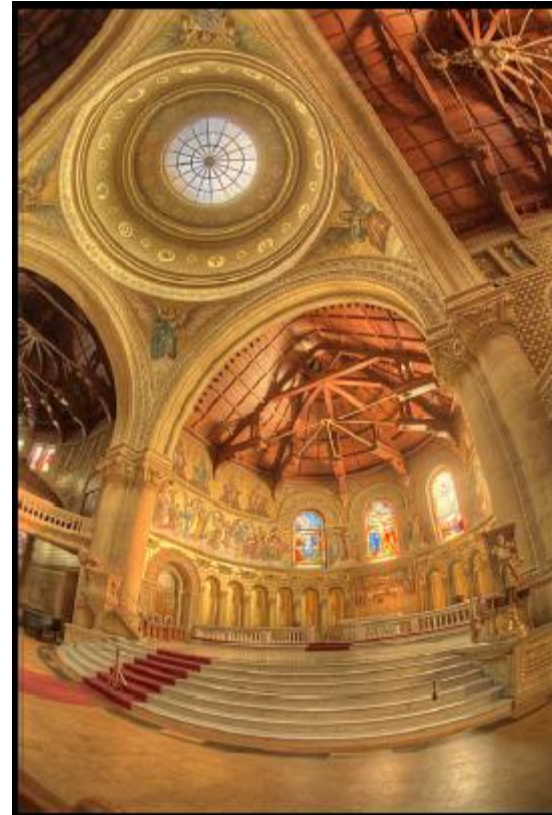

Ellipse Fitting

COMP 4900D

Winter 2012

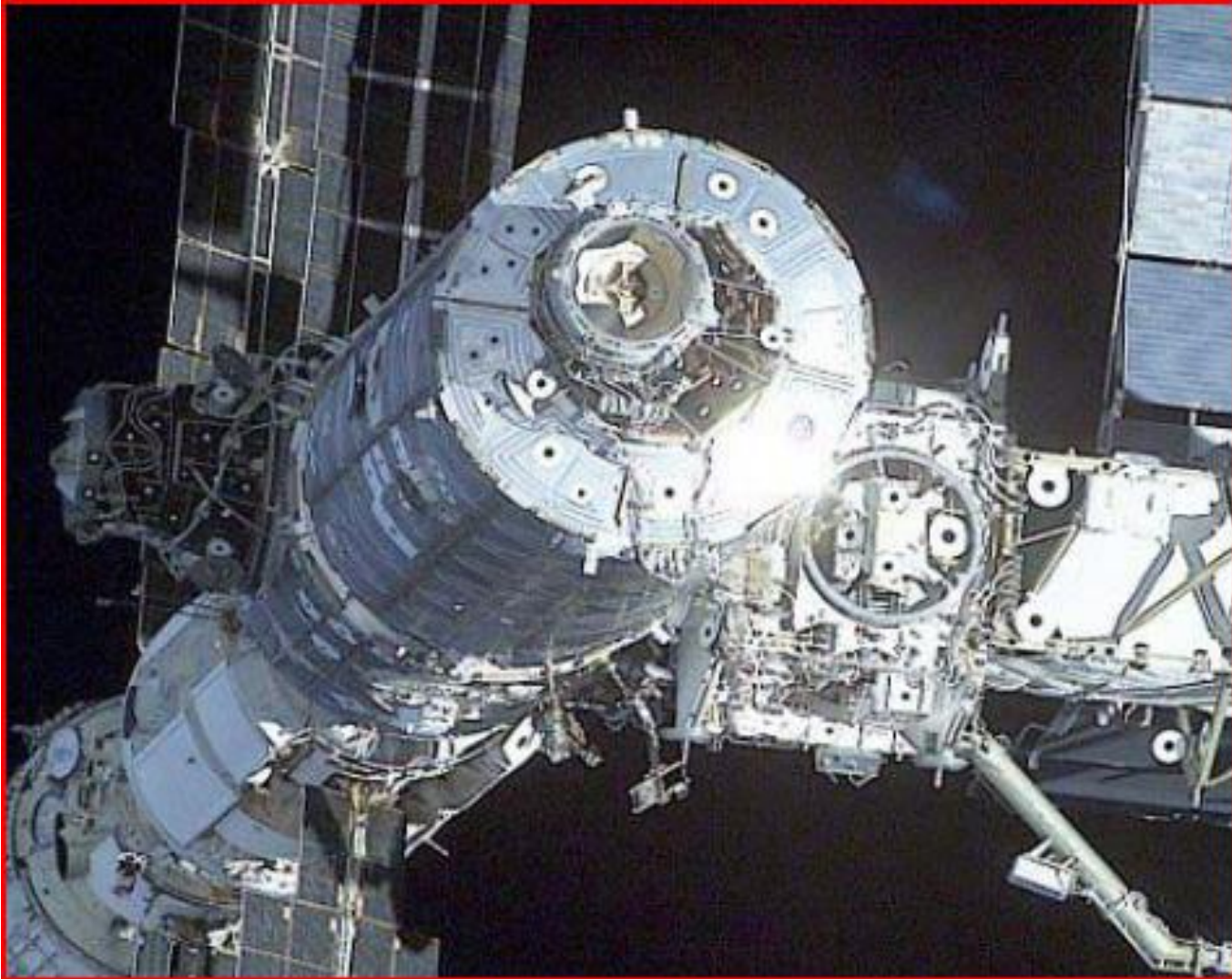
Gerhard Roth

Ellipses in Images

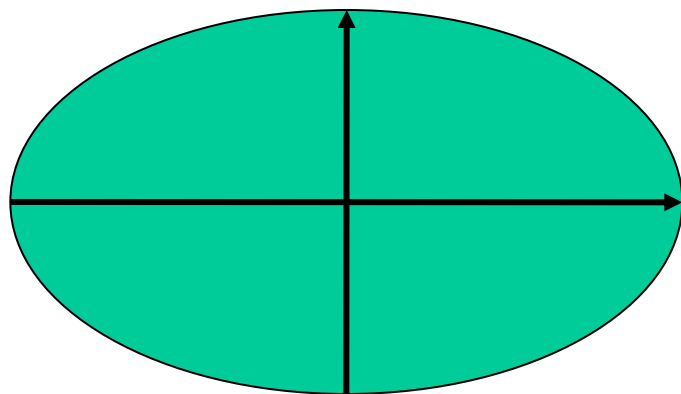


Perspective projection of circles form ellipses in the images.

Ellipses in Images

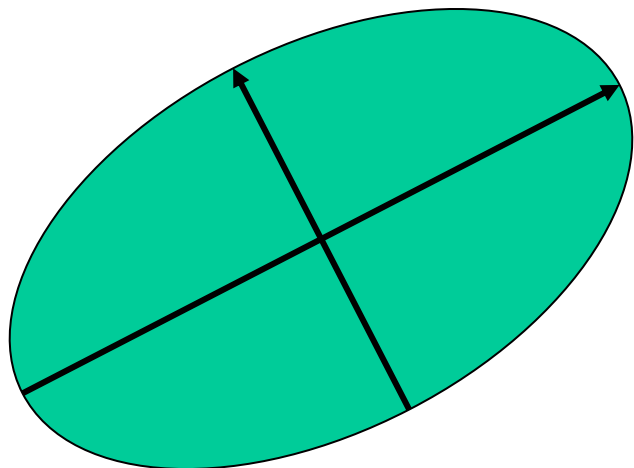


Equations of Ellipse



$$\frac{x^2}{r_1^2} + \frac{y^2}{r_2^2} = 1$$

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$



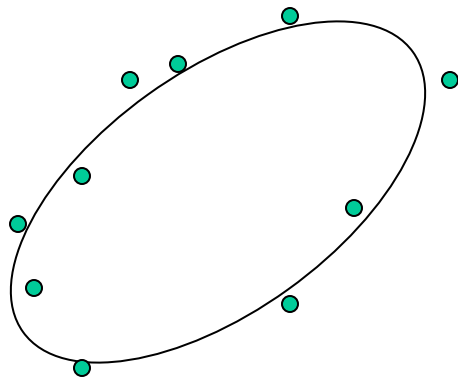
Let $\mathbf{x} = [x^2, xy, y^2, x, y, 1]^T$

$$\mathbf{a} = [a, b, c, d, e, f]^T$$

Then $\mathbf{x}^T \mathbf{a} = 0$

Ellipse Fitting: Problem Statement

Given a set of N image points $\mathbf{p}_i = [x_i, y_i]^T$
find the parameter vector \mathbf{a}_0 such that the ellipse



$$f(\mathbf{p}, \mathbf{a}) = \mathbf{x}^T \mathbf{a} = ax^2 + bxy + cy^2 + dx + ey + f = 0$$

fits \mathbf{p}_i best in the least square sense:

$$\min_{\mathbf{a}} \sum_{i=1}^N [D(\mathbf{p}_i, \mathbf{a})]^2$$

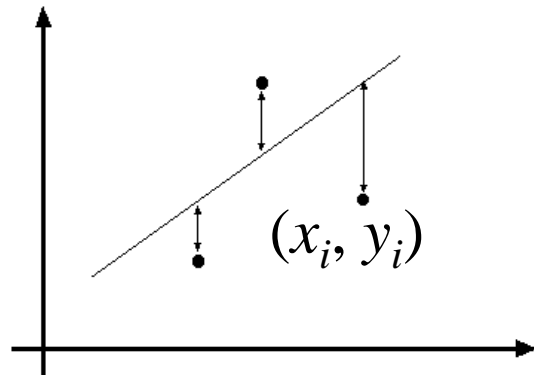
Where $D(\mathbf{p}_i, \mathbf{a})$ is the distance from \mathbf{p}_i to the ellipse.

Ellipse Fitting: Geometric or Algebraic

- Remember that $D(\mathbf{p}_i, \mathbf{a})$ is the distance from point \mathbf{p}_i to the ellipse
- There are two ways to calculate this distance
- First uses the true geometric distance
 - Is most accurate but is more difficult to calculate
 - In this case the ellipse fitting algorithm is iterative
 - But it gets the best possible result
- Second uses the algebraic distance
 - Is less accurate but is easier to calculate
 - In this the ellipse fitting algorithm has only one iteration
 - But does not get best possible result (it is reasonable)

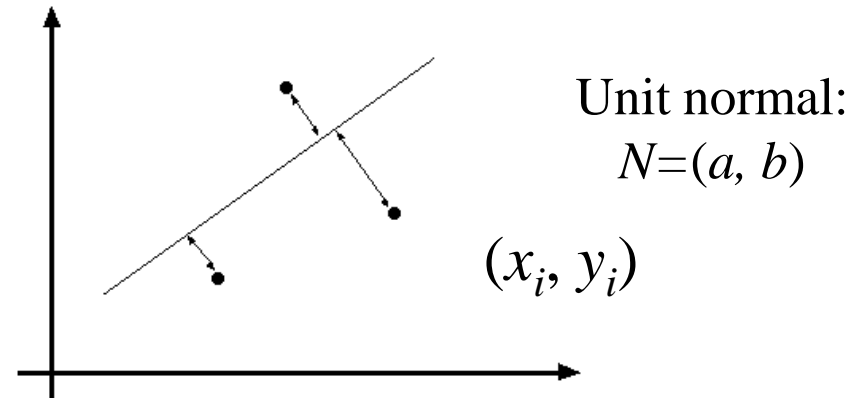
Algebraic Distance/Geometric Distance

Algebraic Distance
for a line



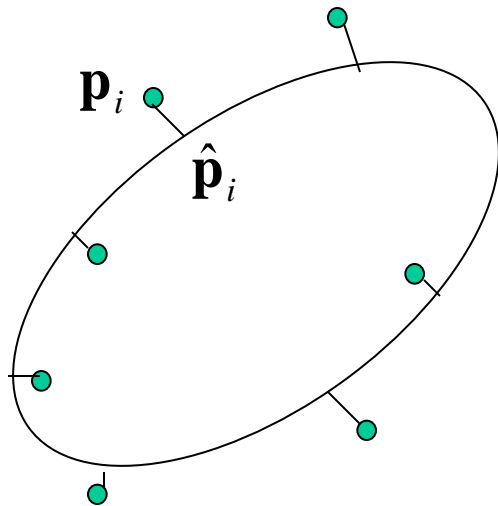
$$y=mx+b$$

Geometric Distance
for a line



$$ax+by=d$$

Euclidean Distance Fit



$$D(\mathbf{p}_i, \mathbf{a}) = \|\hat{\mathbf{p}}_i - \mathbf{p}_i\|$$

$\hat{\mathbf{p}}_i$ is the point on the ellipse that is nearest to \mathbf{p}_i

$$f(\hat{\mathbf{p}}_i, \mathbf{a}) = 0$$

$\hat{\mathbf{p}}_i - \mathbf{p}_i$ is normal to the ellipse at $\hat{\mathbf{p}}_i$

Compute Geometric Distance Function

Computing the distance function is a constrained optimization problem:

$$\min_{\hat{\mathbf{p}}_i} \|\hat{\mathbf{p}}_i - \mathbf{p}_i\|^2 \quad \text{subject to} \quad f(\hat{\mathbf{p}}_i, \mathbf{a}) = 0$$

Using **Lagrange multiplier**, define:

$$L(x, y, \lambda) = \|\hat{\mathbf{p}}_i - \mathbf{p}_i\|^2 - 2\lambda f(\hat{\mathbf{p}}_i, \mathbf{a})$$

where $\hat{\mathbf{p}}_i = [x, y]^T$

Then the problem becomes: $\min_{\hat{\mathbf{p}}_i} L(x, y, \lambda)$

Set $\frac{\partial L}{\partial x} = \frac{\partial L}{\partial y} = 0$ we have $\hat{\mathbf{p}}_i - \mathbf{p}_i = \lambda \nabla f(\hat{\mathbf{p}}_i, \mathbf{a})$

Two Approximations

1. First-order approximation at \mathbf{p}_i

$$f(\hat{\mathbf{p}}_i, \mathbf{a}) \approx f(\mathbf{p}_i, \mathbf{a}) + (\hat{\mathbf{p}}_i - \mathbf{p}_i)^T \nabla f(\mathbf{p}_i, \mathbf{a}) = 0$$

2. Assume \mathbf{p}_i is close to $\hat{\mathbf{p}}_i$, then

$$\nabla f(\hat{\mathbf{p}}_i, \mathbf{a}) \approx \nabla f(\mathbf{p}_i, \mathbf{a})$$

Approximate Distance Function

$$f(\hat{\mathbf{p}}_i, \mathbf{a}) \approx f(\mathbf{p}_i, \mathbf{a}) + (\hat{\mathbf{p}}_i - \mathbf{p}_i)^T \nabla f(\mathbf{p}_i, \mathbf{a}) = 0$$

$$\hat{\mathbf{p}}_i - \mathbf{p}_i = \lambda \nabla f(\hat{\mathbf{p}}_i, \mathbf{a}) \approx \lambda \nabla f(\mathbf{p}_i, \mathbf{a})$$

Solve for λ

$$\lambda = -\frac{f(\mathbf{p}_i, \mathbf{a})}{\|\nabla f(\mathbf{p}_i, \mathbf{a})\|^2}$$

Substitute back

$$\hat{\mathbf{p}}_i - \mathbf{p}_i = -\frac{f(\mathbf{p}_i, \mathbf{a}) \nabla f(\mathbf{p}_i, \mathbf{a})}{\|\nabla f(\mathbf{p}_i, \mathbf{a})\|^2}$$

$$D(\mathbf{p}_i, \mathbf{a}) = \|\hat{\mathbf{p}}_i - \mathbf{p}_i\| = \frac{|f(\mathbf{p}_i, \mathbf{a})|}{\|\nabla f(\mathbf{p}_i, \mathbf{a})\|}$$

Ellipse Fitting with Euclidean Distance

Given a set of N image points $\mathbf{p}_i = [x_i, y_i]^T$
find the parameter vector \mathbf{a}_0 such that

$$\min_{\mathbf{a}} \sum_{i=1}^N \frac{|f(\mathbf{p}_i, \mathbf{a})|^2}{\|\nabla f(\mathbf{p}_i, \mathbf{a})\|^2}$$

This problem can be solved by using a numerical nonlinear optimization system.

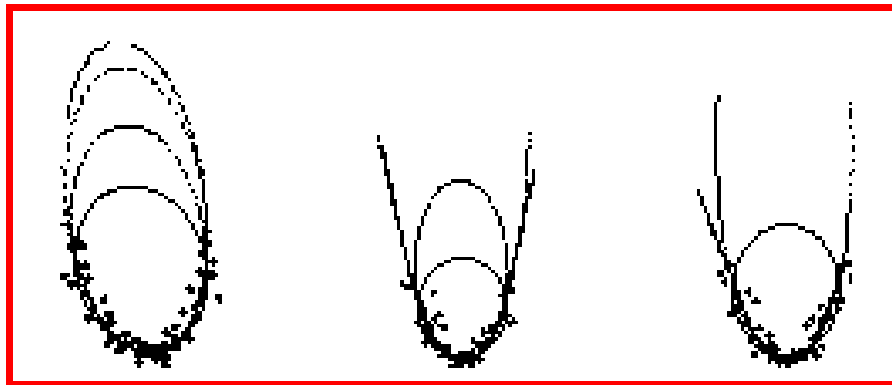
Ellipse Fitting with Algebraic Distance

- The algebraic distance from a point \mathbf{p}_i to a curve defined by $f(\mathbf{p}_i, \mathbf{a}) = 0$ is $|f(\mathbf{p}_i, \mathbf{a})|$
- This is simply putting point \mathbf{p}_i into the function
- However, algebraic distance is not the true geometric distance, only an approximation
- We minimize the function below but to avoid the trivial solution, $\mathbf{a} = 0$, we constrain \mathbf{a} so that $b^2 - 4ac = 0$

$$\min_{\mathbf{a}} \sum_{i=1}^N \left| \begin{pmatrix} x_i^T & a \end{pmatrix} \right|^2$$

Algebraic Fitting of Ellipse

- With this constraint the fitting becomes an eigenvector problem (easy to solve)
- Works directly, in one iteration (real-time)
- But has a bias towards low eccentricity
 - Algebraic ellipse is more “circle like” than true ellipse



Ellipse Fitting

- Given a set of 2d points fit the “best” ellipse
- Can use the algebraic or geometric approach
 - Algebraic method is more commonly used
 - Always produces an ellipse, and is fast
 - Result not perfect but more than good enough
- Issue of geometric versus algebraic is a common problem in all fitting
 - And is more complex for higher order curves and surfaces
- OpenCV has an implementation of algebraic fitting method for ellipses
 - Demo program ellipsefit shows this computation in action