# Epipolar Geometry

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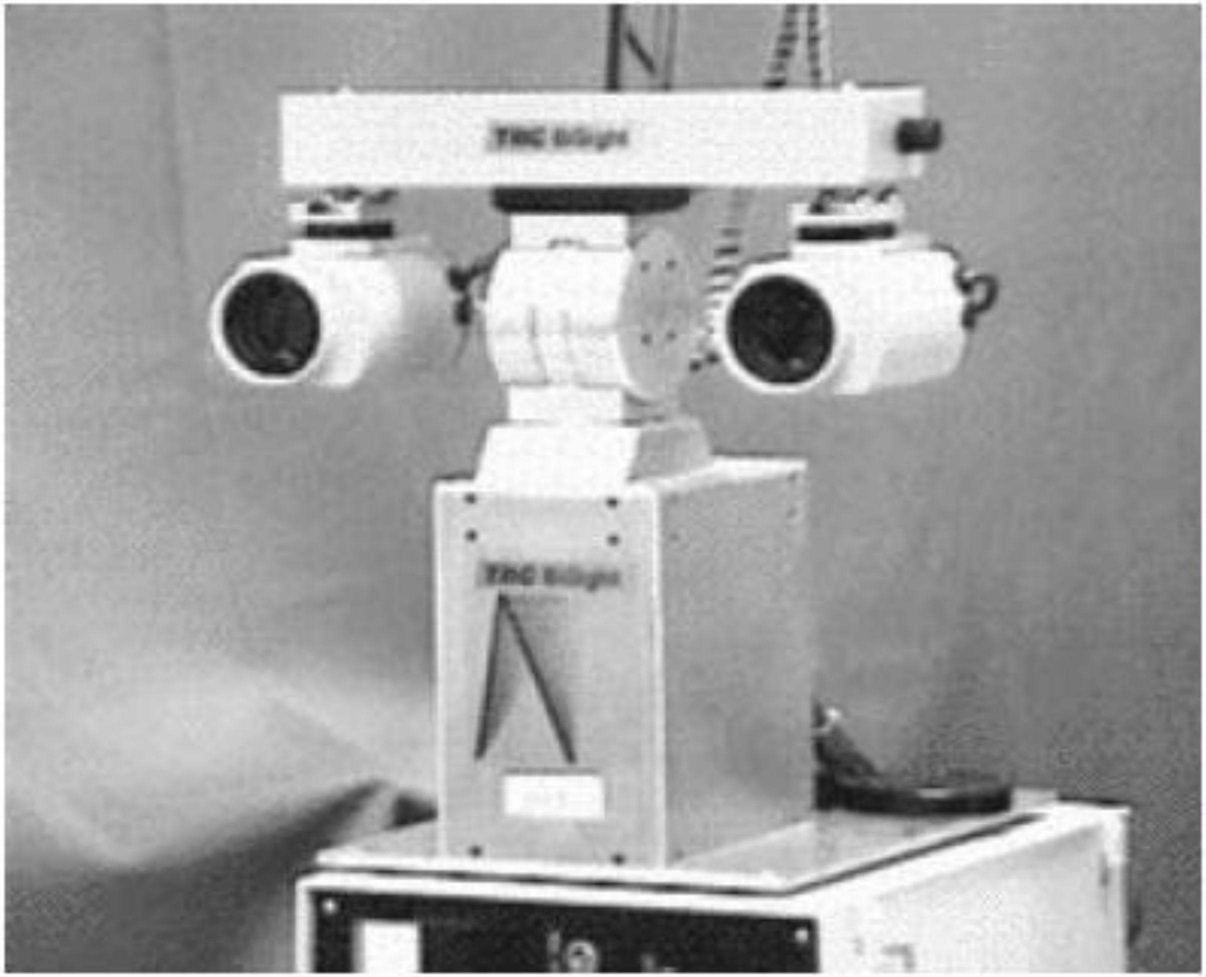
# Problem Definition

- Simple stereo configuration
  - Corresponding points are on same horizontal line
  - This makes correspondence search a 1D search
  - Need only look for matches on same horizontal line
- if two cameras are in an arbitrary location is there a similar constraint to make search 1D?
  - Yes, called epipolar constraint
  - Based on epipolar geometry
  - We will derive this constraint
  - Consider two cameras that can see a single point P
  - They are in an arbitrary positions and orientation
    - One camera is rotated and translated relative to the other camera
    - Must be some overlap for correspondence and reconstruction!

# Controllable Stereo Head



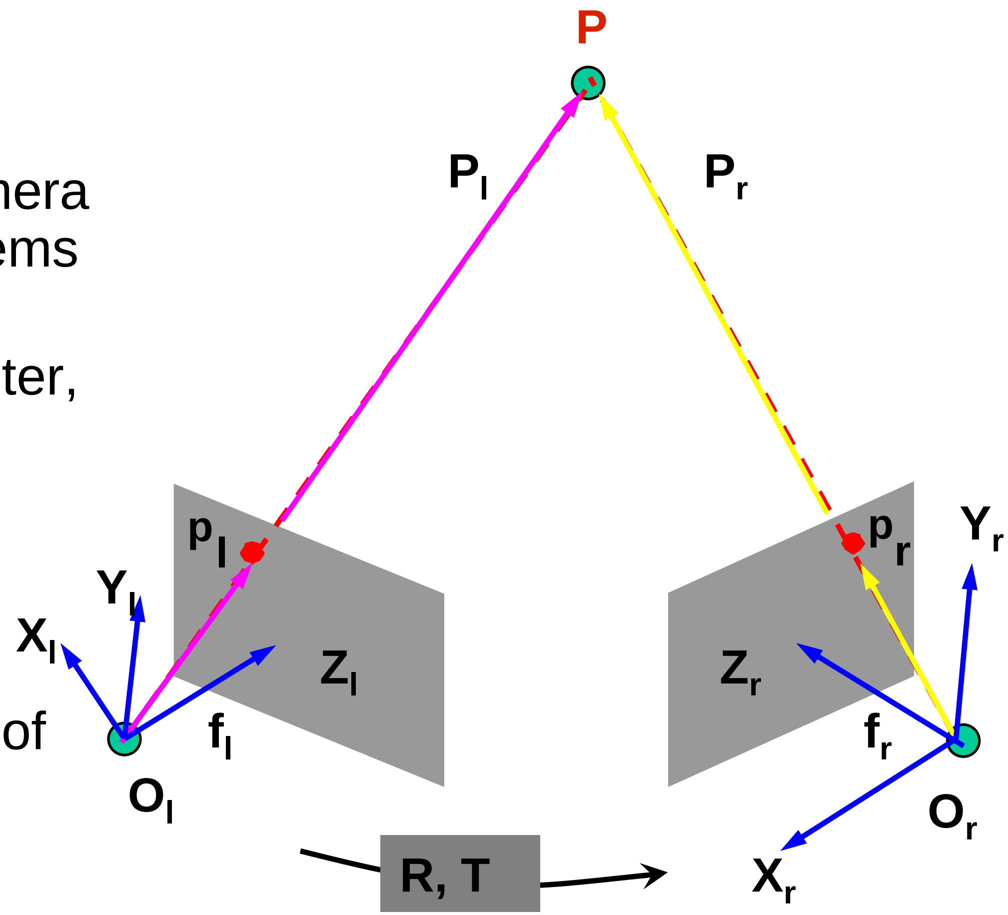
# Parameters of a Stereo System

## Intrinsic Parameters

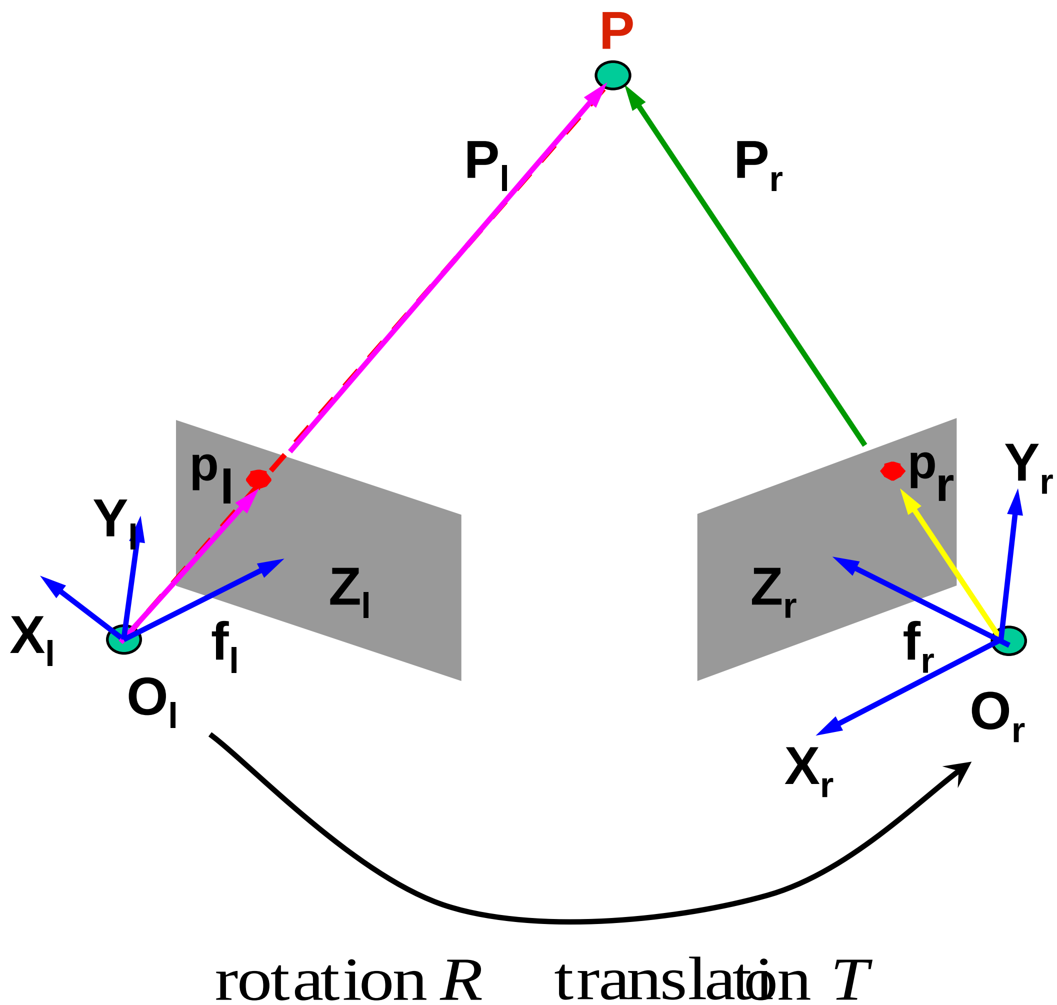
- Characterize the transformation from camera to pixel coordinate systems of each camera
- Focal length, image center, aspect ratio

## Extrinsic parameters

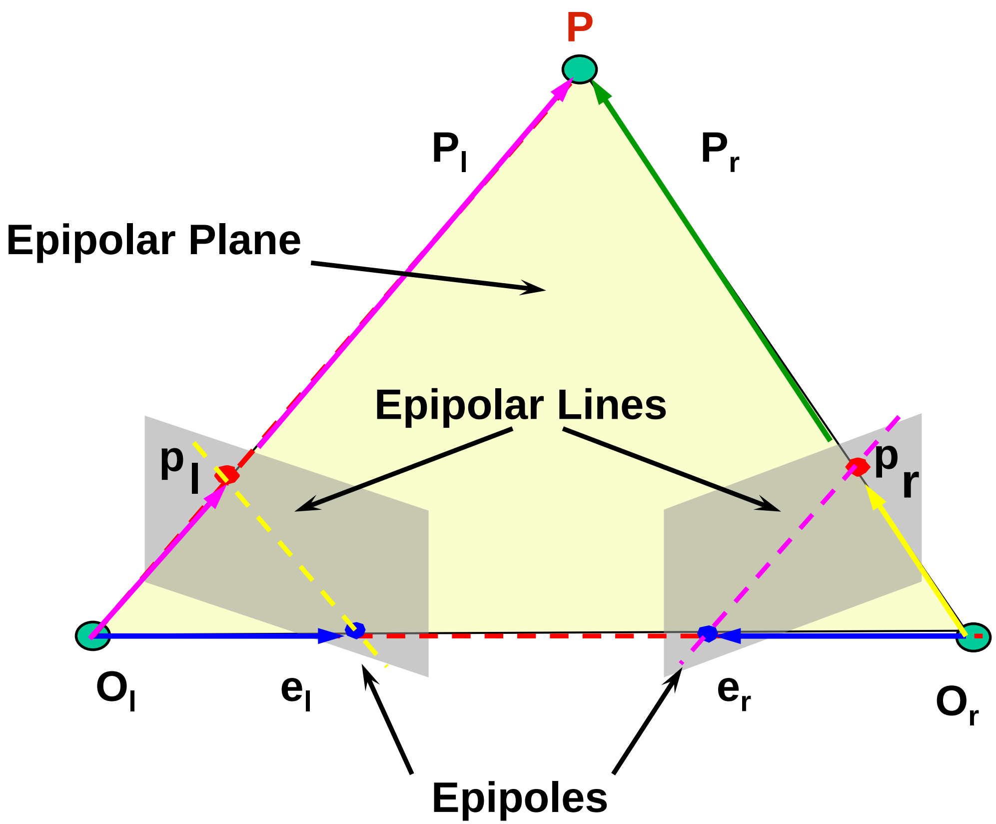
- Describe the relative position and orientation of the two cameras
- Rotation matrix R and translation vector T



# Epipolar Geometry



# Epipolar Geometry

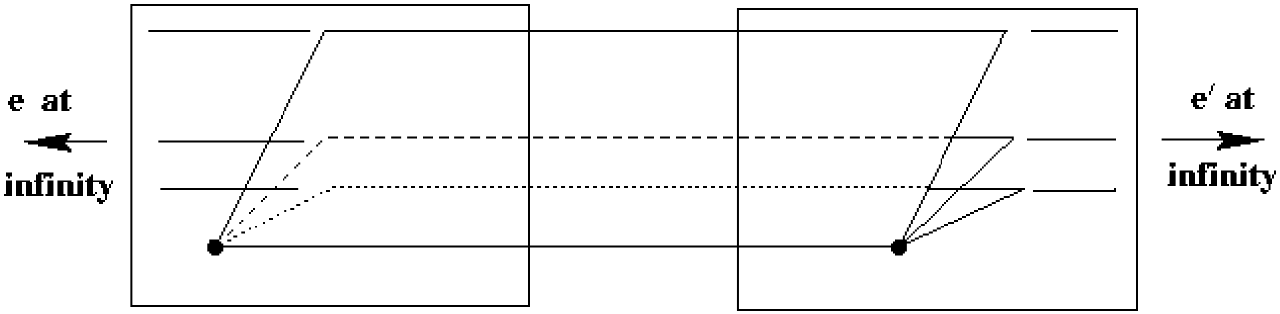


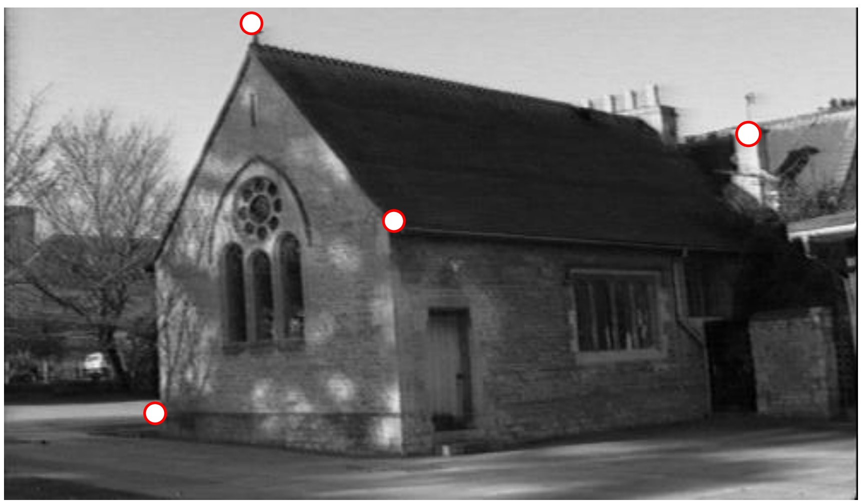

$$\mathbf{P_r} = \mathbf{R}(\mathbf{P_l} - \mathbf{T})$$

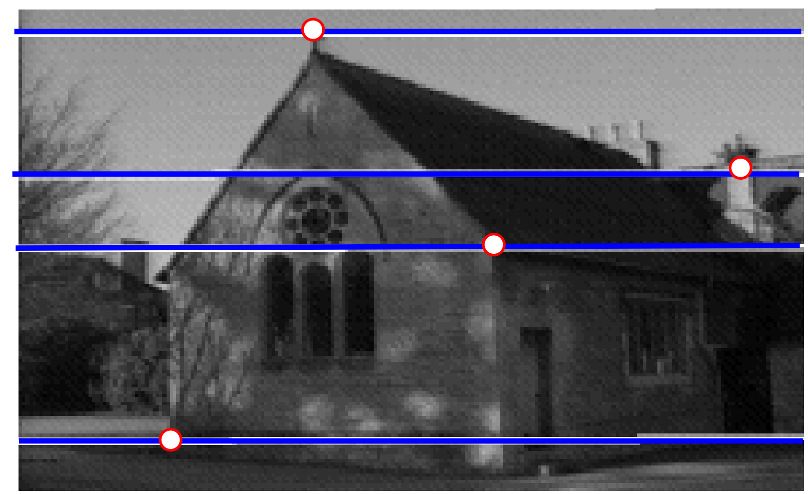
# Shape of epipolar lines

- Translating the cameras in the x, y plane without any camera rotation then the epipolar lines are parallel
- Translating the cameras in the direction of the camera y axis (horizontal) you get the simple stereo configuration of horizontal epipolar lines
- Translating the cameras with forward motion in the z axis produces epipolar lines that emanate from the epipole (sometimes called focus of projection)

# Epipolar geometry : parallel cameras





Epipolar geometry depends only on the relative pose (position and orientation) and internal parameters of the two cameras, i.e. the position of the camera centres and image planes. It does not depend on the scene structure (3D points external to the camera).

# Epipolar geometry : forward motion

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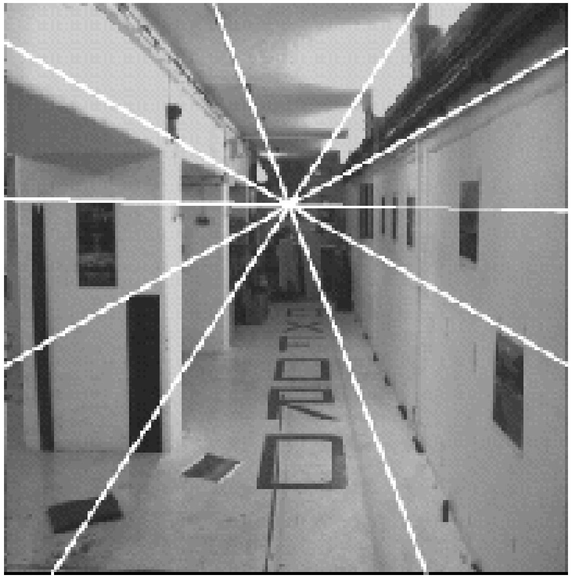
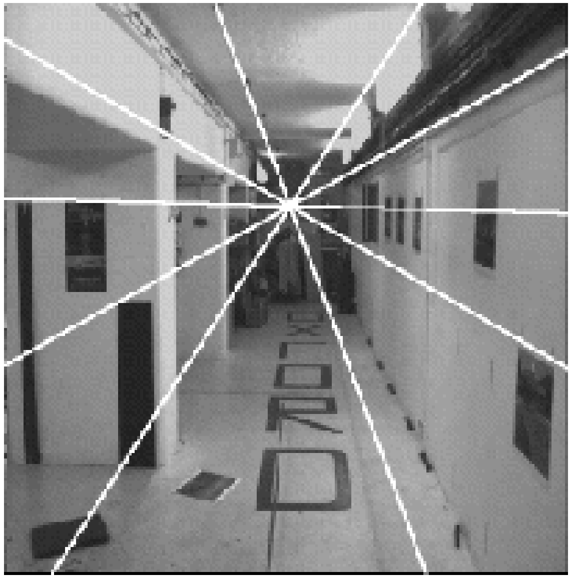
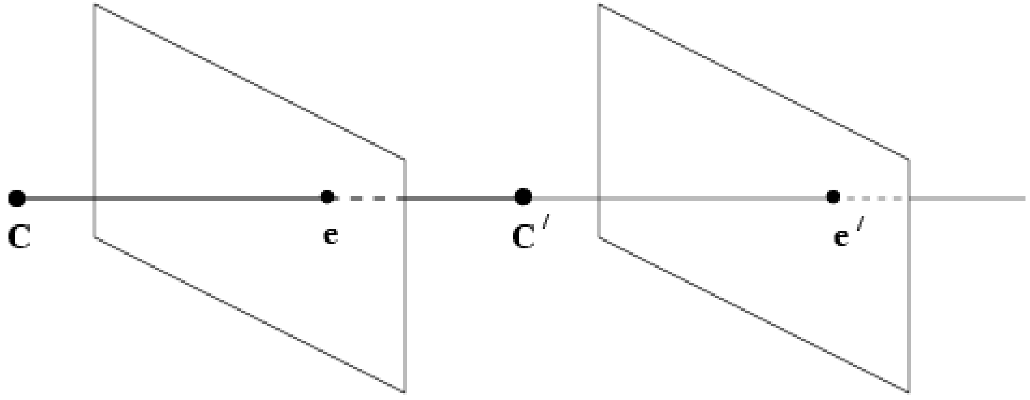
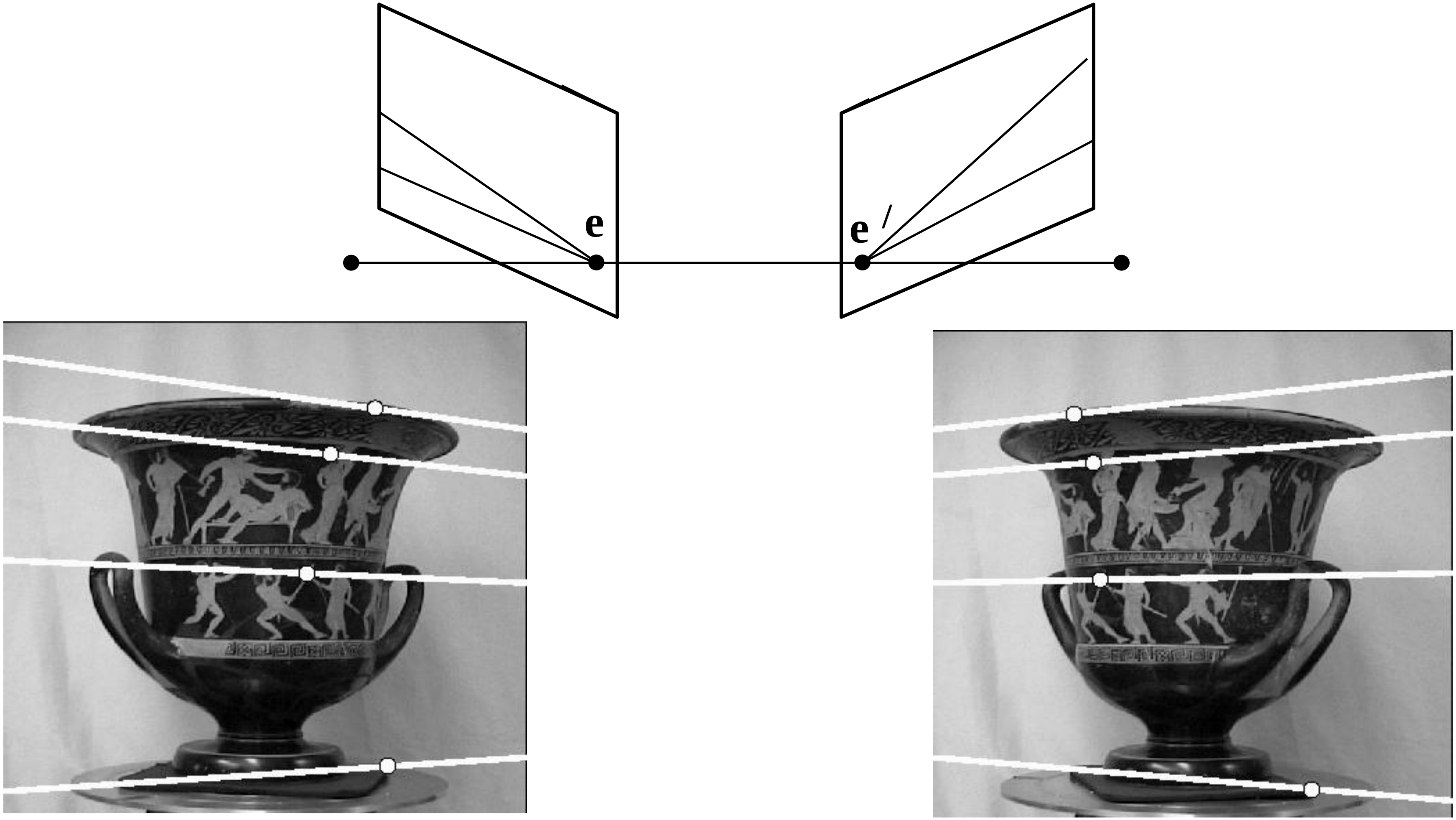






# Epipolar geometry : converging cameras




So this means that epipolar lines are in general not parallel (only for certain cases)

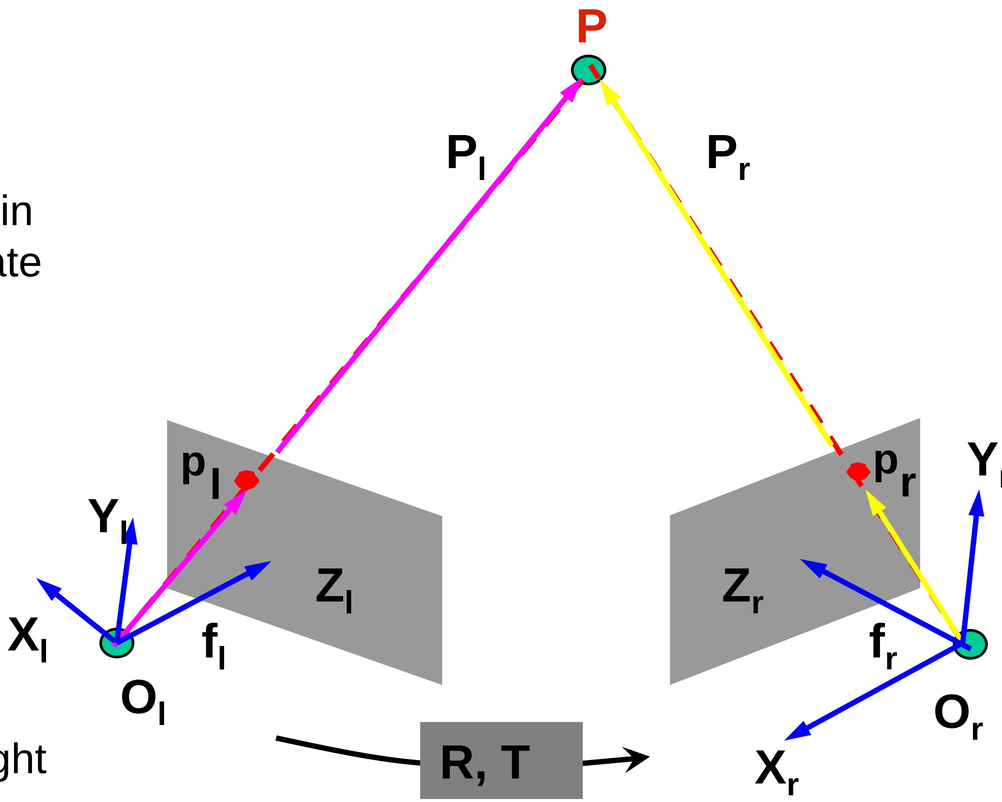
# Epipolar Geometry

## Notations

- $P_l = (X_l, Y_l, Z_l)$, $P_r = (X_r, Y_r, Z_r)$
  - Vectors of the same 3-D point P, in the left and right camera coordinate systems respectively
- Extrinsic Parameters
  - Translation Vector $T = (O_r - O_l)$
  - Rotation Matrix R

$$\mathbf{P_r} = \mathbf{R}(\mathbf{P_l} - \mathbf{T})$$

- $p_l = (x_l, y_l, z_l)$, $p_r = (x_r, y_r, z_r)$
  - Projections of P on the left and right image plane respectively
  - For all image points, we have $z_l = f_l$, $z_r = f_r$




$$\mathbf{p}_l = \frac{f_l}{Z_l}\mathbf{P_l} \qquad \mathbf{p}_r = \frac{f_r}{Z_r}\mathbf{P}_r$$

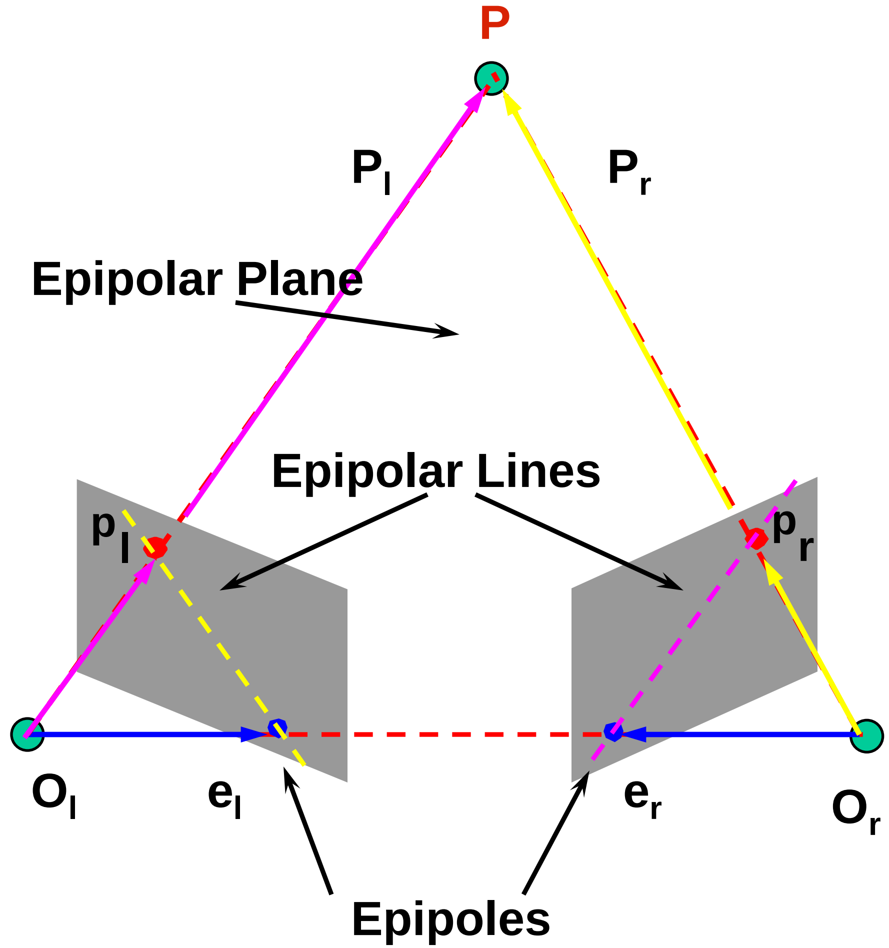
# Epipolar Geometry

Motivation: where to search correspondences?

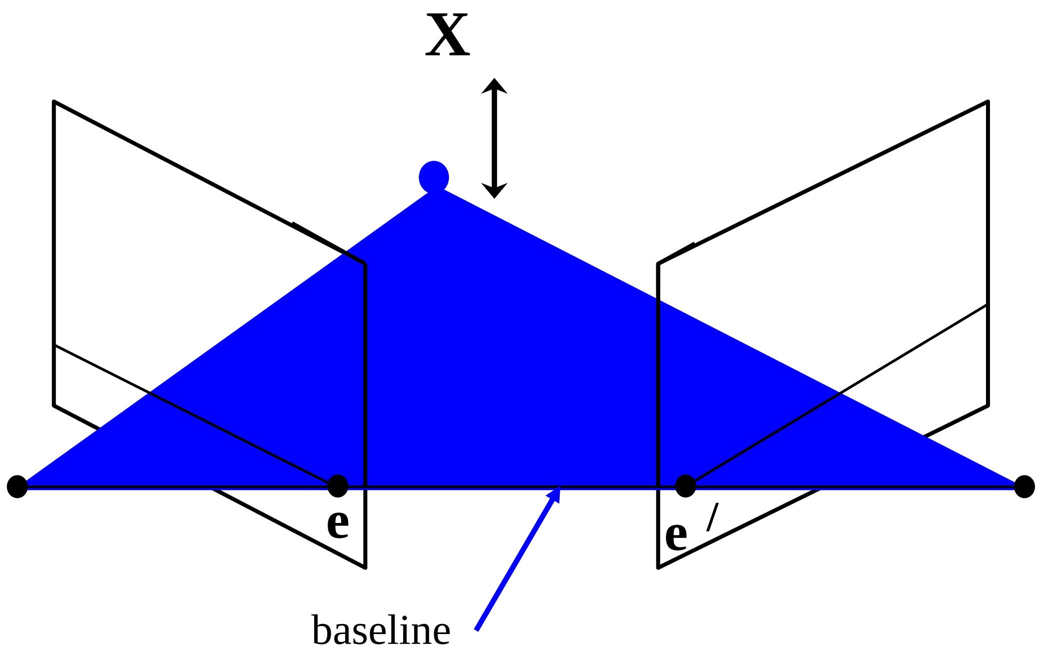
- Epipolar Plane
  - A plane going through point P and the centers of projection (COPs) of the two cameras
- Epipolar Lines
  - Lines where epipolar plane intersects the image planes
- Epipoles
  - The image in one camera of the COP of the other

## Epipolar Constraint

- Corresponding points must lie on epipolar lines
- True for EVERY camera configuration, not depend on the geometry of the scene!



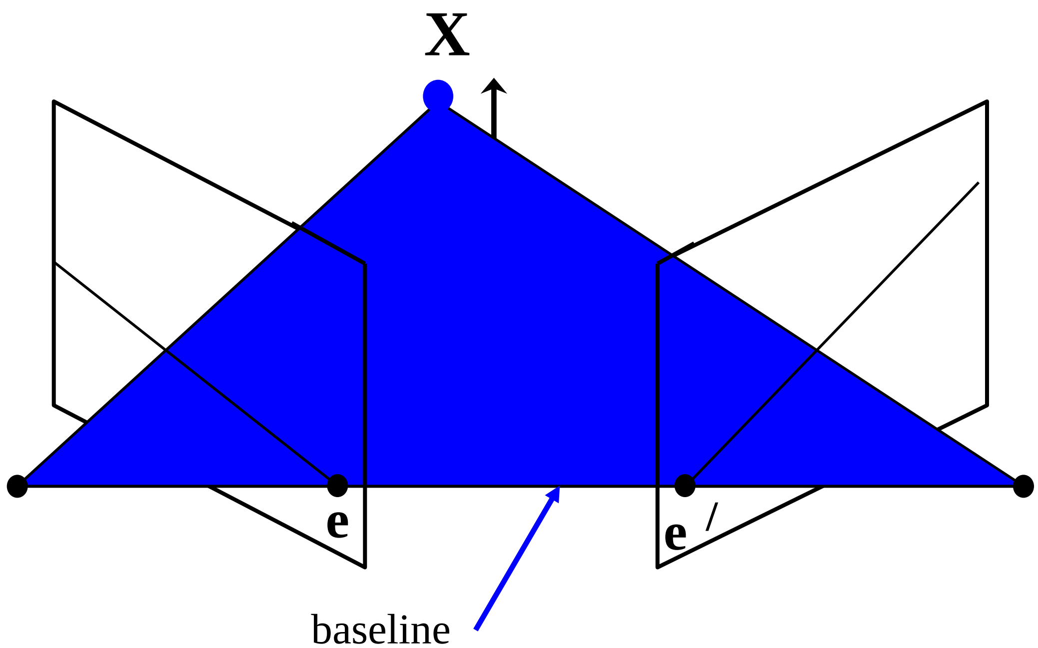
# The epipolar pencil




As the position of the 3D point **X** varies, the epipolar planes “rotate” about the baseline. This family of planes is known as an epipolar pencil. All epipolar lines intersect at the epipole.

(a pencil is a one parameter family)

# The epipolar pencil




As the position of the 3D point **X** varies, the epipolar planes “rotate” about the baseline. This family of planes is known as an epipolar pencil. All epipolar lines intersect at the epipole.

(a pencil is a one parameter family)

# Epipolar Geometry

Epipolar plane: plane going through point P and the centers of projection (COPs) of the two cameras

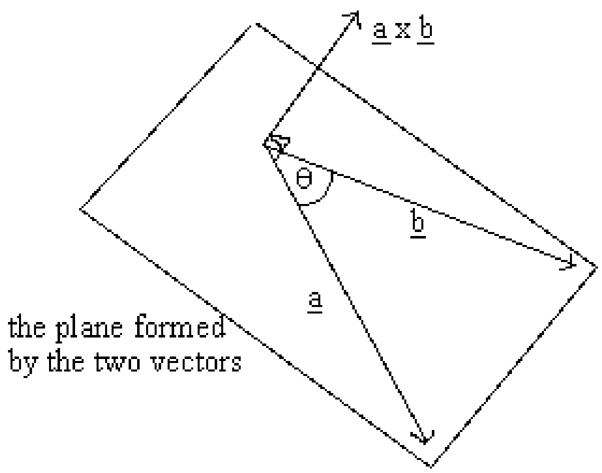
Epipolar lines: where this epipolar plane intersects the two image planes

Epipoles: The image in one camera of the COP of the other

Epipolar Constraint: Corresponding points between the two images must lie on epipolar lines

# Cross product

- Consider two vectors in 3D space
  - (a1,a2,a3) and (b1,b2,b3)
- **a** x **b** = **n** a b sin q
- Cross product is at 90 degrees to both vectors
  - Normal to the plane defined by the two vectors



# Cross product

- Two possible normal directions
  - We use the right hand rule to compute direction
- $\underline{\mathbf{a}} \times \underline{\mathbf{b}} =$

  $( a1\, \underline{\mathbf{i}} + a2\, \underline{\mathbf{j}} + a3\, \underline{\mathbf{k}}) \times (b1\, \underline{\mathbf{i}} + b2\, \underline{\mathbf{j}} + b3\, \underline{\mathbf{k}})$

- $\underline{\mathbf{a}} \times \underline{\mathbf{b}} =$

  $(a2b3 - a3b2)\underline{\mathbf{i}} + (a3b1 - a1b3)\underline{\mathbf{j}} + (a1b2 - a2b1)\underline{\mathbf{k}}$
- Cross product can also be written as multiplication by a matrix
- $\underline{\mathbf{a}} \times \underline{\mathbf{b}} = S\ \underline{\mathbf{b}}$

# Cross product as matrix multiplication

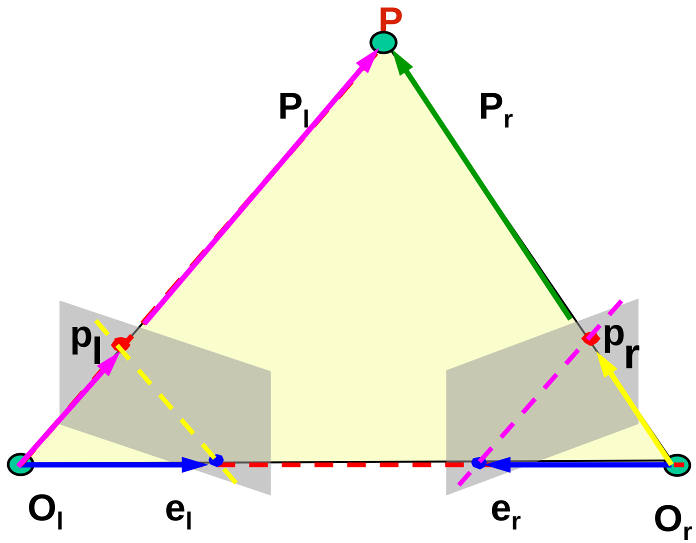
•Define matrix S as

$$\begin{bmatrix} 0 & -a3 & a2 \\ a3 & 0 & -a1 \\ -a2 & a1 & 0 \end{bmatrix}$$

•**a** x **b** = S **b**

- Try the program cross1.ch on the course web site

# Essential Matrix




$$T \times P_l = SP_l$$

$$S = \begin{bmatrix} 0 & -T_z & T_y \\ T_z & 0 & -T_x \\ -T_y & T_x & 0 \end{bmatrix}$$

Coordinate Transformation: $P_r = R(P_l - T)$

$T, P_l, P_l - T$ are coplanar $(P_l - T)^T T \times P_l = 0$

Resolves to

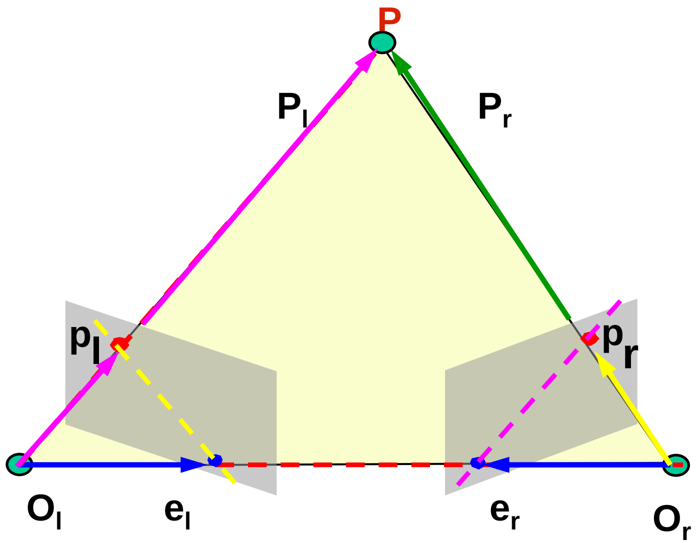
$$(R^T P_r)^T T \times P_l = 0$$

$$(R^T P_r)^T SP_l = 0$$

$$P_r^T RSP_l = 0$$

Essential Matrix $E = RS$ $P_r^T EP_l = 0$

# Essential Matrix




$$P_r^T E P_l = 0 \quad \Rightarrow \quad p_r^T E p_l = 0$$

$$\mathbf{P_r}^{\mathbf{T}}\mathbf{E}\mathbf{P_l} = 0$$

Big P are points
In 3d space

$$\mathbf{p}_l = \frac{f_l}{Z_l}\mathbf{P_l} \qquad \mathbf{p}_r = \frac{f_r}{Z_r}\mathbf{P}_r$$

Little p are points
In image plane
(camera co-ords)

$$\mathbf{p_r}^{\mathbf{T}}\mathbf{E}\mathbf{p_l} = 0$$

# Essential Matrix

## Essential Matrix E = RS

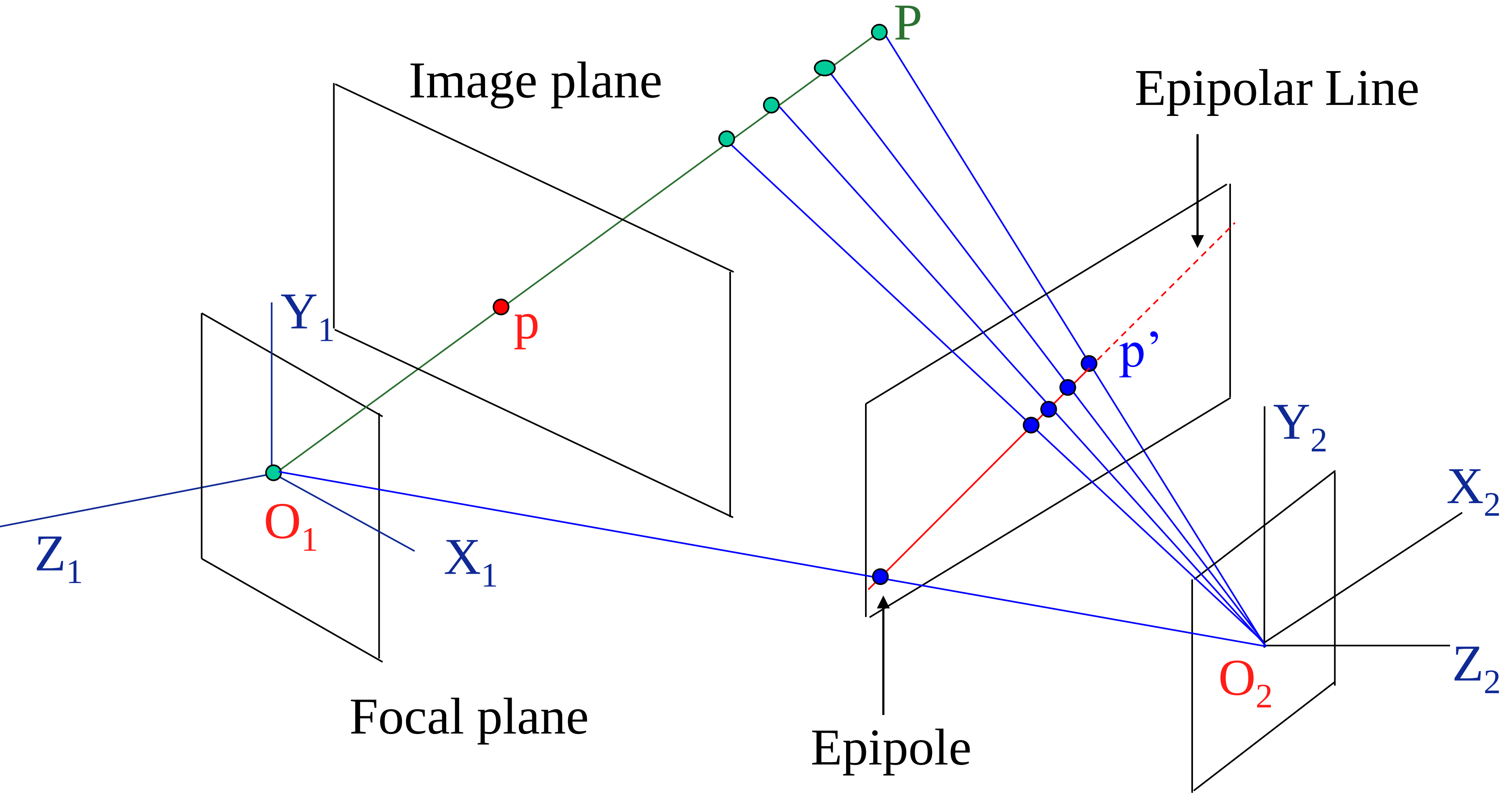
$$\mathbf{p_r}^{\mathbf{T}}\mathbf{E}\mathbf{p_l} = 0$$

- A natural link between the stereo point pair and the extrinsic parameters of the stereo system
  - One correspondence -> a linear equation of 9 entries
  - Given 8 pairs of (pl, pr) -> E
- Mapping between points and epipolar lines we are looking for
  - Given $p_l$, E -> $p_r$ on the projective line in the right plane
  - Equation represents the epipolar line of either pr (or pl) in the right (or left) image

## Note:

- pl, pr are in the camera coordinate system, not pixel coordinates that we can more easily measure
- Notation in next slide: p, p’ is equivalent to pr, and pl

# What does Essential Matrix Mean?



# Essential Matrix

## Essential Matrix $E = RS$

- 3x3 matrix constructed from R and T (extrinsic only)
  - Rank (E) = 2, two equal nonzero singular values

$$\mathbf{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \qquad S = \begin{bmatrix} 0 & -T_z & T_y \\ T_z & 0 & -T_x \\ -T_y & T_x & 0 \end{bmatrix}$$

**Rank (R) =3** **Rank (S) =2**

- E has five degrees of freedom (3 rotation, 2 translation)
- If we know R and T it is easy to compute E
  - use the camera calibration method of Ch. 6 for two cameras
- But if we just have correspondences between the two stereo cameras then computing E is not as simple to compute

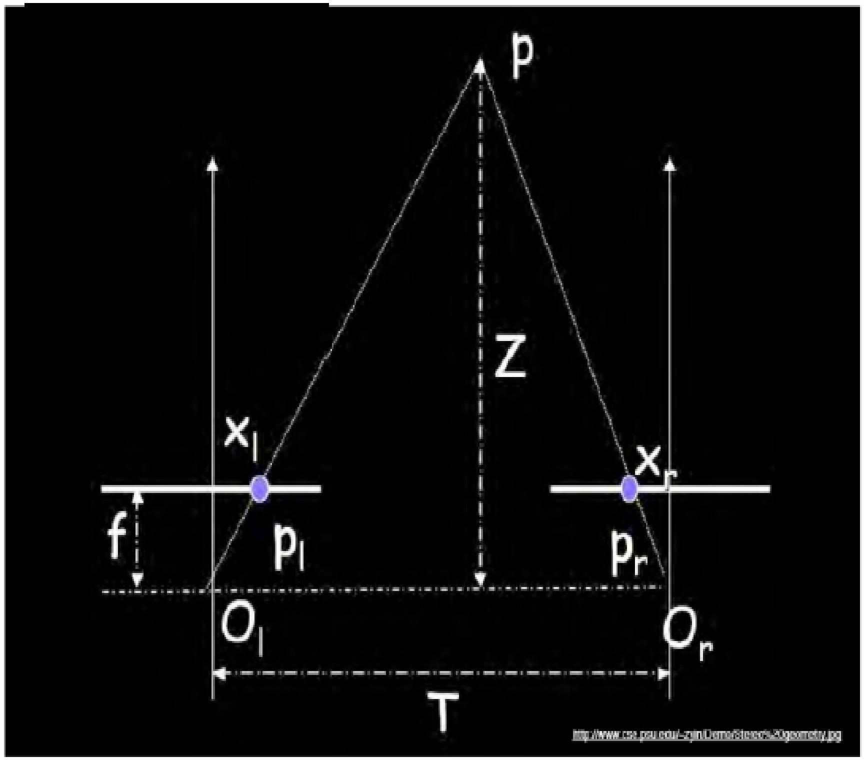
# Projective Geometry

- Projective Plane - $P^2$
  - Set of equivalence classes of triplets of real numbers
  - $p = [x,y,z]^T$ and $p' = [x',y',z']^T$ are equivalent if and only if there is a real number k such that $[x,y,z]^T = k [x',y',z']^T$
  - Each projective point p in $P^2$ corresponds to a line through the origin in $P^3$
  - So points in $P^2$, the projective plane, and lines in $P^3$, ordinary space, are in a one to one correspondence
  - A line in the projective plane is called a projective line represented by $u = [ux, uy, uz]^T$
  - Set of points p that satisfy the relation $u^T \bullet p = 0$
  - A projective line u can be represented by a 3d plane through the origin, that is the line defined by the equation $u^T \bullet p = 0$
  - p is either a point lying on the line u, or a line going through the point u

# Projective Line

- If we have a point in one image then this means the 3D point P is on the line from the origin through that point in the image plane
  - So in the other image the corresponding point must be on the epipolar line
- What is the meaning of $Ep_l$ ?
  - the line in the right plane that goes through $p_r$ and the epipole $e_r$
- Therefore the essential matrix is a mapping between points and epipolar lines
- $Ep_l$ defines the equation of the epipolar line in the right image for point $p_r$ in the right image
- $E^T p_r$ defines the equation of the epipolar line in the left image for point $p_l$ in the left image
- The relationship $\mathbf{p_r}^{\mathbf{T}}\mathbf{E}\mathbf{p_l} = 0$ maps a point to a line

# Essential Matrix for Parallel Cameras

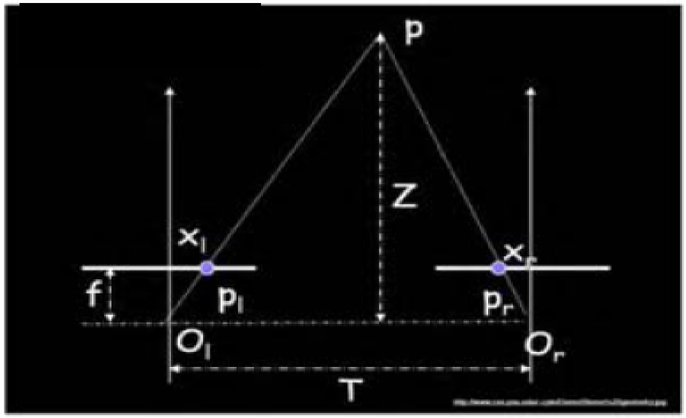



$$\mathbf{R} = \mathbf{I}$$

$$\mathbf{T} = [-T, 0, 0]^{\mathrm{T}}$$

$$\mathbf{E} = [\mathbf{T}_x]\mathbf{R} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & T \\ 0 & -T & 0 \end{pmatrix}$$

# Essential Matrix for Parallel Cameras




$$\mathbf{R} = \mathbf{I}$$

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$$\mathbf{E} = [\mathbf{T}_\mathbf{x}]\mathbf{R} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & T \\ 0 & -T & 0 \end{pmatrix}$$

$$\mathbf{p}^{\mathrm{T}}\mathbf{E}\mathbf{p}' = 0$$

$$\begin{bmatrix} x & y & f \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & T \\ 0 & -T & 0 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ f \end{bmatrix} = 0$$

$$\Leftrightarrow \begin{bmatrix} x & y & f \end{bmatrix} \begin{bmatrix} 0 \\ Tf \\ -Ty' \end{bmatrix} = 0$$

Image of any point must lie on same horizontal line in each image plane!

$$\Leftrightarrow y = y'$$

# Fundamental Matrix

Same as Essential Matrix but points are in pixel coordinates and not camera coordinates

$$p_r^T E p_l = 0$$

$$\Downarrow$$

$$\bar{p}_r^T F \, \bar{p}_l = 0$$

Pixel coordinates

$$F = K_r^{-T} E K_l^{-1}$$

Intrinsic parameters

Book uses M and not K

# Fundamental Matrix

Mapping between points and epipolar lines in the pixel coordinate systems

- With no prior knowledge of the stereo system parameters

From Camera to Pixels: Matrices of intrinsic parameters (M matrix in book)

$$\mathrm{K}_{int} = \begin{bmatrix} -f_x & 0 & o_x \\ 0 & -f_y & o_y \\ 0 & 0 & 1 \end{bmatrix}$$

**Rank ($\mathbf{K}_{int}$) =3**

$$\mathbf{p_r}^{\mathbf{T}}\mathbf{E}\mathbf{p_l} = 0$$

$$\mathbf{p_l} = \mathrm{K}_\mathbf{l}^{-\mathbf{1}}\overline{\mathbf{p}}_\mathbf{l}$$

$$\mathbf{p}_r = \mathrm{K}_r^{-\mathbf{1}}\overline{\mathbf{p}}_r$$

$$\overline{\mathbf{p}}_\mathbf{r}^{\mathbf{T}}\mathbf{F}\overline{\mathbf{p}}_\mathbf{l} = 0$$

$$\mathbf{F} = \mathrm{K}_r^{-\mathbf{T}}\mathbf{E}\mathrm{K}_l^{-1}$$

# Fundamental Matrix

## Fundamental Matrix

$$\mathbf{F} = \mathrm{K}_r^{-\mathbf{T}} \mathbf{E} \mathrm{K}_l^{-1}$$

- Rank (F) = 2
- Encodes info on both intrinsic and extrinsic parameters

- Enables full reconstruction of the epipolar geometry
- In pixel coordinate systems without any knowledge of the intrinsic and extrinsic parameters
- Linear equation of the 9 entries of F but only 8 degrees of freedom because of homogeneous nature of equations

$$\overline{\mathbf{p}}_{\mathbf{r}}^{\mathbf{T}} \mathbf{F} \overline{\mathbf{p}}_{\mathbf{l}} = 0 \Rightarrow (x_{im}^{(l)} \quad y_{im}^{(l)} \quad 1) \begin{bmatrix} f11 & f12 & f13 \\ f21 & f22 & f23 \\ f31 & f32 & f33 \end{bmatrix} \begin{pmatrix} x_{im}^{(r)} \\ y_{im}^{(r)} \\ 1 \end{pmatrix} = 0$$

# Computing E and F

- If we known extrinsic parameters of stereo system we can compute E = R S
- Using E and intrinsic parameters we can compute epipolar lines
- If we do not know intrinsic and extrinsic parameters can we compute F?
- Yes, for at least 8 correspondences
  - Choose 8 correct correspondences manually
- Then compute F matrix
  - Then use F to guide search by computing the epipolar lines
  - This will simply the process of finding correspondences

# Computing F: The Eight-point Algorithm

Input: n point correspondences ( n >= 8)

- Construct homogeneous system Ax= 0 from $\bar{\mathbf{p}}_{\mathbf{r}}^{\mathbf{T}} \mathbf{F} \bar{\mathbf{p}}_{\mathbf{l}} = 0$
  - x = ($f_{11}$,$f_{12}$, ,$f_{13}$, $f_{21}$,$f_{22}$,$f_{23}$ $f_{31}$,$f_{32}$, $f_{33}$) : entries in F
  - Each correspondence give one equation
  - A is a nx9 matrix
- Obtain estimate F^ by Eigenvector with smallest eigenvalue
  - x (up to a scale) is column of V corresponding to the least singular value
- Enforce singularity constraint: since Rank (F) = 2
  - Compute SVD of F^ $\hat{\mathbf{F}} = \mathbf{U}\mathbf{D}\mathbf{V}^T$
  - Set the smallest singular value to 0: D -> D'
  - Correct estimate of F : $\mathbf{F'} = \mathbf{U}\mathbf{D'}\mathbf{V}^T$

Output: the fundamental matrix, F'

can then compute E given intrinsic parameters

# Estimating Fundamental Matrix

The 8-point algorithm

$$u^T F u' = 0$$

Each point correspondence can be expressed as a linear equation

$$\begin{bmatrix} u & v & 1 \end{bmatrix} \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{bmatrix} \begin{bmatrix} u' \\ v' \\ 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} uu' & uv' & u & u'v & vv' & v & u' & v' & 1 \end{bmatrix} \begin{bmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \\ F_{33} \end{bmatrix} = 0$$

# Homogeneous System

- M linear equations of form A**x** = 0
- If we have a given solution x1, s.t. Ax1 = 0 then c * x1 is also a solution A(c* x1) = 0
- Need to add a constraint on **x,**
  - Basically make **x** a unit vector $\mathrm{x}^{T}\mathrm{x} = 1$
- Can prove that the solution is the eigenvector correponding to the single zero eigenvalue of that matrix $\mathrm{A}^{T}\mathrm{A}$
  - This can be computed using eigenvector routine
  - Then finding the zero eigenvalue
  - Returning the associated eigenvector

# Singular Value Decomposition

- Any m by n matrix A can be written as product of three matrices $A = UDV^T$
- The columns of the m by m matrix U are mutually orthogonal unit vectors, as are the columns of the n by n matrix V
- The m by n matrix D is diagonal, and the diagonal elements, $\sigma_i$ are called the singular values
- It is the case that $\sigma_1 \geq \sigma_2 \geq \ldots \sigma_n \geq 0$
- Singular values squared are eigenvlaues (square mat.)
- The rank of a square matrix is the number of linearly independent rows or columns
- For a square matrix (m = n) then the number of non-zero singular values equals the rank of the matrix

# Essential/Fundamental Matrix

- Essential and fundamental matrix differ
- Relate different quantities
  - Essential matrix is defined in terms of camera co-ordinates
  - Fundamental matrix defined in terms of pixel co-ordinates
- Need different things to calculate them
  - Essential matrix requires camera calibration and knowledge of correspondences
    - known intrinsic parameters, unknown extrinsic parameters
  - Fundamental matrix does not require any camera calibration, just knowledge of correspondences
    - Unknown intrinsic and unknown extrinsic
- Essential and fundamental matrix are related by the camera calibration parameters

# Essential/Fundamental matrix

- Essential matrix E = RS
  - Encodes information on the extrinsic parameters only
  - Has rank 2, since S has rank 2, and R is full rank
  - This means one singular value (eigenvalue) is zero
  - Its two non-zero singular values (eigenvalues) are equal
- Fundamental matrix $F = K_r^{-T} E K_l^{-1}$
  - Encodes information on both intrinsic and extrinsic params
  - Has rank 2, since Kr and Kl have full rank
  - This means one singular value (eigenvalue) is zero
  - Its two non-zero singular values (eigenvalues) are not necessarily equal

# Essential/Fundamental Matrix

- We compute the fundamental matrix from the 2d pixel co-ordinates of correspondences between the left and right image
- If we have the fundamental matrix it is possible to compute the essential matrix if we know the camera calibration
- But we can still compute the epipolar lines using the fundamental matrix
- Therefore if we have the fundamental matrix this limits correspondence search to 1D search for general stereo camera positions in same way as for simple stereo

# Locating the Epipoles from F

$$\bar{\mathbf{p}}_{\mathbf{r}}{}^{\mathbf{T}}\mathbf{F}\bar{\mathbf{p}}_{\mathbf{l}} = 0$$

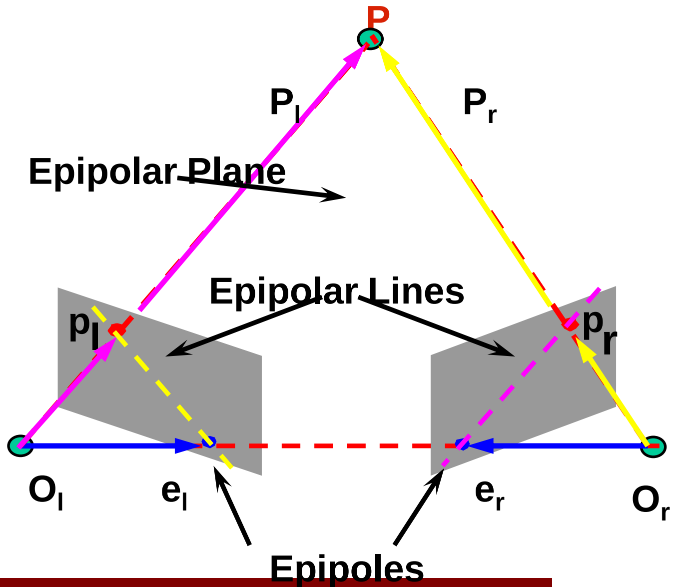
$e_l$ lies on all the epipolar lines of the left image

$$\bar{\mathbf{p}}_{\mathbf{r}}{}^{\mathbf{T}}\mathbf{F}\bar{\mathbf{e}}_{\mathbf{l}} = 0$$

**For every $p_r$**

F is not identically zero

$$\mathbf{F}\bar{\mathbf{e}}_{\mathbf{l}} = 0$$




Input: Fundamental Matrix F

$$\mathbf{F} = \mathbf{U}\mathbf{D}\mathbf{V}^T$$

- Find the SVD of F
- The epipole $e_l$ is the column of V corresponding to the null singular value (as shown above)
- The epipole $e_r$ is the column of U corresponding to the null singular value

Output: Epipole $e_l$ and $e_r$

# Fundamental and Essential matrix

Both F and E are 3 x 3 matrices, facts below apply to both

**Transpose :** If F is a fundamental matrix of cameras (P, P') then $F^T$ is a fundamental matrix of camera (P',P)

**Epipolar lines:** Think of *p* and *p'* as points in the projective plane then *F p* is projective line in the right image.

That is $l' = F\,p \qquad l = F^T p'$

**Epipole:** Since for any p the epipolar line $l' = F\,p$ contains the epipole *e'*. Thus $(e'^T F)\,p = 0$ for a all *p* .
Thus $e'^T F = 0$ and $F\,e = 0$

# Epipolar Geometry

- Basic constraint used to help correspondence
- Makes search for matching points into a 1D search along epipolar lines
- If you have intrinsic and extrinsic parameters
  - Then compute essential matrix and find epipolar lines
- If you do not have intrinsic or extrinsic parameters but have at least 8 correct correspondences then
  - Compute fundamental matrix and find epipolar lines
- Can also compute the epipoles using SVD