Ellipse Fitting

COMP 4900D

Winter 2011

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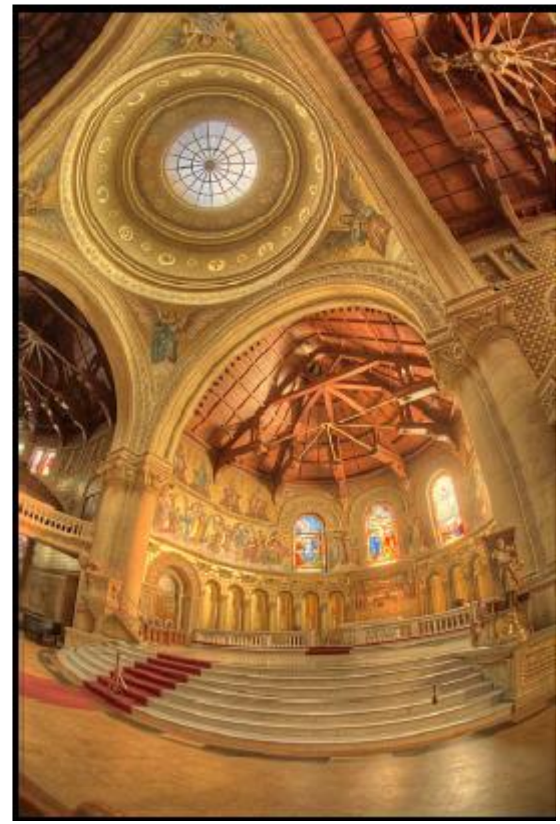
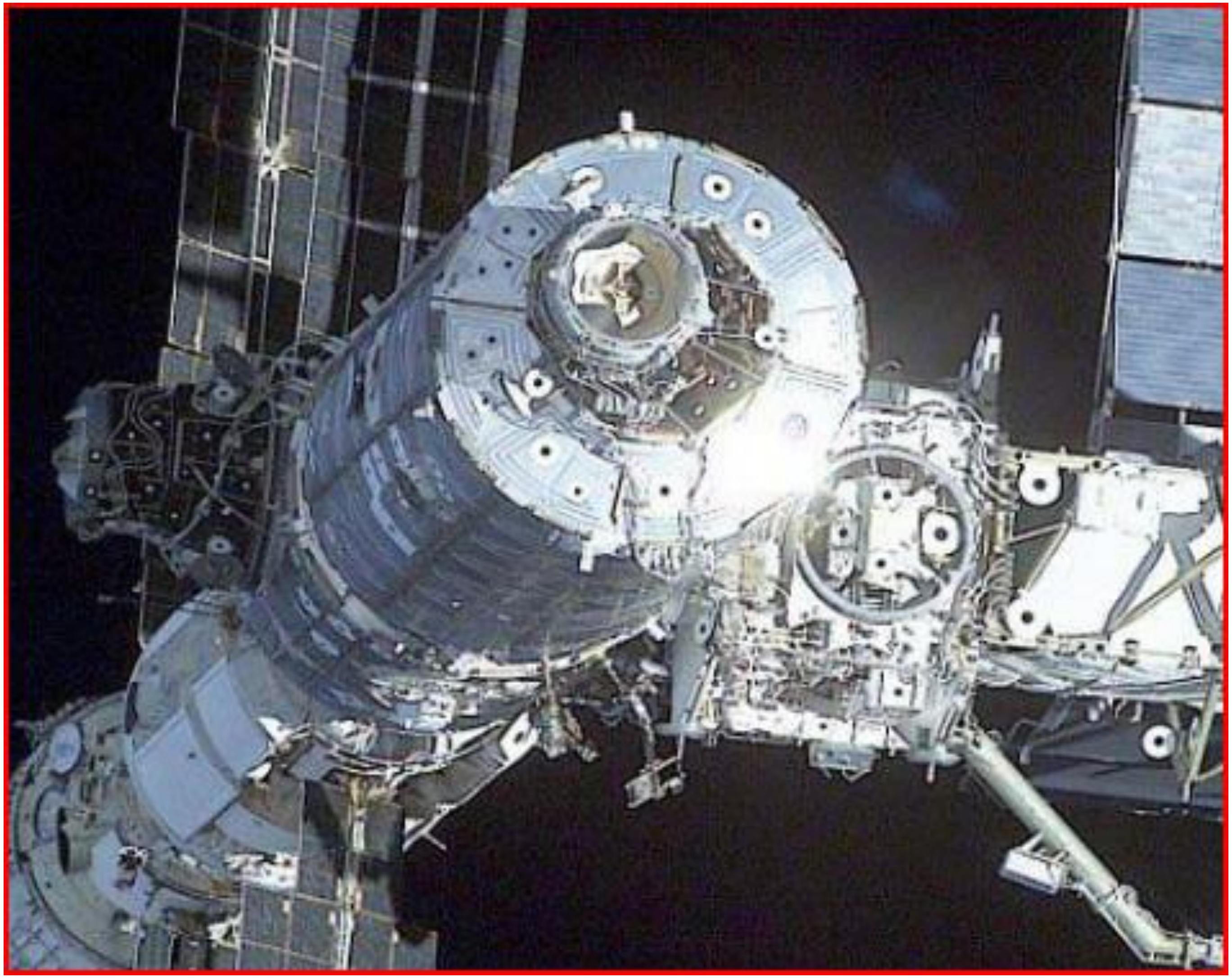
Ellipses in Images

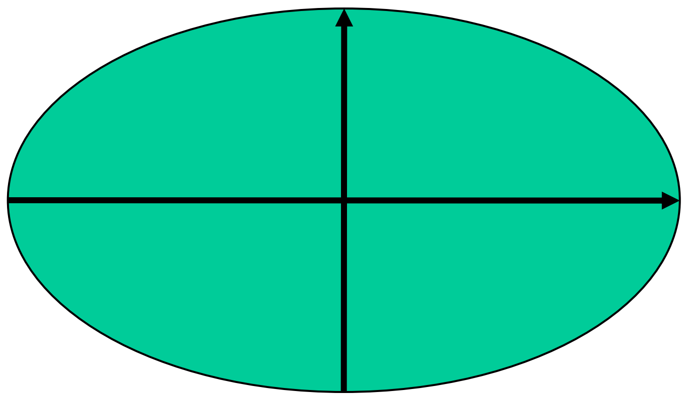


Perspective projection of circles form ellipses in the images.

Ellipses in Images

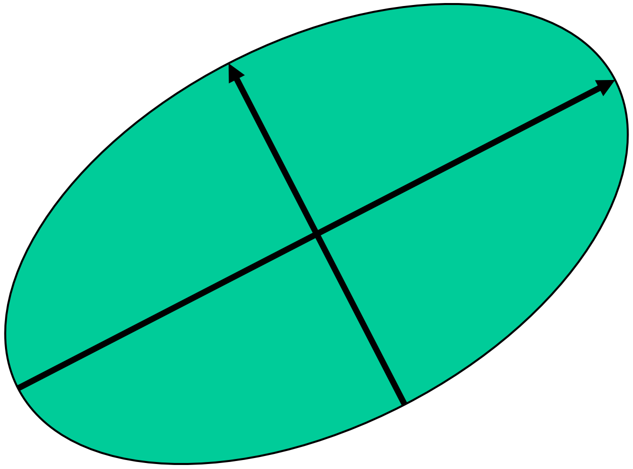


Equations of Ellipse



$$\frac{x^2}{r_1^2} + \frac{y^2}{r_2^2} = 1$$

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

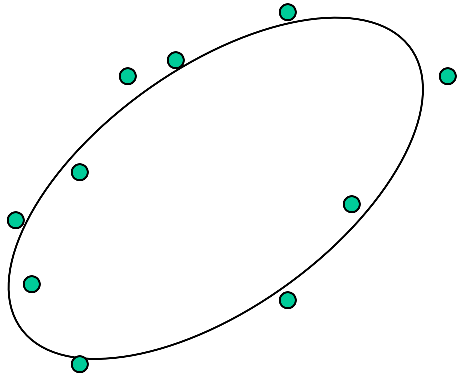


Let $\mathbf{x} = [x^2, xy, y^2, x, y, 1]^T$

$\mathbf{a} = [a, b, c, d, e, f]^T$

Then $\mathbf{x}^T \mathbf{a} = 0$

Ellipse Fitting: Problem Statement



Given a set of N image points $\mathbf{p}_i = [x_i, y_i]^T$
find the parameter vector $\mathbf{a_0}$ such that the ellipse

$$f(\mathbf{p}, \mathbf{a}) = \mathbf{x}^T \mathbf{a} = ax^2 + bxy + cy^2 + dx + ey + f = 0$$

fits $\mathbf{p}_i$ best in the least square sense:

$$\min_{\mathbf{a}} \sum_{i=1}^{N} \left[D(\mathbf{p}_i, \mathbf{a}) \right]^2$$

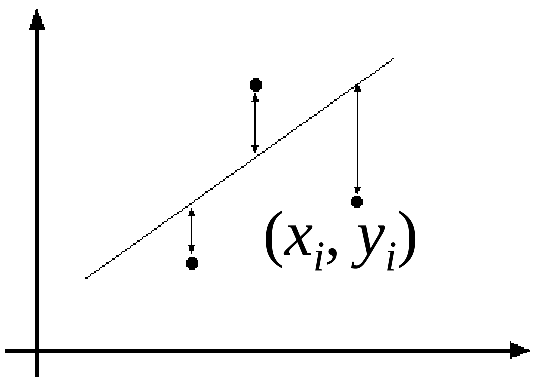
Where $D(\mathbf{p}_i, \mathbf{a})$ is the distance from $\mathbf{p}_i$ to the ellipse.

Ellipse Fitting: Geometric or Algebraic

- Remember that $D(\mathbf{p}_i, \mathbf{a})$ is the distance from point $\mathbf{p}_i$ to the ellipse
- There are two ways to calculate this distance
- First uses the true geometric distance
  - Is most accurate but is more difficult to calculate
  - In this case the ellipse fitting algorithm is iterative
  - But it gets the best possible result
- Second uses the algebraic distance
  - Is less accurate but is easier to calculate
  - In this the ellipse fitting algorithm has only one iteration
  - But does not get best possible result (it is reasonable)

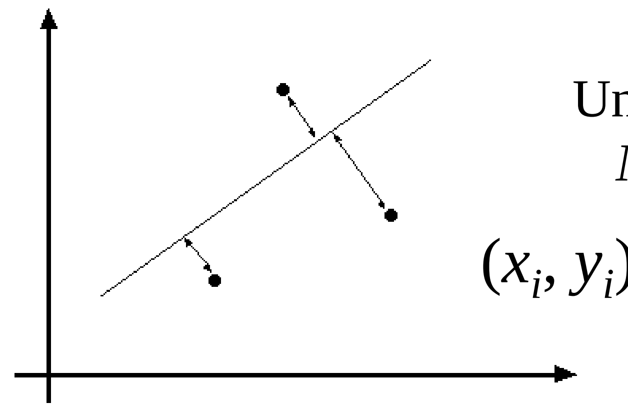
Algebraic Distance/Geometric Distance

Algebraic Distance for a line



$y=mx+b$

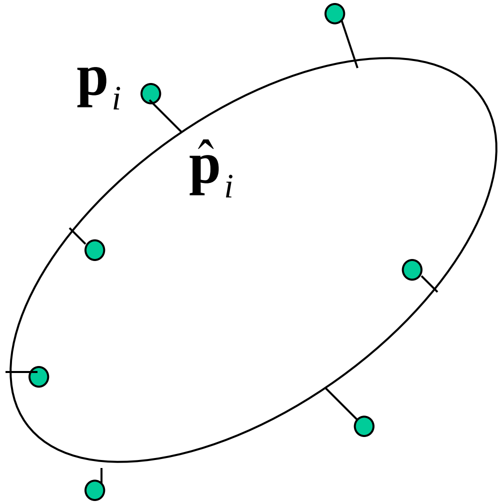
Geometric Distance for a line



Unit normal:
$N=(a, b)$

$ax+by=d$

Euclidean Distance Fit



$$D(\mathbf{p}_i, \mathbf{a}) = \left\| \hat{\mathbf{p}}_i - \mathbf{p}_i \right\|$$

$\hat{\mathbf{p}}_i$ is the point on the ellipse that is nearest to $\mathbf{p}_i$

$$f(\hat{\mathbf{p}}_i, \mathbf{a}) = 0$$

$\hat{\mathbf{p}}_i - \mathbf{p}_i$ is normal to the ellipse at $\hat{\mathbf{p}}_i$

Compute Geometric Distance Function

Computing the distance function is a constrained optimization problem:

$$\min_{\hat{\mathbf{p}}_i} \|\hat{\mathbf{p}}_i - \mathbf{p}_i\|^2 \qquad \text{subject to} \qquad f(\hat{\mathbf{p}}_i, \mathbf{a}) = 0$$

Using Lagrange multiplier, define:

$$L(x, y, \lambda) = \|\hat{\mathbf{p}}_i - \mathbf{p}_i\|^2 - 2\lambda f(\hat{\mathbf{p}}_i, \mathbf{a})$$

where $\hat{\mathbf{p}}_i = [x, y]^T$

Then the problem becomes: $\min_{\hat{\mathbf{p}}_i} L(x, y, \lambda)$

Set $\frac{\partial L}{\partial x} = \frac{\partial L}{\partial y} = 0$ we have $\hat{\mathbf{p}}_i - \mathbf{p}_i = \lambda \nabla f(\hat{\mathbf{p}}_i, \mathbf{a})$

Two Approximations

1. First-order approximation at $\mathbf{p}_i$

$$f(\hat{\mathbf{p}}_i, \mathbf{a}) \approx f(\mathbf{p}_i, \mathbf{a}) + (\hat{\mathbf{p}}_i - \mathbf{p}_i)^T \nabla f(\mathbf{p}_i, \mathbf{a}) = 0$$

2. Assume $\mathbf{p}_i$ is close to $\hat{\mathbf{p}}_i$, then

$$\nabla f(\hat{\mathbf{p}}_i, \mathbf{a}) \approx \nabla f(\mathbf{p}_i, \mathbf{a})$$

Approximate Distance Function

$$f(\hat{\mathbf{p}}_i, \mathbf{a}) \approx f(\mathbf{p}_i, \mathbf{a}) + (\hat{\mathbf{p}}_i - \mathbf{p}_i)^T \nabla f(\mathbf{p}_i, \mathbf{a}) = 0$$

$$\hat{\mathbf{p}}_i - \mathbf{p}_i = \lambda \nabla f(\hat{\mathbf{p}}_i, \mathbf{a}) \approx \lambda \nabla f(\mathbf{p}_i, \mathbf{a})$$

Solve for λ

$$\lambda = -\frac{f(\mathbf{p}_i, \mathbf{a})}{\left\| \nabla f(\mathbf{p}_i, \mathbf{a}) \right\|^2}$$

Substitute back

$$\hat{\mathbf{p}}_i - \mathbf{p}_i = -\frac{f(\mathbf{p}_i, \mathbf{a}) \nabla f(\mathbf{p}_i, \mathbf{a})}{\left\| \nabla f(\mathbf{p}_i, \mathbf{a}) \right\|^2}$$

$$D(\mathbf{p}_i, \mathbf{a}) = \left\| \hat{\mathbf{p}}_i - \mathbf{p}_i \right\| = \frac{\left| f(\mathbf{p}_i, \mathbf{a}) \right|}{\left\| \nabla f(\mathbf{p}_i, \mathbf{a}) \right\|}$$

Ellipse Fitting with Euclidean Distance

Given a set of N image points $\mathbf{p}_i = [x_i, y_i]^T$
find the parameter vector $\mathbf{a_0}$ such that

$$\min_{\mathbf{a}} \sum_{i=1}^{N} \frac{\left| f(\mathbf{p}_i, \mathbf{a}) \right|^2}{\left\| \nabla f(\mathbf{p}_i, \mathbf{a}) \right\|^2}$$

This problem can be solved by using a numerical nonlinear optimization system.

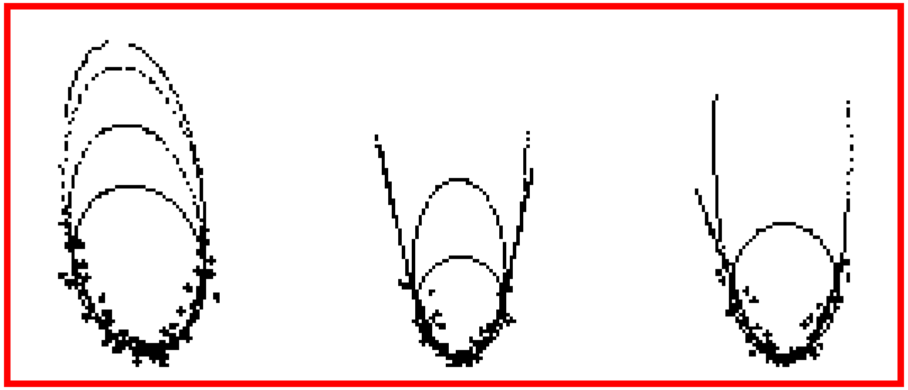
Ellipse Fitting with Algebraic Distance

- The algebraic distance from a point $\mathbf{p}_i$ to a curve defined by $f(\mathbf{p}_i, \mathbf{a}) = 0$ is $\left| f(\mathbf{p}_i, \mathbf{a}) \right|$
- This is simply putting point $\mathbf{p}_i$ into the function
- However, algebraic distance is not the true geometric distance, only an approximation
- We minimize the function below but to avoid the trivial solution, **a** = 0, we constrain **a** so that $b^2 - 4ac = 0$

$$\min_{\mathbf{a}} \sum_{i=1}^{N} \left| \left(x_i^T a \right) \right|^2$$

Algebraic Fitting of Ellipse

- With this constraint the fitting becomes an eigenvector problem (easy to solve)
- Works directly, in one iteration (real-time)
- But has a bias towards low eccentricity
  - Algebraic ellipse is more “circle like” than true ellipse



Ellipse Fitting

- Given a set of 2d points fit the “best” ellipse
- Can use the algebraic or geometric approach
  - Algebraic method is more commonly used
  - Always produces an ellipse, and is fast
  - Result not perfect but more than good enough
- Issue of geometric versus algebraic is a common problem in all fitting
  - And is more complex for higher order curves and surfaces
- OpenCV has an implementation of algebraic fitting method for ellipses
  - Demo program ellipsefit shows this computation in action