Geometric Model of Camera

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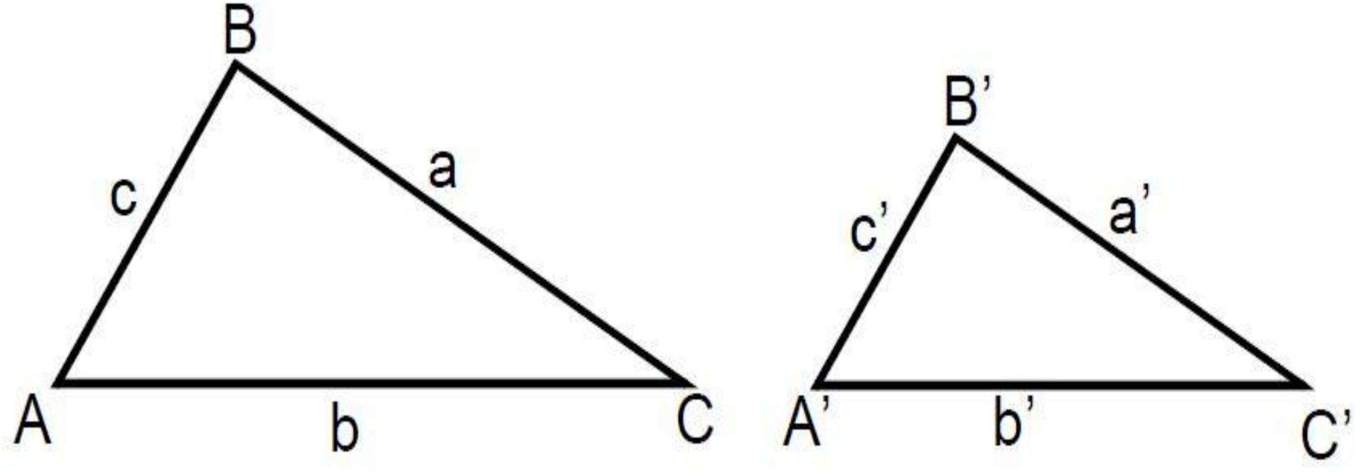
Similar Triangles



property (i): corresponding angles are equal

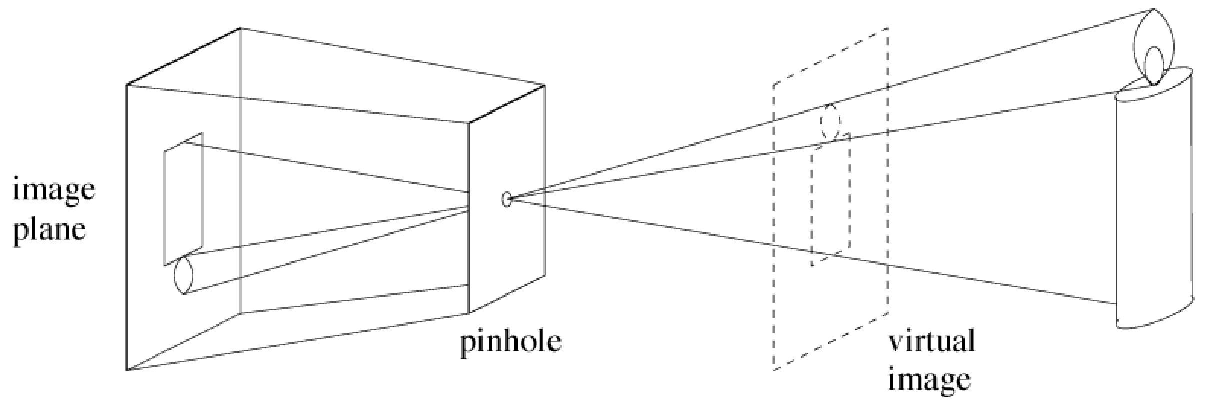
(A = A' and B = B' and C = C')

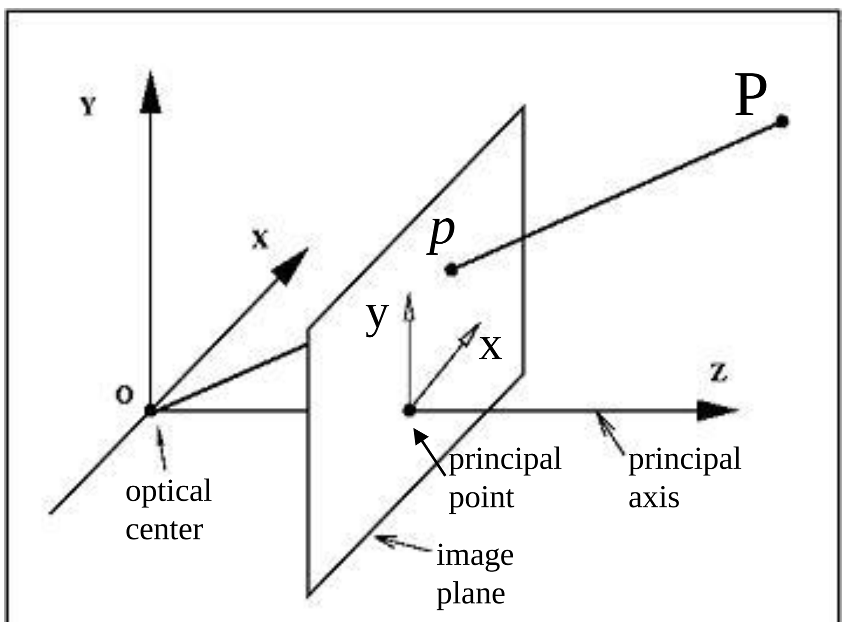
property (ii): corresponding sides have proportional lengths

$$\left(\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'} \right)$$

Geometric Model of Camera

Perspective projection





$$P(X,Y,Z) \longrightarrow p(x,y)$$

$$x = f\,\frac{X}{Z} \qquad y = f\,\frac{Y}{Z}$$

Parallel lines aren't…

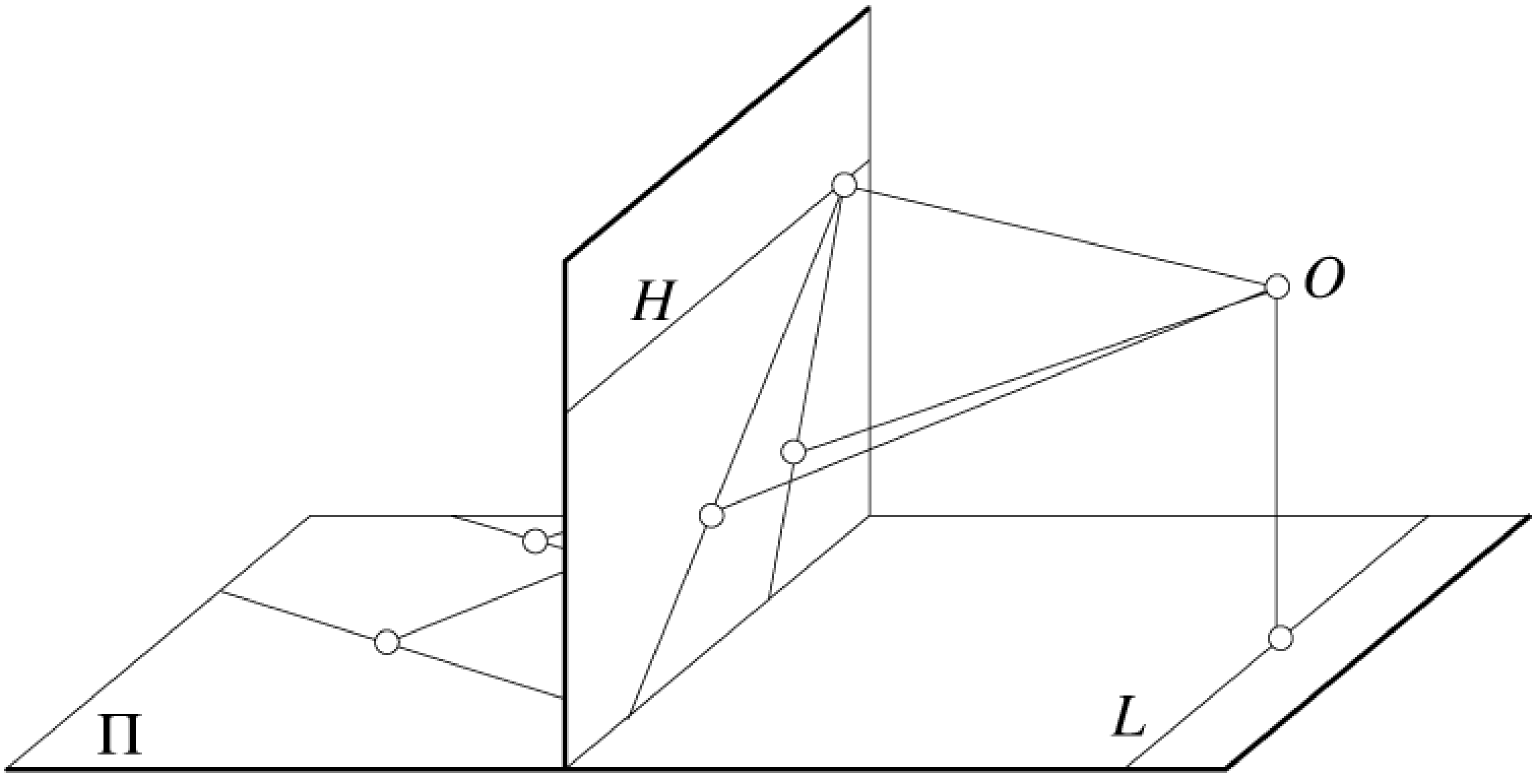
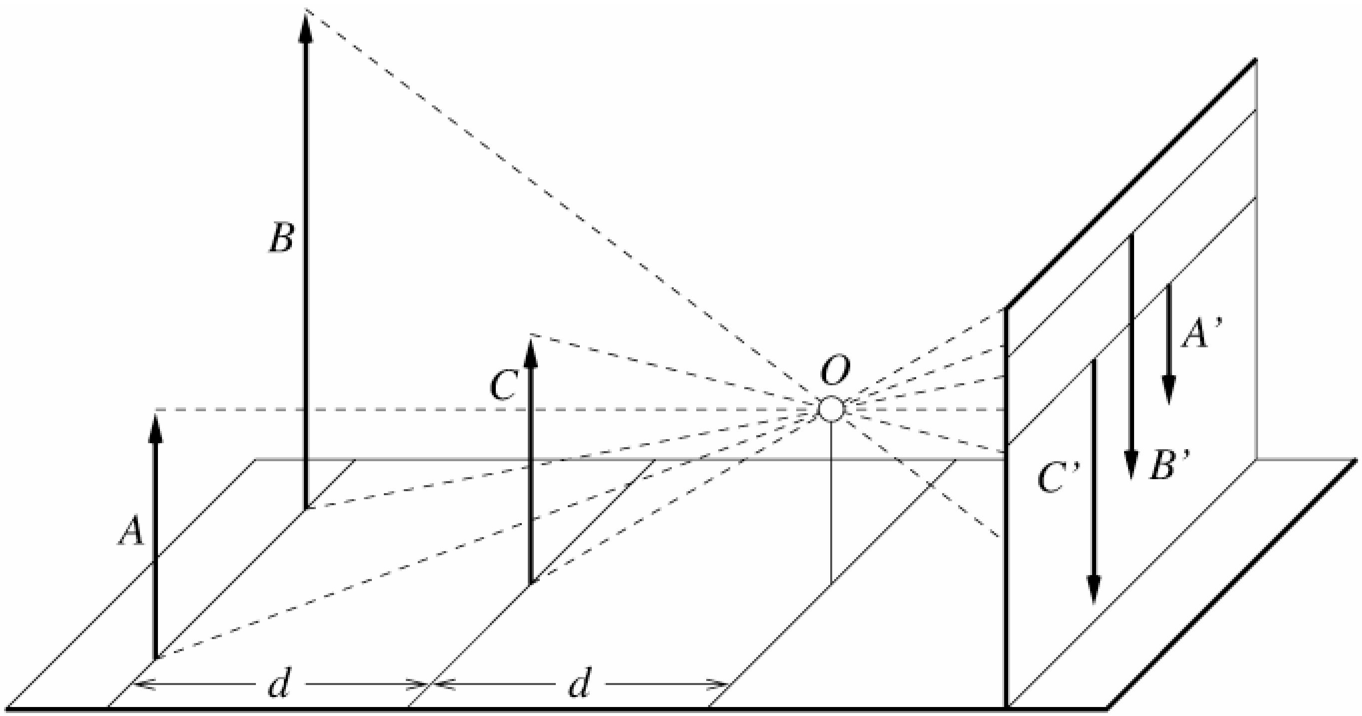
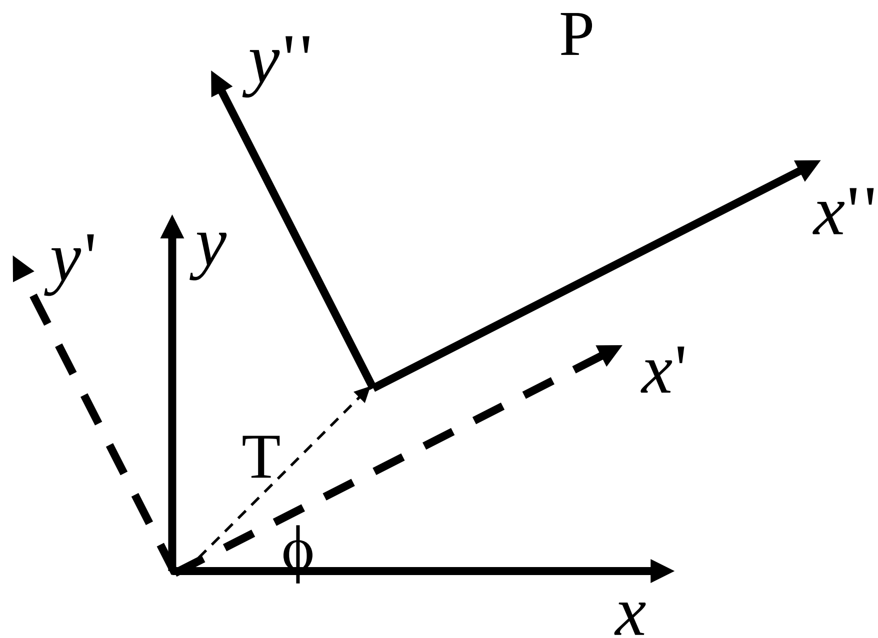


Figure by David Forsyth

Lengths can’t be trusted...



Coordinate Transformation – 2D



Rotation and Translation

$$p' = \begin{bmatrix} p_x' \\ p_y' \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} p_x \\ p_y \end{bmatrix}$$

$$p'' = \begin{bmatrix} p_x' \\ p_y' \end{bmatrix} + \begin{bmatrix} T_x \\ T_y \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} p_x \\ p_y \end{bmatrix} + \begin{bmatrix} T_x \\ T_y \end{bmatrix}$$

Homogeneous Coordinates

Go one dimensional higher:

$$\begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} wx \\ wy \\ w \end{bmatrix} \qquad \begin{bmatrix} x \\ y \\ z \end{bmatrix} \rightarrow \begin{bmatrix} wx \\ wy \\ wz \\ w \end{bmatrix}$$

w is an arbitrary non-zero scalar, usually we choose 1.

From homogeneous coordinates to Cartesian coordinates:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \rightarrow \begin{bmatrix} x_1 / x_3 \\ x_2 / x_3 \end{bmatrix} \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \rightarrow \begin{bmatrix} x_1 / x_4 \\ x_2 / x_4 \\ x_3 / x_4 \end{bmatrix}$$

2D Transformation with Homogeneous Coordinates

2D coordinate transformation:

$$p'' = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} p_x \\ p_y \end{bmatrix} + \begin{bmatrix} T_x \\ T_y \end{bmatrix}$$

2D coordinate transformation using homogeneous coordinates:

$$\begin{bmatrix} p_x'' \\ p_y'' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & T_x \\ -\sin\phi & \cos\phi & T_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ 1 \end{bmatrix}$$

3D Rotation Matrix

Rotate around each coordinate axis:

$$R_1(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix} \quad R_2(\beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix} \quad R_3(\gamma) = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Combine the three rotations:

$$R = R_1 R_2 R_3$$

3D rotation matrix has three parameters,
no matter how it is specified.

Rotation Matrices

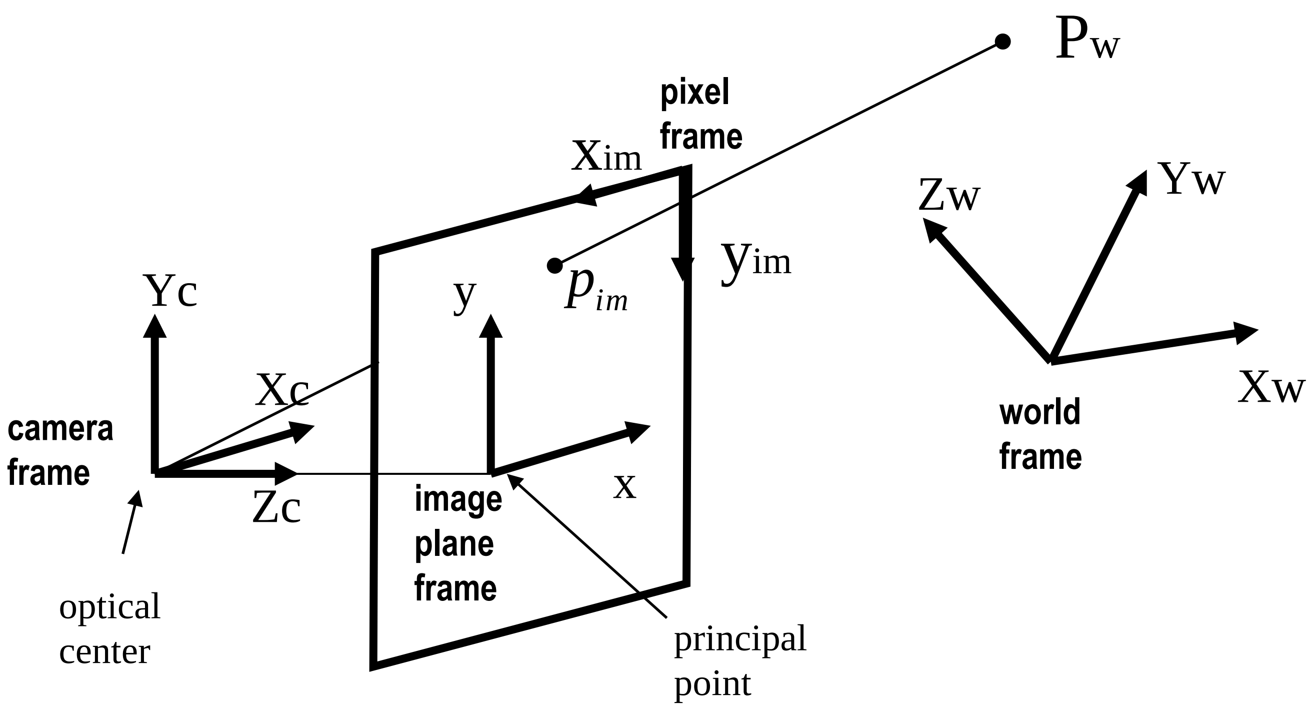
- Both 2d and 3d rotation matrices have two characteristics
- They are orthogonal (also called orthonormal)

$$R^T R = I \qquad\qquad R^T = R^{-1}$$

- Their determinant is 1
- Matrix below is orthogonal but not a rotation matrix because the determinate is not 1

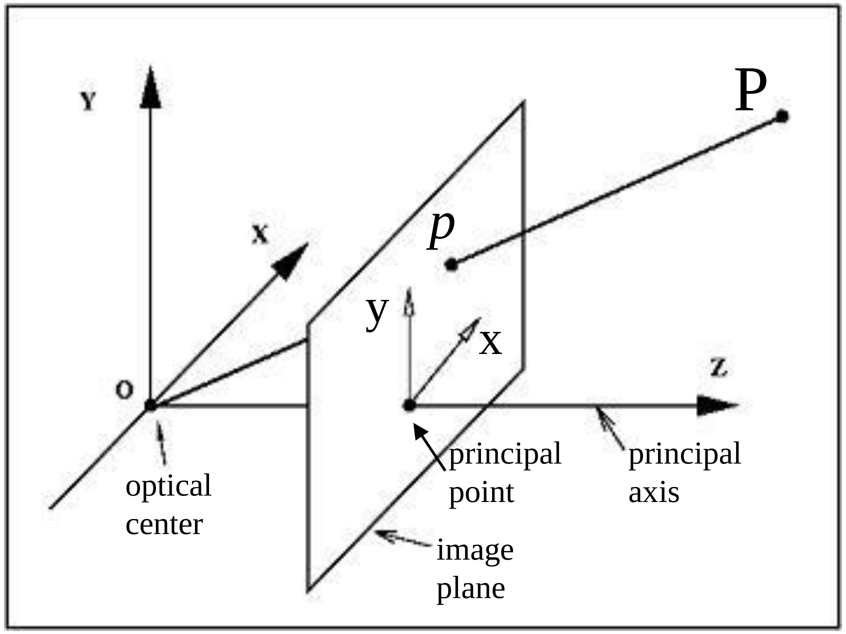
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \longleftarrow \text{this is a reflection matrix}$$

Four Coordinate Frames



Camera model: $$p_{im} = \begin{bmatrix} transformation \\ matrix \end{bmatrix} P_w$$

Perspective Projection



$$x = f\frac{X}{Z} \quad y = f\frac{Y}{Z}$$

These are *nonlinear*.

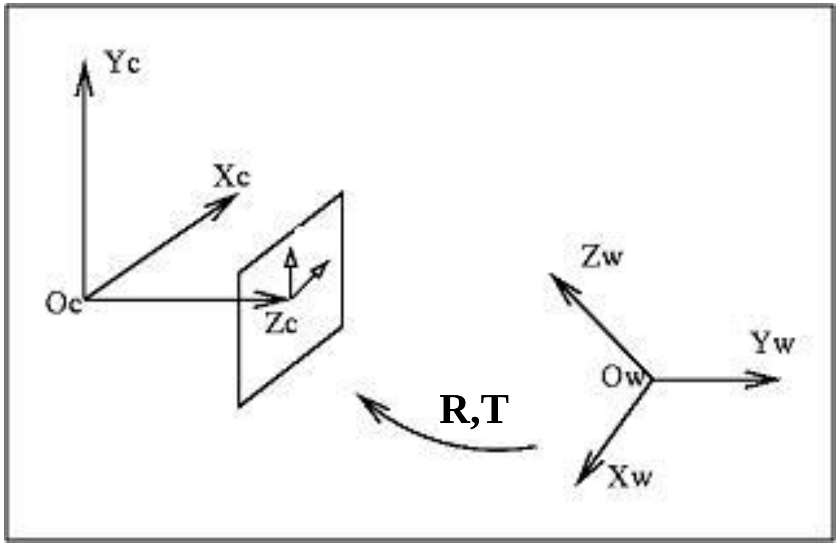
Using homogenous coordinate, we have a *linear* relation:

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

$$x = u / w \qquad y = v / w$$

World to Camera Coordinate

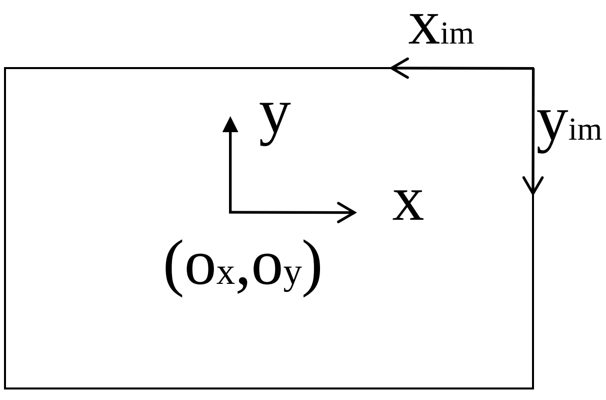
Transformation between the camera and world coordinates:



$$\mathbf{X}_c = \mathbf{R}\mathbf{X}_w + \mathbf{T}$$

$$\begin{bmatrix} X_c \\ Y_c \\ Z_c \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{R} & \mathbf{T} \\ \mathbf{0}^{\mathrm{T}} & 1 \end{bmatrix} \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix}$$

Camera Coordinates to Pixel Coordinates



$$x = (o_x - x_{im})s_x \quad y = (o_y - y_{im})s_y$$

s_x, s_y : pixel sizes in millimeters per pixel

$$\begin{bmatrix} x_{im} \\ y_{im} \\ 1 \end{bmatrix} = \begin{bmatrix} -1/s_x & 0 & o_x \\ 0 & -1/s_y & o_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Camera co-ordinates *x*, and *y* are in millimetres

Image co-ordinates x_{im}, y_{im}, are in pixels

Center of projection o_x, o_y is in pixels

Put All Together – World to Pixel

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1/s_x & 0 & o_x \\ 0 & -1/s_y & o_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

$$= \begin{bmatrix} -1/s_x & 0 & o_x \\ 0 & -1/s_y & o_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} X_c \\ Y_c \\ Z_c \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1/s_x & 0 & o_x \\ 0 & -1/s_y & o_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} R & T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -f/s_x & 0 & o_x \\ 0 & -f/s_y & o_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} R & T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix} = K \begin{bmatrix} R & T \end{bmatrix} \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix}$$

$$x_{im} = x_1 / x_3 \quad y_{im} = x_2 / x_3$$

Camera Parameters

- Extrinsic parameters define the location and orientation of the camera reference frame with respect to a world reference frame
  - Depend on the external world, so they are extrinsic
- Intrinsic parameters link pixel co-ordinates in the image with the corresponding co-ordinates in the camera reference frame
  - An intrinsic characteristic of the camera
- Image co-ordinates are in pixels
- Camera co-ordinates are in millimetres

Intrinsic Camera Parameters

$$K = \begin{bmatrix} -f/s_x & 0 & o_x \\ 0 & -f/s_y & o_y \\ 0 & 0 & 1 \end{bmatrix}$$

K is a 3x3 upper triangular matrix, called the **Camera Calibration Matrix**.

There are five intrinsic parameters:

(a) The pixel sizes in x and y directions s_x, s_y in millimeters/pixel

(b) The focal length f in millimeters

(c) The principal point (o_x,o_y) in pixels, which is the point where the optic axis intersects the image plane.

Camera intrinsic parameters

- Can write three of these parameters differently by letting f/sx = fx and f/sy = fy
  - Then intrinsic parameters are ox,oy,fx,fy
  - The units of these parameters are pixels!
- In practice pixels are square (sx = sy) so that means fx should equal fy for most cameras
  - However, every explicit camera calibration process (using calibration objects) introduces some small errors
  - These calibration errors make fx not exactly equal to fy
- So in OpenCV the intrinsic camera parameters are the four following ox,oy,fx,fy
  - However fx is usually very close to fy and if this is not the case then there is a problem

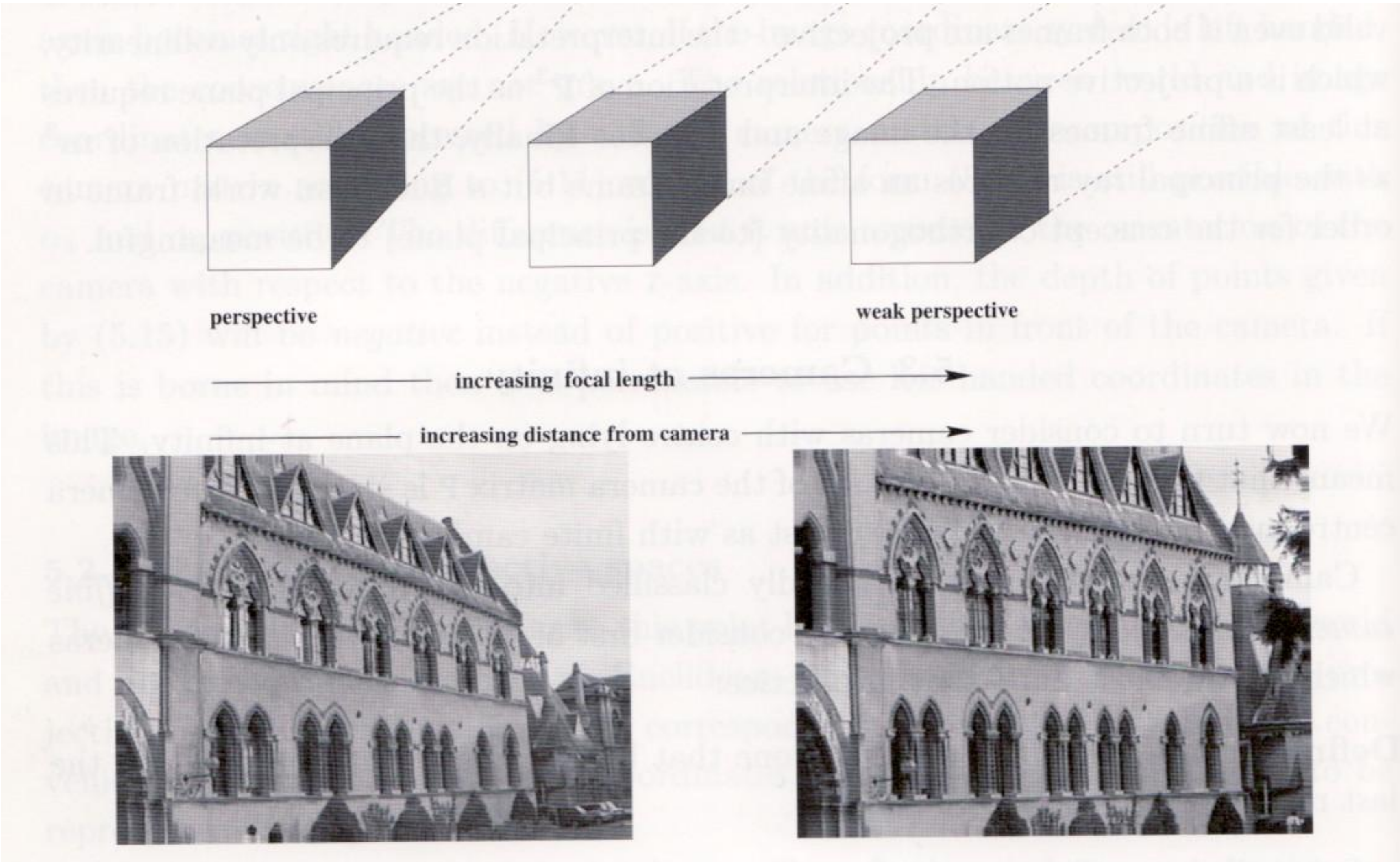
Extrinsic Parameters

$$p_{im} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = K\begin{bmatrix} R & T \end{bmatrix} \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix} = M \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix}$$

[R|T] defines the **extrinsic parameters**.
The 3x4 matrix M = K[R|T] is called the **projection matrix**.
It takes 3d points in the world co-ordinate system and maps them to the appropriate image co-ordinates in pixels

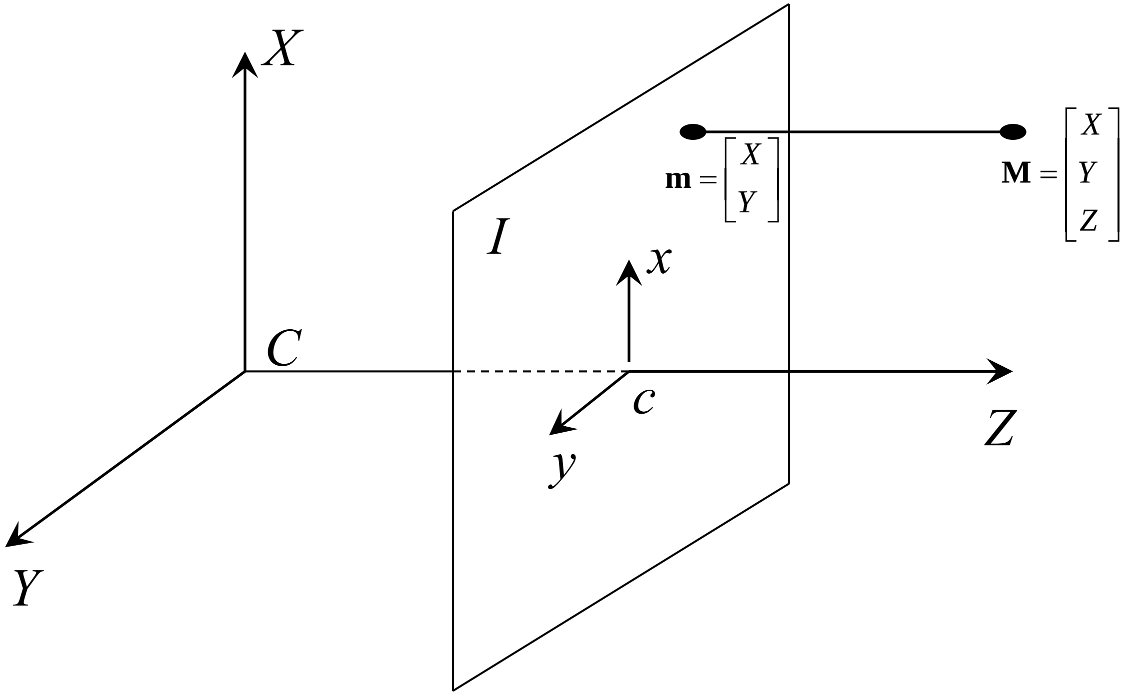
Effect of change in focal length

Small f is wide angle, large f is telescopic



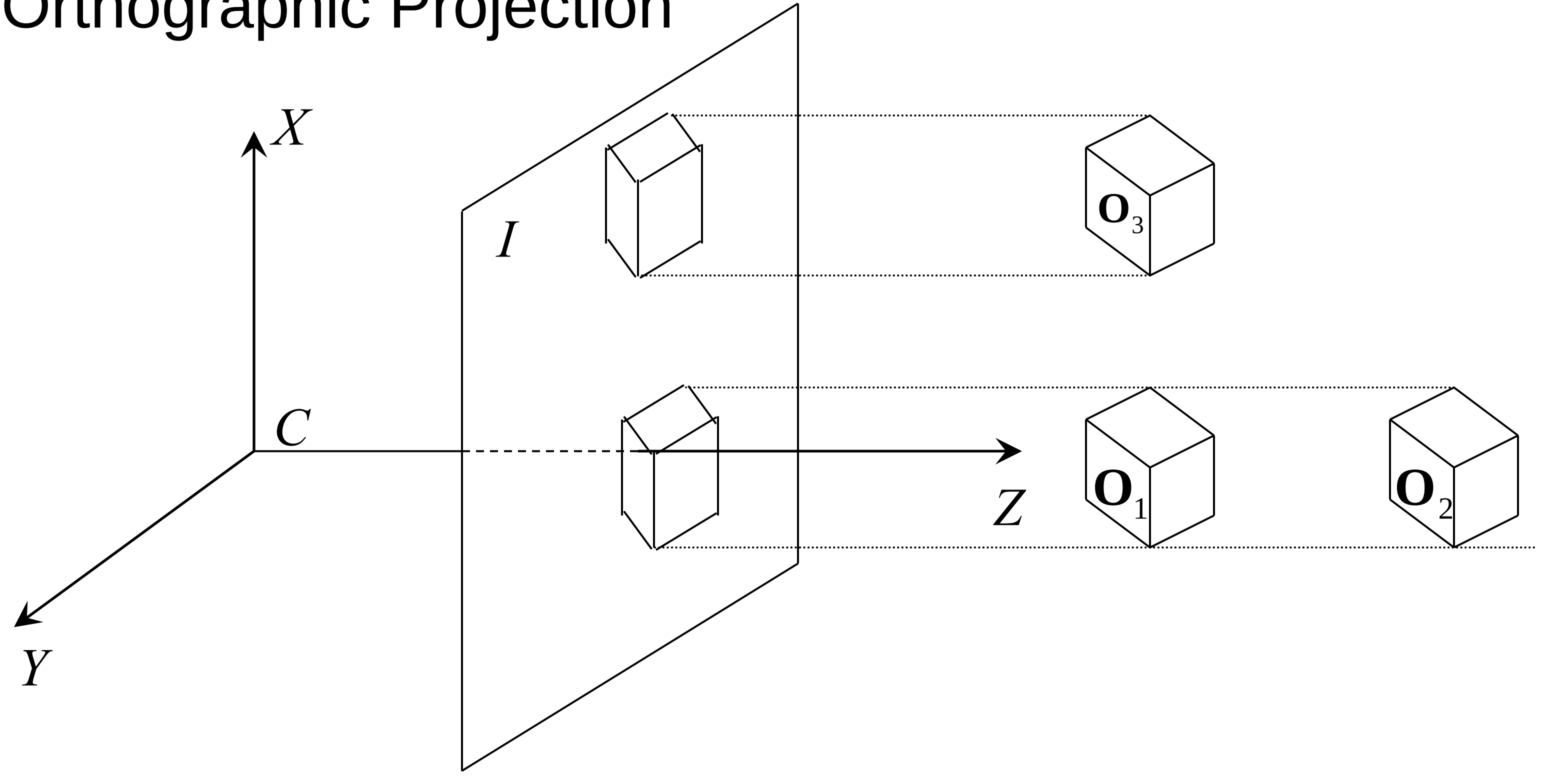
Orthographic Projection

Orthographic Projection



Orthographic Projection

Orthographic Projection



Weak Perspective Model

Assume the relative distance between any two points along the principal axis is much smaller (1/20th at most) than the average distance $\bar{Z}$

Then the camera projection can be approximated as:

$$x = f\frac{X}{Z} \approx \frac{f}{\bar{Z}}X$$

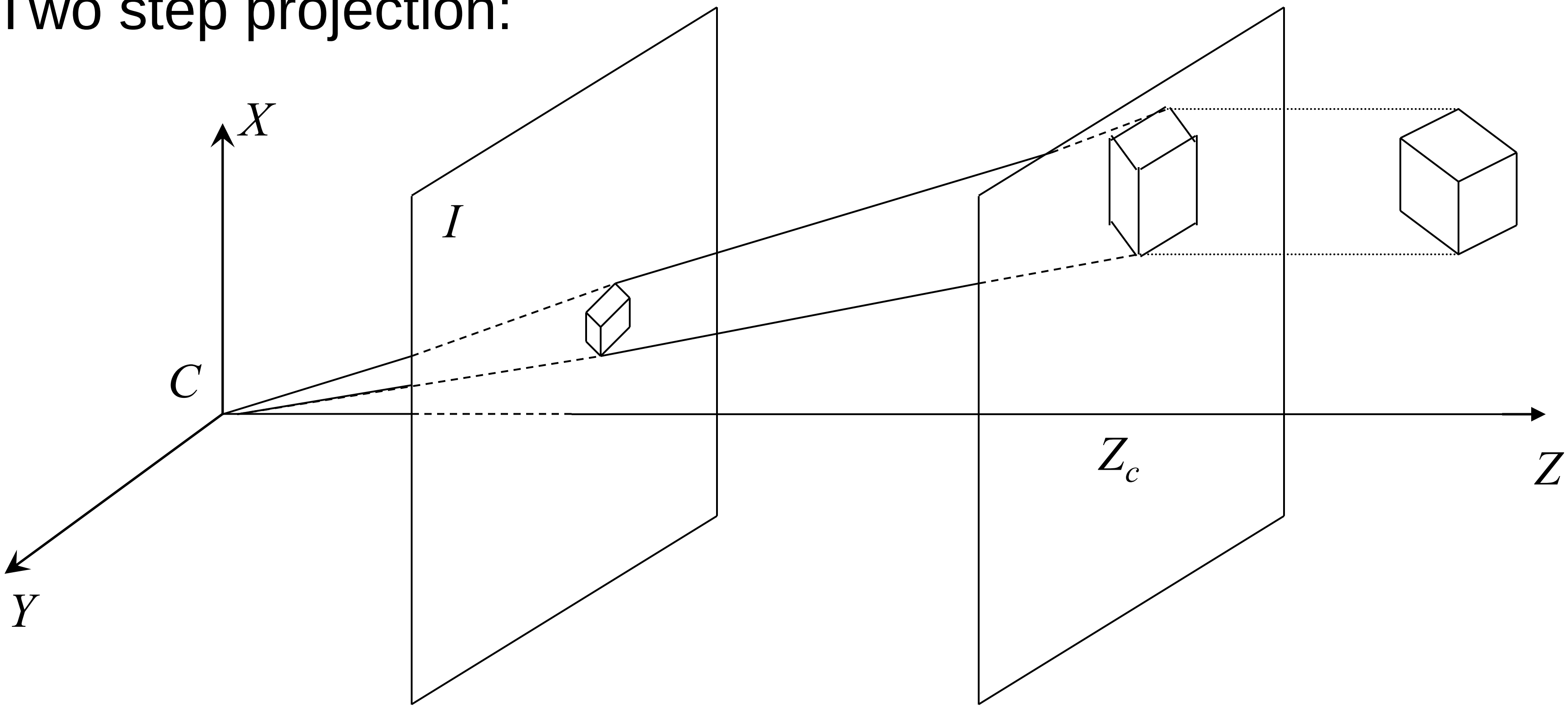
$$y = f\frac{Y}{Z} \approx \frac{f}{\bar{Z}}Y$$

This is the **weak-perspective** camera model.

Sometimes called scaled orthography.

Weak Perspective Projection

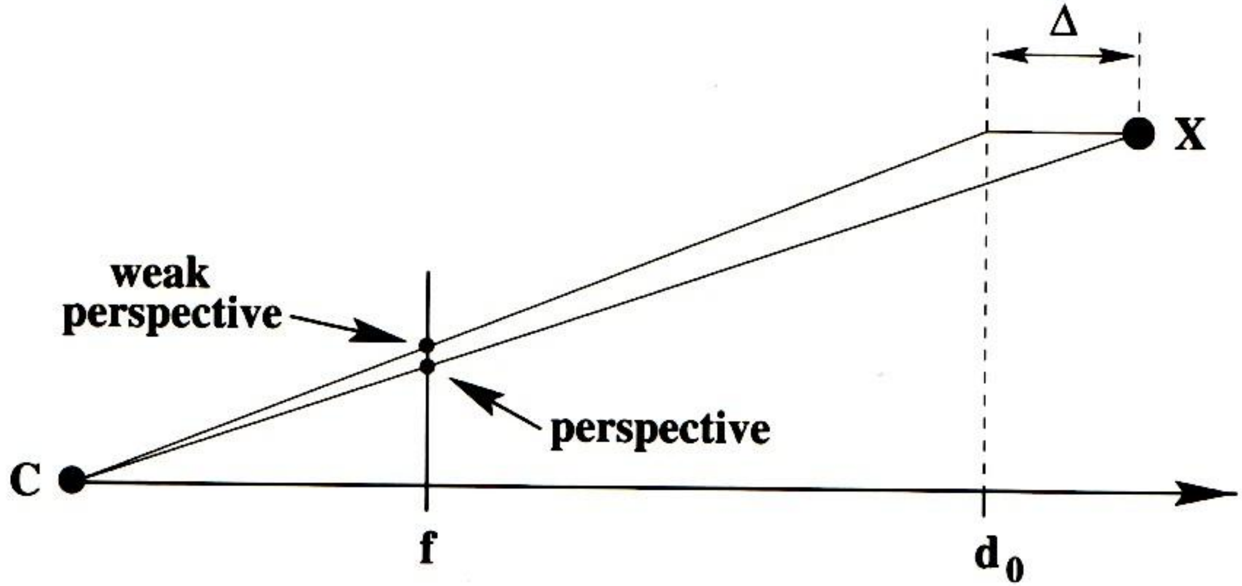
Two step projection:



Z_c

image plane   average depth plane

Weak Perspective

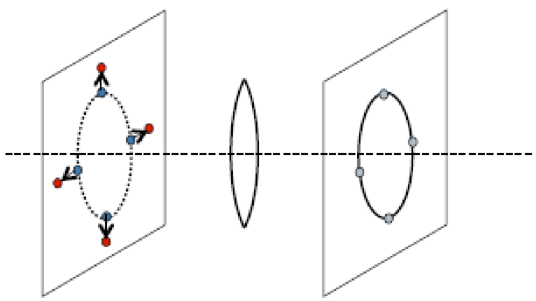


Impact of different projections

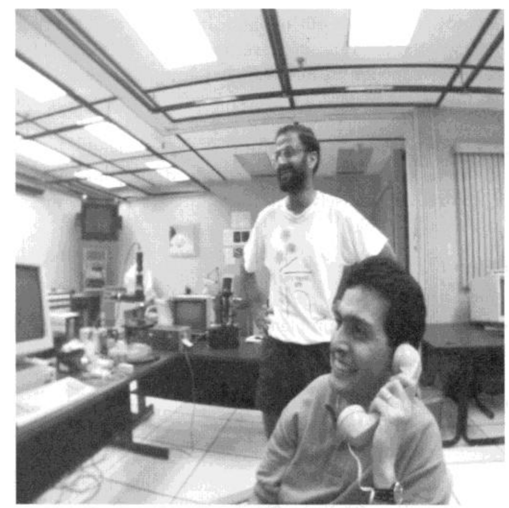
- Perspective projections
  - Parallel lines in world are not parallel in the image
  - Object projection gets smaller with distance from camera
- Weak perspective projection
  - Parallel lines in the world are parallel in the image
  - Object projection gets smaller with distance from camera
- Orthographic projection
  - Parallel lines in the world are parallel in the image
  - Object projection is unchanged with distance from camera

Image distortion due to optics

- Radial distortion which depends on radius r, distance of each point from center of image
- $r^2 = (x - o_x)^2 + (y - o_y)^2$



Radial distortion



$$x_{corrected} = x(1 + k_1 r^2 + k_2 r^4 + k_3 r^6)$$
$$y_{corrected} = y(1 + k_1 r^2 + k_2 r^4 + k_3 r^6)$$

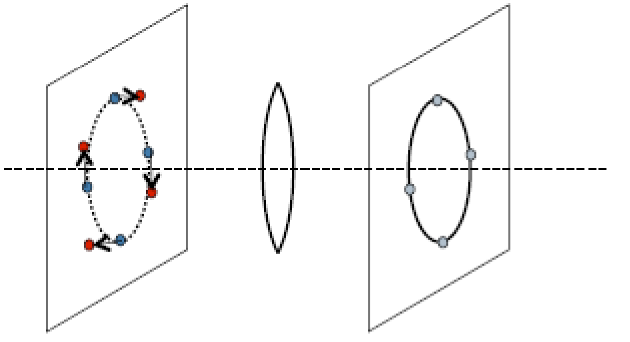
Correction uses three parameters, k_1, k_2, k_3

Correcting Radial Distortions

Tangential Distortion

- Lens not exactly parallel to the image plane

$$x_{\text{corrected}} = x + [2p_1 y + p_2(r^2 + 2x^2)]$$
$$y_{\text{corrected}} = y + [p_1(r^2 + 2y^2) + 2p_2 x]$$



Tangential distortion

- Correction uses two parameter p_1, p_2
- Both types of distortion are removed (image is un-distorted) and only then does standard calibration matrix K apply to the image
- Camera calibration computes both K and these five distortion parameters

How to find the camera parameters

- Can use the EXIF tag for any digital image
  - Has focal length f in millimeters but not the pixel size
  - But you can get the pixel size from the camera manual
  - There are only a finite number of different pixels sizes because number of sensing element sizes is limited
  - If there is not a lot of image distortion due to optics then this approach is sufficient (only linear calibration)
- Can perform explicit camera calibration
  - Put a calibration pattern in front of the camera
  - Take a number of different pictures of this pattern
  - Now run the calibration algorithm (different types)
  - Result is intrinsic camera parameters (linear and non-linear) and the extrinsic camera parameters of all the images