatrix}$$

$$u = \frac{u'}{w'}, \quad v = \frac{v'}{w'}$$

Using the projection matrix - example

$$\begin{bmatrix} P_{11} & P_{12} & P_{13} & P_{14} \\ P_{21} & P_{22} & P_{23} & P_{24} \\ P_{31} & P_{32} & P_{33} & P_{34} \end{bmatrix} = \begin{bmatrix} 1000 & 0 & 320 & -19600 \\ 0 & 1000 & 240 & -27200 \\ 0 & 0 & 1 & -30 \end{bmatrix}$$

$$u = \frac{u'}{w'} \quad v = \frac{v'}{w'}$$

- Where would the point (20,50,200) project to in the image?

Projection matrix

$$\begin{bmatrix} u' \\ v' \\ w' \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} & P_{13} & P_{14} \\ P_{21} & P_{22} & P_{23} & P_{24} \\ P_{31} & P_{32} & P_{33} & P_{34} \end{bmatrix} \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix} = \begin{bmatrix} 1000 & 0 & 320 & -19600 \\ 0 & 1000 & 240 & -27200 \\ 0 & 0 & 1 & -30 \end{bmatrix} \begin{bmatrix} 20 \\ 50 \\ 200 \\ 1 \end{bmatrix}$$

$$u = \frac{u'}{w'} \quad v = \frac{v'}{w'}$$

$$\begin{bmatrix} u' \\ v' \\ w' \end{bmatrix} = \begin{bmatrix} 64400 \\ 70800 \\ 170 \end{bmatrix}$$

u=u’/w’ = 64400/170 = 378.8
v=v’/w’ = 70800/170 = 416.5

- World point (20,50,200) project to pixel With co-ordinates of (379,417)

Using the projection matrix - example

$$\begin{bmatrix} P_{11} & P_{12} & P_{13} & P_{14} \\ P_{21} & P_{22} & P_{23} & P_{24} \\ P_{31} & P_{32} & P_{33} & P_{34} \end{bmatrix} = \begin{bmatrix} 1000 & 0 & 320 & -19600 \\ 0 & 1000 & 240 & -27200 \\ 0 & 0 & 1 & -30 \end{bmatrix}$$

$$u = \frac{u'}{w'}, \quad v = \frac{v'}{w'}$$

- Where would the point (10,20,200) project to in the image?

Projection matrix

$$\begin{bmatrix} u' \\ v' \\ w' \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} & P_{13} & P_{14} \\ P_{21} & P_{22} & P_{23} & P_{24} \\ P_{31} & P_{32} & P_{33} & P_{34} \end{bmatrix} \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix} = \begin{bmatrix} 1000 & 0 & 320 & -19600 \\ 0 & 1000 & 240 & -27200 \\ 0 & 0 & 1 & -30 \end{bmatrix} \begin{bmatrix} 10 \\ 20 \\ 200 \\ 1 \end{bmatrix}$$

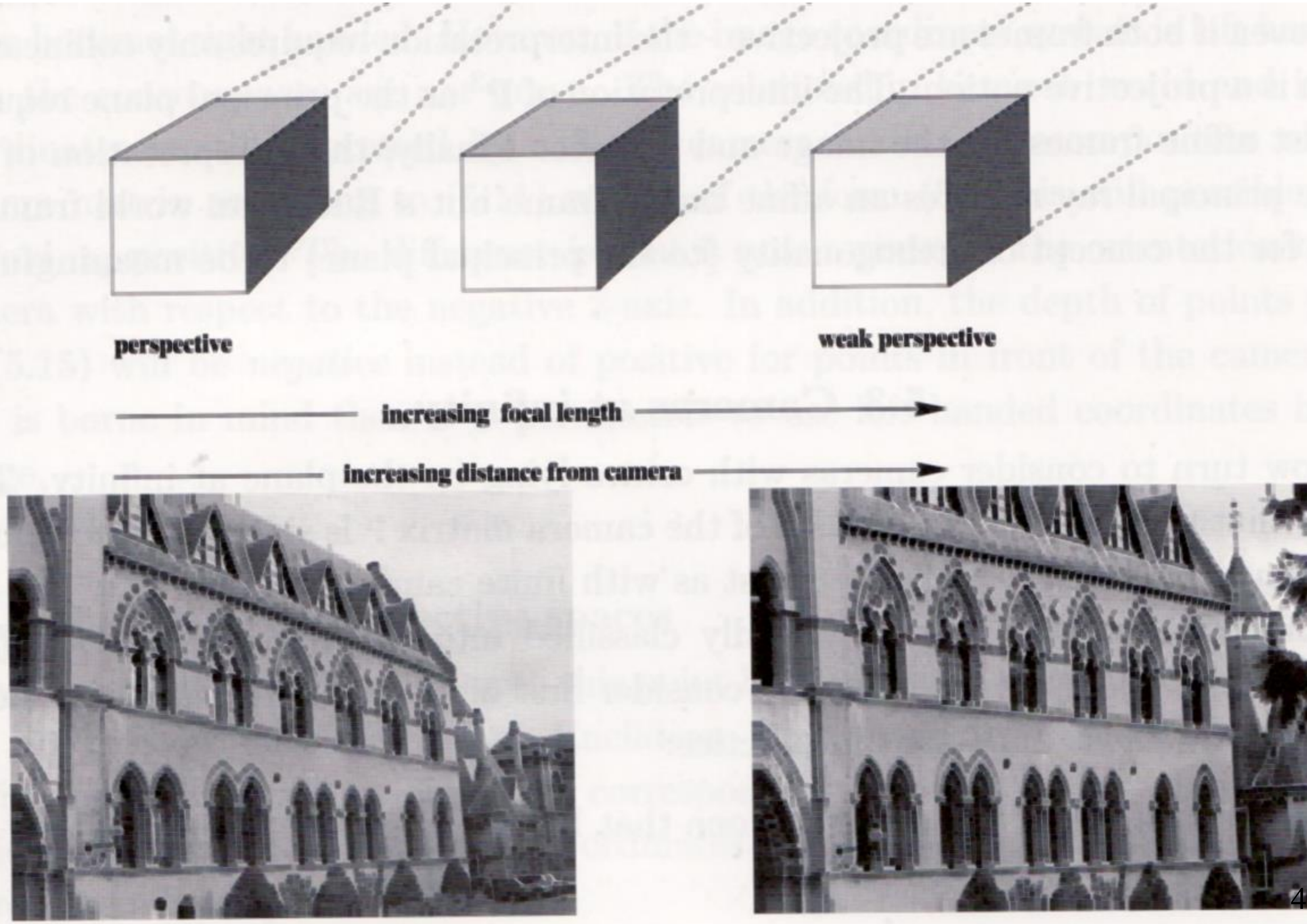
$$u = \frac{u'}{w'}, \quad v = \frac{v'}{w'}$$

$$\begin{bmatrix} u' \\ v' \\ w' \end{bmatrix} = \begin{bmatrix} 54400 \\ 40400 \\ 170 \end{bmatrix}$$

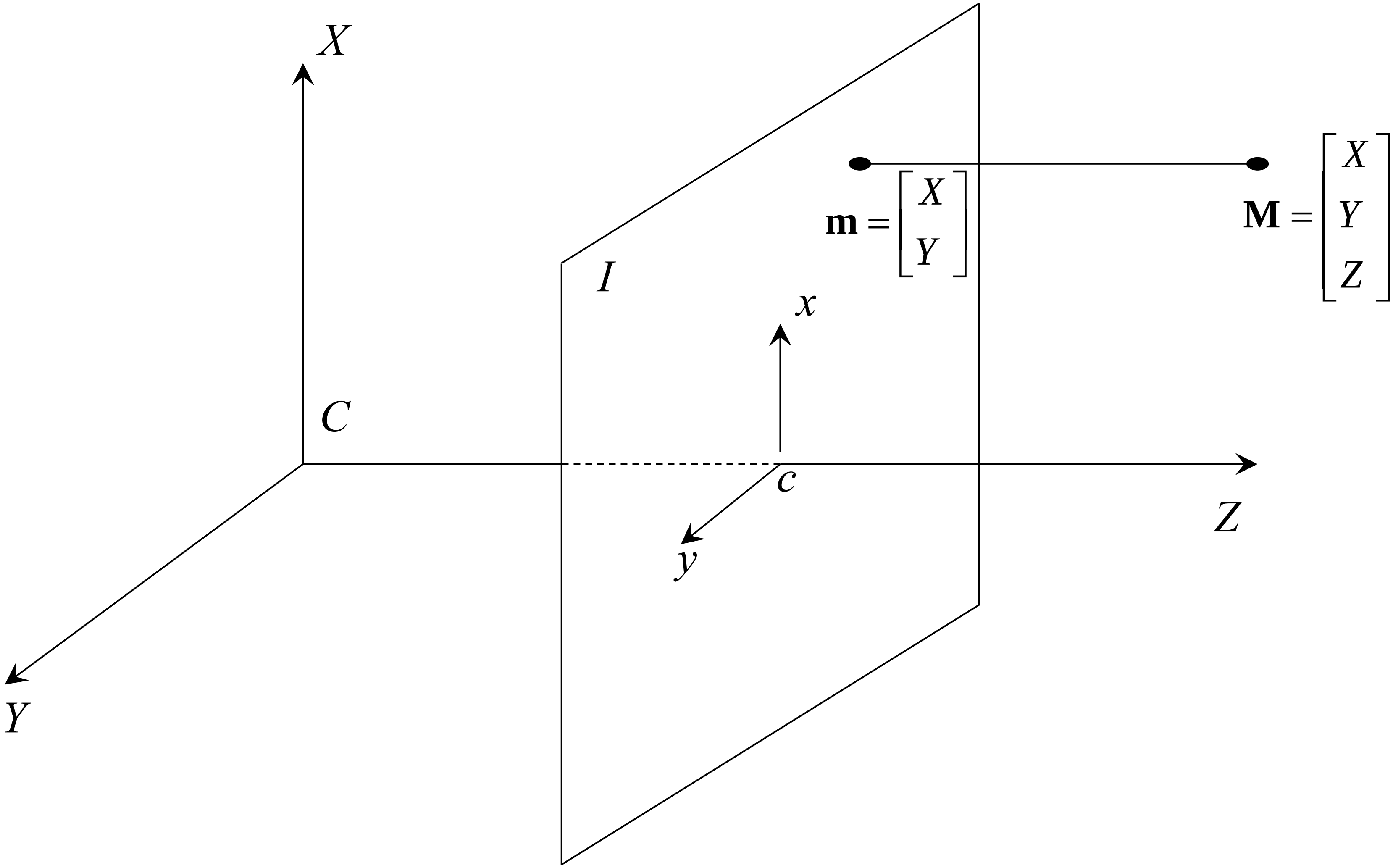
$u=u'/w' = 54400/170 = 320$
$v=v'/w' = 40800/170 = 240$

Effect of change in focal length

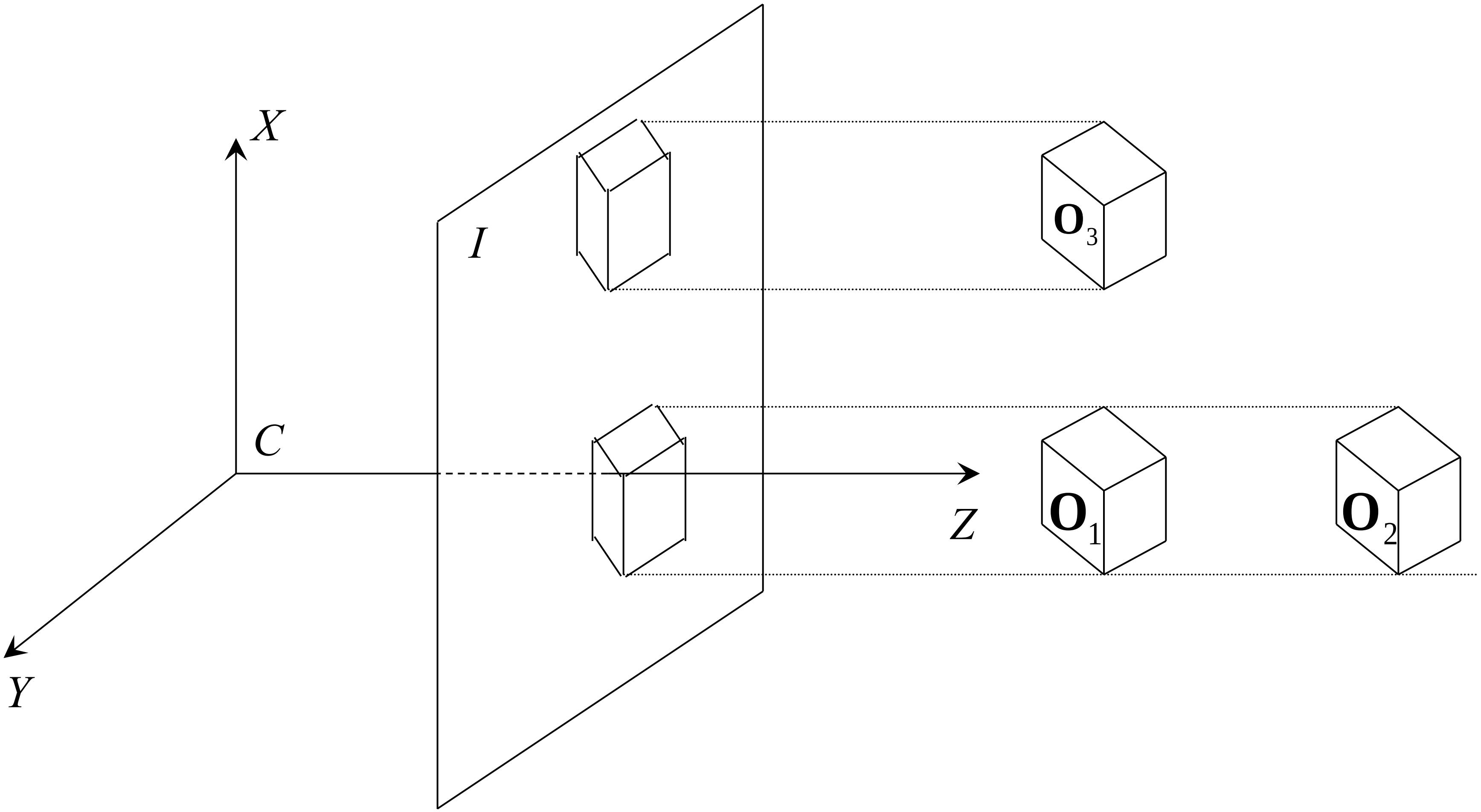
Small f is wide angle, large f is telescopic



Orthographic Projection



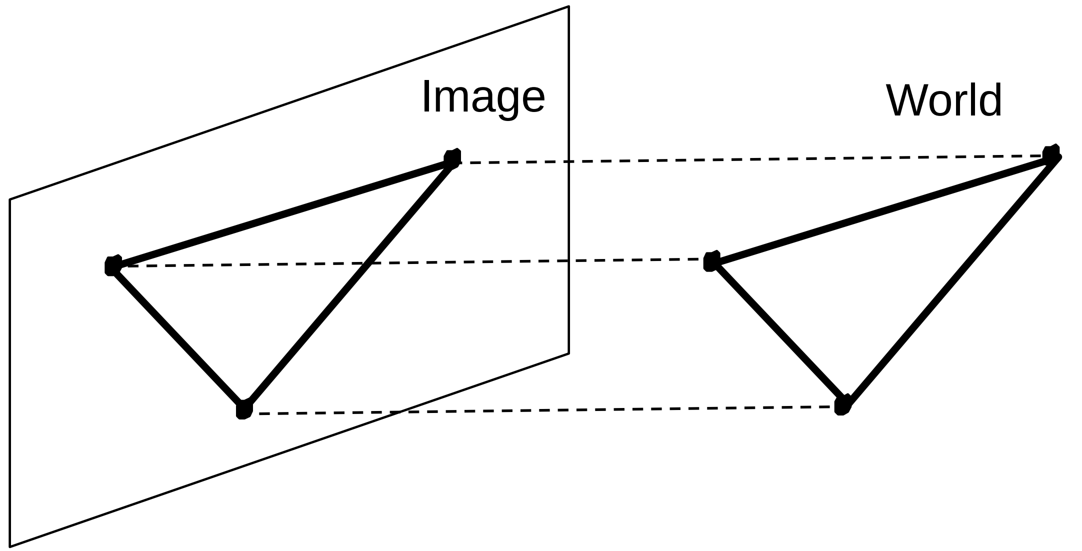
Orthographic Projection



Orthographic Projection

Special case of perspective projection

- Distance from center of projection to image plane is infinite



- Also called “parallel projection”
- What’s the projection matrix?

Slide by Steve Seitz

Weak Perspective Model

Assume the relative distance between any two points in an object along the principal axis is much smaller (1/20th at most) than the $\bar{Z}$ average distance of the object. Then the camera projection can be approximated as:

$$x = f\frac{X}{Z} \approx \frac{f}{\bar{Z}}X \qquad y = f\frac{Y}{Z} \approx \frac{f}{\bar{Z}}Y$$

This is the **weak-perspective** camera model. Sometimes called scaled orthography.

Weak Perspective Projection

Two step projection:

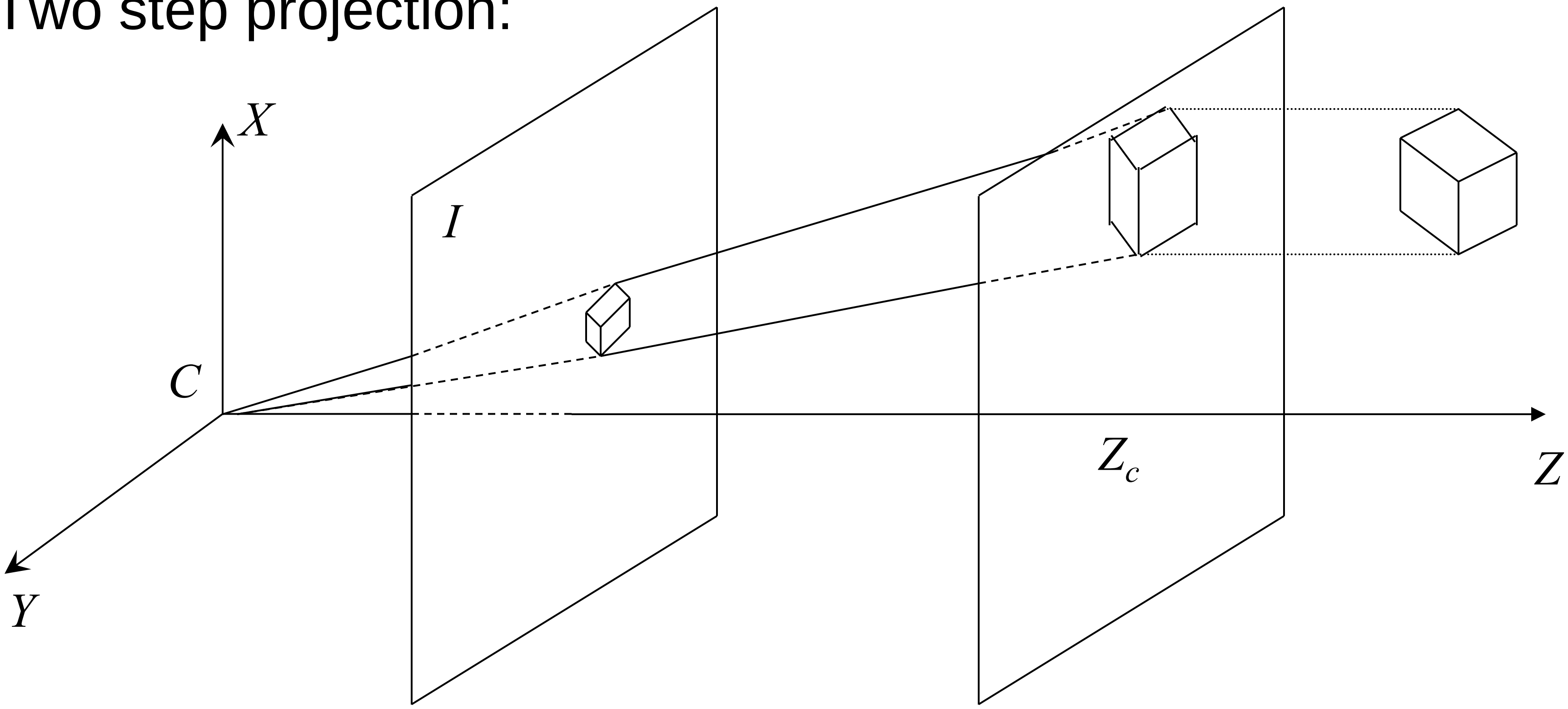
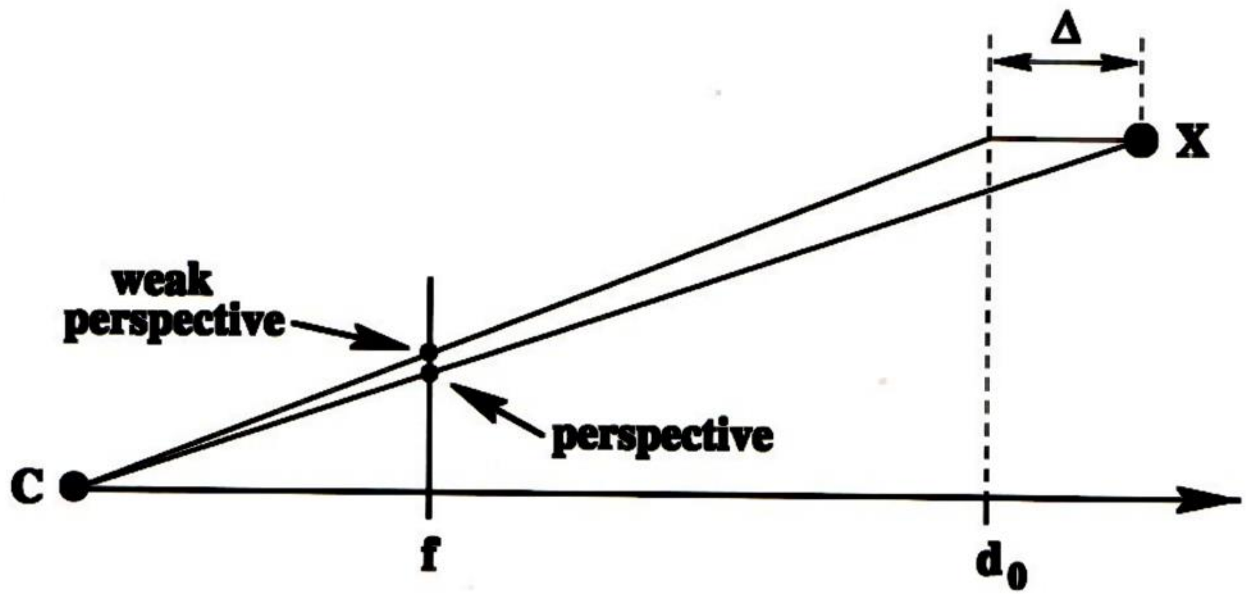


image plane average depth plane

Weak Perspective



Impact of different projections

- Perspective projections
  - Parallel lines in world are not parallel in the image
  - Object projection gets smaller with distance from camera
- Weak perspective projection
  - Parallel lines in the world are parallel in the image
  - Object projection gets smaller with distance from camera
- Orthographic projection
  - Parallel lines in the world are parallel in the image
  - Object projection is unchanged with distance from camera

Ordinary Perspective

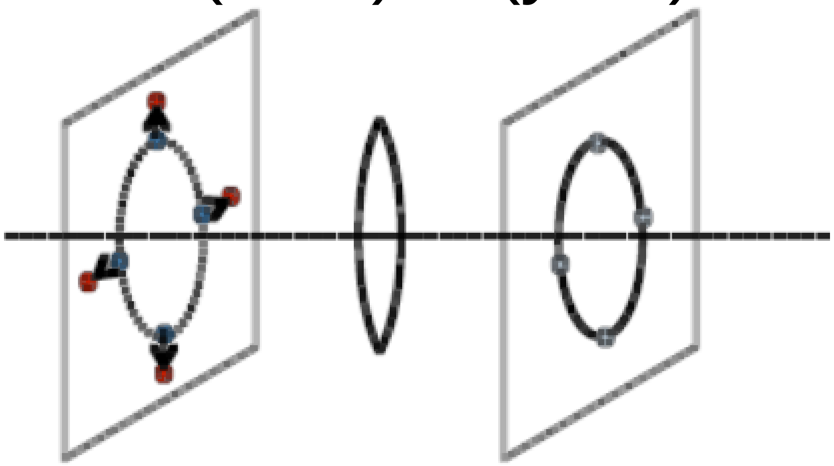
- Parallel lines in world - not parallel in image

Weak Perspective

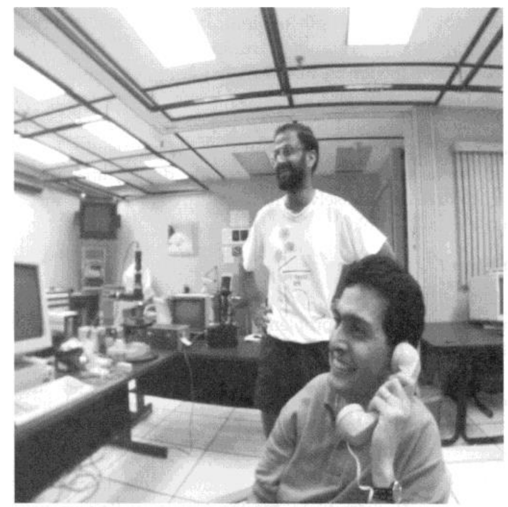
- Parallel lines in world – parallel in image

Image distortion due to optics

- Radial distortion which depends on radius r, distance of each point from center of image
- $r^2 = (x - o_x)^2 + (y - o_y)^2$



Radial distortion

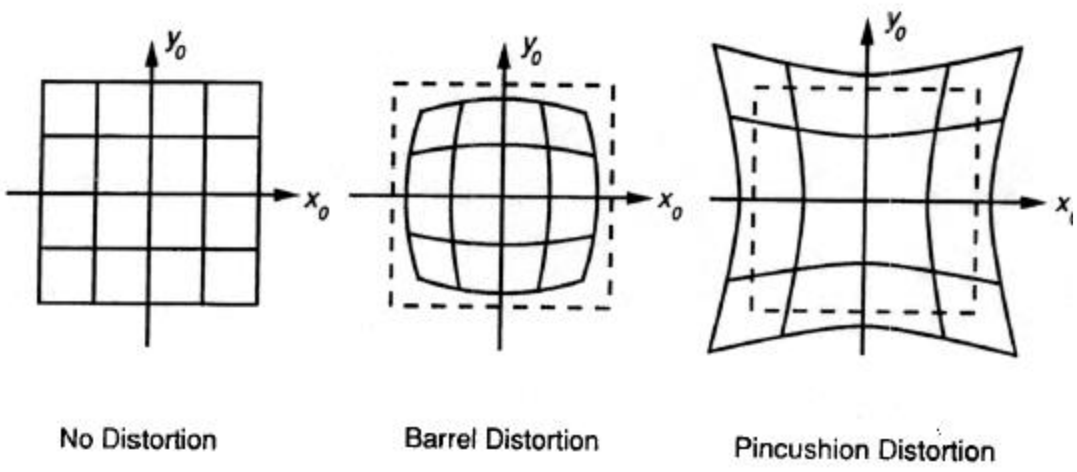


$$x_{corrected} = x(1 + k_1 r^2 + k_2 r^4 + k_3 r^6)$$
$$y_{corrected} = y(1 + k_1 r^2 + k_2 r^4 + k_3 r^6)$$

orrection uses three
arameters, k_1, k_2, k_3

Radial Distortion

- Error is proportional to distance of pixel from the camera center (the radius of the point)

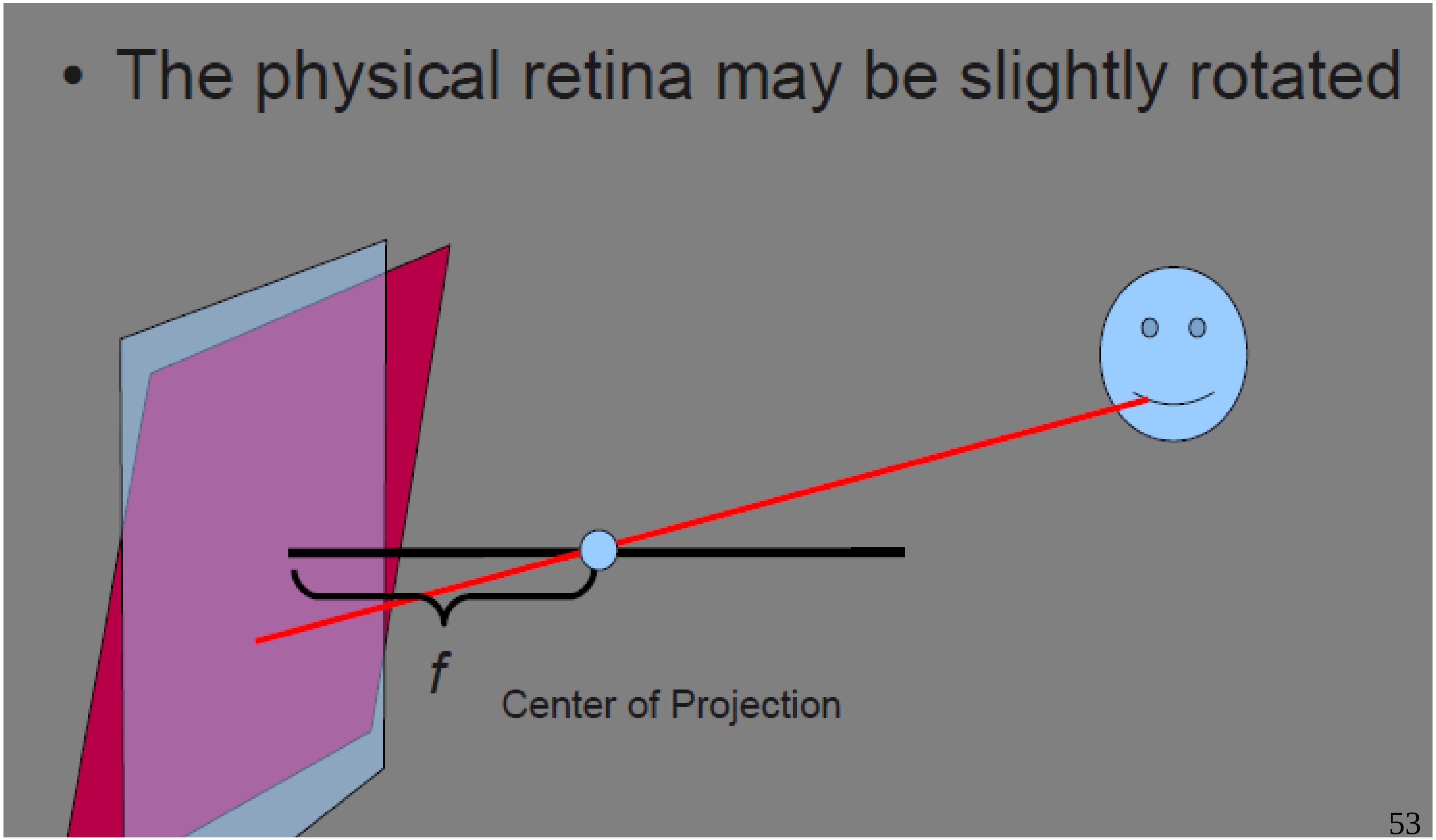


Barrel – too far, Pincushion – too close

Correcting Radial Distortions

Tangential Distortion

- (Ox, Oy) – center of projection) is not the center of image
- Also causes a more complex distortion of the image

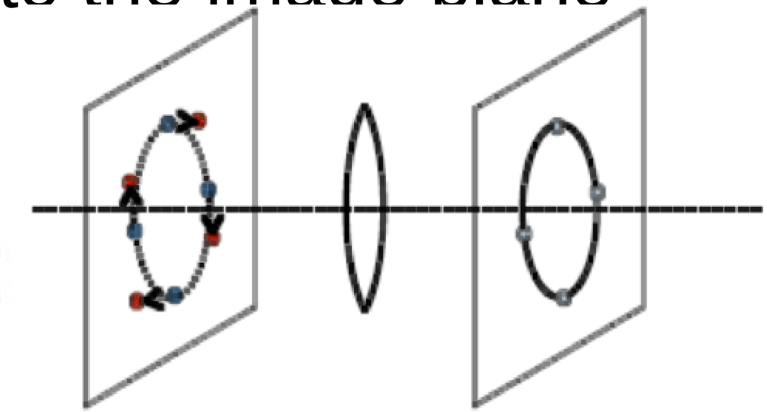


Tangential Distortion

- Lens not exactly parallel to the image plane

$$x_{\text{corrected}} = x + [2p_1 y + p_2(r^2 + 2x^2)]$$

$$y_{\text{corrected}} = y + [p_1(r^2 + 2y^2) + 2p_2 x]$$



Tangential distortion

- Correction uses two parameter p_1, p_2
- Both types of distortion are removed (image is un-distorted) and only then does standard calibration matrix K apply to the image
- Camera calibration computes both K and these five distortion parameters

How to find the camera parameters K

- Can use the EXIF tag for any digital image
  - Has focal length f in millimeters but not the pixel size
  - But you can get the pixel size from the camera manual
  - There are only a finite number of different pixels sizes because the number of sensing element sizes is limited
  - If there is not a lot of image distortion due to optics then this approach is sufficient (this is only a linear calibration)
- Can perform explicit camera calibration
  - Put a calibration pattern in front of the camera
  - Take a number of different pictures of this pattern
  - Now run the calibration algorithm (different types)
  - Result is intrinsic camera parameters (linear and non-linear) and the extrinsic camera parameters of all the images

Summary Questions

Why do we use homogeneous co-ordinates for image projection?

Write the projection equation in a linear form using a matrix with homogeneous co-ordinates.

What are the units of image co-ordinates and camera co-ordinates? How do you convert between them?

Why do the names extrinsic, and intrinsic parameters of a lens make sense?

What are the units of f/sx, and f/sy? Implications?

What are characteristics of perspective, weak perspective and orthographic projection?

What are parameters of models for non-linear lens characteristics?