
Camera Calibration

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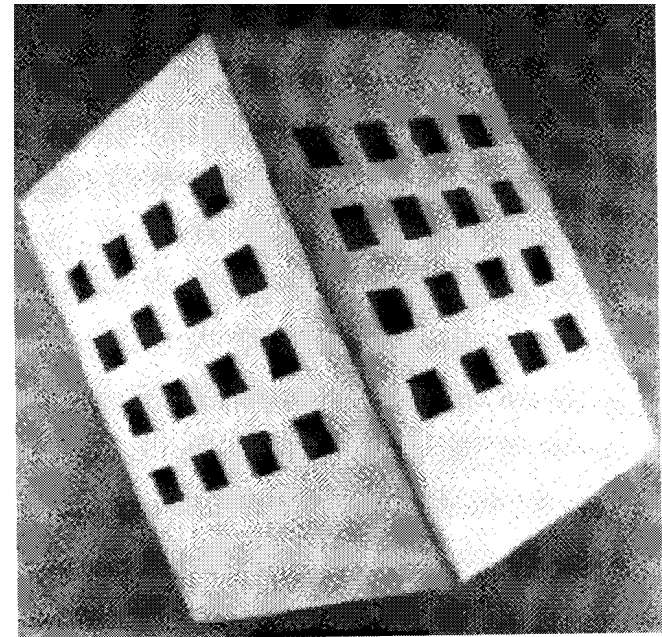
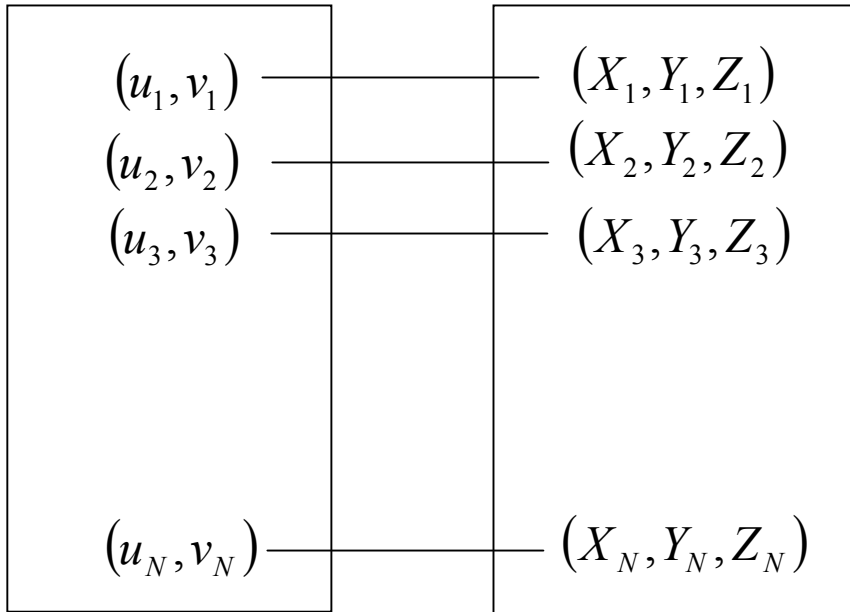
Problem statement

- Use a calibration pattern with known 3d geometry (often a box)
- Write projection equations linking known coordinates of a set of 3-D points and their projections and solve for camera parameters
- Given a set of one or more images of the calibration pattern estimate
 - Intrinsic camera parameters
 - (depend only on camera characteristics)
 - Extrinsic camera parameters
 - (depend only on position camera)

Estimating camera parameters

- Projection matrix

Calibration pattern



Camera parameters

- Intrinsic parameters (K matrix)
 - There are 5 intrinsic parameters
 - Focal length f
 - Pixel size in x and y directions, s_x and s_y
 - Principal point o_x, o_y
- But they are not independent
 - Focal length $f_x = f / s_x$ and $f_y = f / s_y$
 - Principal point o_x, o_y
 - This makes four intrinsic parameters
- Extrinsic parameters [R| T]
 - Rotation matrix and translation vector of camera
 - Relations camera position to a known frame
 - [R|T] are the extrinsic parameters
- Projection matrix
 - 3 by 4 matrix $P = K [R | T]$ is called projection matrix

Projection Equations

Projective Space

- Add fourth coordinate
 - $P_w = (X_w, Y_w, Z_w, 1)^T$
- Define $(u, v, w)^T$ such that
 - $u/w = x_{im}, v/w = y_{im}$

$$\begin{pmatrix} x_{im} \\ y_{im} \end{pmatrix} = \begin{pmatrix} u/w \\ v/w \end{pmatrix}$$



$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \mathbf{M}_{int} \mathbf{M}_{ext} \begin{pmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{pmatrix}$$

3x4 Matrix \mathbf{E}_{ext}

- Only extrinsic parameters
- World to camera

$$\mathbf{M}_{ext} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & T_x \\ r_{21} & r_{22} & r_{23} & T_y \\ r_{31} & r_{32} & r_{33} & T_z \end{bmatrix} = \begin{bmatrix} \mathbf{R}_1^T & T_x \\ \mathbf{R}_2^T & T_y \\ \mathbf{R}_3^T & T_z \end{bmatrix}$$

3x3 Matrix \mathbf{E}_{int}

- Only intrinsic parameters
- Camera to frame

$$\mathbf{M}_{int} = \begin{bmatrix} -f_x & 0 & o_x \\ 0 & -f_y & o_y \\ 0 & 0 & 1 \end{bmatrix}$$

Simple Matrix Product! Projective Matrix

$$\mathbf{M} = \mathbf{M}_{int} \mathbf{M}_{ext}$$

- $(X_w, Y_w, Z_w)^T \rightarrow (x_{im}, y_{im})^T$
- Linear Transform from projective space to projective plane
- \mathbf{M} defined up to a scale factor – 11 independent entries

Two different calibration methods

- Both use a set of 3d points and 2d projections
- Direct approach (called Tsai method)
 - Write projection equations in terms of all the parameters
 - That is all the unknown intrinsic and extrinsic parameters
 - Solve for these parameters using non-linear equations
- Projection matrix approach
 - Compute the projection matrix (the 3x4 matrix M)

$$\begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \end{bmatrix}$$

- Compute camera parameters as closed-form functions of M

• Two different calibration methods

- Both approaches work with same data
 - Projection matrix approach is simpler to explain than the direct approach
- Direct approach requires an extra step
 - There are also other calibration methods
- But all calibration methods
 - Attempt to compute camera parameters (intrinsic/extrinsic)
 - Use patterns with known geometry or shape
 - Take multiple views of these patterns
- Perform some mathematics to calculate the intrinsic and extrinsic camera parameters
 - Usually linear algebra or non-linear optimization

Estimating the projection matrix

World – Frame Transform

- Drop “im” and “w”
- N pairs $(x_i, y_i) \leftrightarrow (X_i, Y_i, Z_i)$

$$x_i = \frac{u_i}{w_i} = \frac{m_{11}X_i + m_{12}Y_i + m_{13}Z_i + m_{14}}{m_{31}X_i + m_{32}Y_i + m_{33}Z_i + m_{34}}$$
$$y_i = \frac{u_i}{w_i} = \frac{m_{21}X_i + m_{22}Y_i + m_{23}Z_i + m_{24}}{m_{31}X_i + m_{32}Y_i + m_{33}Z_i + m_{34}}$$

Linear equations of m

- 2N equations, 11 independent variables
- $N \geq 6$, SVD \Rightarrow m up to a unknown scale

$$\mathbf{A}\mathbf{m} = \mathbf{0}$$

$$\mathbf{A} = \begin{bmatrix} X_1 & Y_1 & Z_1 & 1 & 0 & 0 & 0 & 0 & -x_1X_1 & -x_1Y_1 & -x_1Z_1 & -x_1 \\ 0 & 0 & 0 & 0 & X_1 & Y_1 & Z_1 & 1 & -y_1X_1 & -y_1Y_1 & -y_1Z_1 & -y_1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & & & & \end{bmatrix}$$

$$\mathbf{m} = [m_{11} \quad m_{12} \quad m_{13} \quad m_{14} \quad m_{21} \quad m_{22} \quad m_{23} \quad m_{24} \quad m_{31} \quad m_{32} \quad m_{33} \quad m_{34}]^T$$

Homogeneous System

- M linear equations of form $A\mathbf{x} = 0$
- If we have a given solution \mathbf{x}_1 , s.t. $A\mathbf{x}_1 = 0$ then $c * \mathbf{x}_1$ is also a solution $A(c * \mathbf{x}_1) = 0$
- Need to add a constraint on \mathbf{x} ,
 - Basically make \mathbf{x} a unit vector $\mathbf{x}^T \mathbf{x} = 1$
- Can prove that the solution is the eigenvector corresponding to the single zero eigenvalue of that matrix $A^T A$
 - This can be computed using eigenvector of SVD routine
 - Then finding the zero eigenvalue
 - Returning the associated eigenvector

Decompose projection matrix

- 3x4 Projection Matrix M
 - Both intrinsic (4) and extrinsic (6) – 10 parameters

$$\mathbf{M} = \begin{bmatrix} -f_x r_{11} + o_x r_{31} & -f_x r_{12} + o_x r_{32} & -f_x r_{13} + o_x r_{33} & -f_x T_x + o_x T_z \\ -f_y r_{21} + o_y r_{31} & -f_y r_{22} + o_y r_{32} & -f_y r_{23} + o_y r_{33} & -f_y T_y + o_y T_z \\ r_{31} & r_{32} & r_{33} & T_z \end{bmatrix}$$

From \mathbf{M} to parameters (p134-135)

- Find scale $|\gamma|$ by using unit vector \mathbf{R}_3^T
- Determine T_z and sign of γ from m_{34} (i.e. q_{43})
- Obtain \mathbf{R}_3^T
- Find (O_x, O_y) by dot products of Rows q_1, q_3, q_2, q_3 , using the orthogonal constraints of \mathbf{R}
- Determine f_x and f_y from q_1 and q_2 All the rests: $\mathbf{R}_1^T, \mathbf{R}_2^T, T_x, T_y$
- Enforce orthognoality on \mathbf{R} by using SVD

Rotation Matrices

- Are special 3 by matrices R

$$RR^T = R^T R = I \quad \det(R) = 1$$

- Rows and columns of R are mutually orthogonal unit vectors
- Each row and column has unit norm
- Dot product of each row with others is zero
- Dot product of each column with others is zero

Ensuring the orthogonality of R

- The computation of R does not take into account explicitly the orthogonality constraints on R
- The estimate \hat{R} of R cannot be expected to be orthogonal:
- We can "enforce" the orthogonality on \hat{R} using SVD:

$$\hat{R}\hat{R}^T = I, \quad \hat{R} = UDV^T$$

- Replace D (*singular values*) with Identity Matrix I :

$$\hat{R}' = UIV^T \quad (\hat{R}'\hat{R}'^T = I)$$

Direct Approach

- Main steps

(1) Assuming that σ_x and σ_y are known, estimate all the remaining parameters.

(2) Estimate σ_x and σ_y

Projection Equations (book notation)

- Projection equations (camera and world)

$$x = f \frac{X_C}{Z_C} \quad y = f \frac{Y_C}{Z_C}$$

- Calibration equations

$$x = -\frac{(x_{im} - o_x)}{s_x} \quad y = -\frac{(y_{im} - o_y)}{s_y}$$

- Using the projection equations and camera calibration information we get:

$$x_{im} = -\frac{f}{s_x} \frac{X^C}{Z^C} + o_x \quad y_{im} = -\frac{f}{s_y} \frac{Y^C}{Z^C} + o_y$$

Projection Equations (book notation)

- 3D point P in world reference frame is $[X^w, Y^w, Z^w]$
- $[X^c, Y^c, Z^c]$ are the co-ordinates of P in camera frame
- Extrinsic parameters, apply rotation then translation

$$\begin{bmatrix} X^c \\ Y^c \\ Z^c \end{bmatrix} = R \begin{bmatrix} X^w \\ Y^w \\ Z^w \end{bmatrix} + T$$

$$X^c = r_{11}X^w + r_{12}Y^w + r_{13}Z^w + T_x$$

$$Y^c = r_{21}X^w + r_{22}Y^w + r_{23}Z^w + T_y$$

$$Z^c = r_{31}X^w + r_{32}Y^w + r_{33}Z^w + T_z$$

Projection Equations (book notation)

• Now let $f_x = f / s_x$, $f_y = f / s_y$, $\alpha = s_y / s_x$

$$x - o_x = -f_x \frac{r_{11}X^{(w)} + r_{12}Y^{(w)} + r_{13}Z^{(w)} + T_X}{r_{31}X^{(w)} + r_{32}Y^{(w)} + r_{33}Z^{(w)} + T_Z}$$

$$y - o_y = -f_y \frac{r_{21}X^{(w)} + r_{22}Y^{(w)} + r_{23}Z^{(w)} + T_Y}{r_{31}X^{(w)} + r_{32}Y^{(w)} + r_{33}Z^{(w)} + T_Z}$$

• Now assume that O_x , and O_y are zero then

$$x_i f_y (r_{21}X_i + r_{22}Y_i^{(w)} + r_{23}Z_i^{(w)} + T_Y) = y_i f_x (r_{11}X_i^{(w)} + r_{12}Y_i^{(w)} + r_{13}Z_i^{(w)} + T_X)$$

$$x_i X_i^{(w)} r_{21} + x_i Y_i^{(w)} r_{22} + x_i Z_i^{(w)} r_{23} + x_i T_Y - y_i X_i^{(w)} \alpha r_{11} - y_i Y_i^{(w)} \alpha r_{12} - y_i Z_i^{(w)} \alpha r_{13} - y_i \alpha T_X = 0$$

Projection Equations (book notation)

For each corresponding pair (u_i, v_i) and (X_i, Y_i, Z_i)

$$x_i X_i^{(w)} v_1 + x_i Y_i^{(w)} v_2 + x_i Z_i^{(w)} v_3 + x_i v_4 - y_i X_i^{(w)} v_5 - y_i Y_i^{(w)} v_6 + y_i Z_i^{(w)} v_7 + y_i v_8 = 0$$

$$v = [v_1 \quad v_2 \quad v_3 \quad v_4 \quad v_5 \quad v_6 \quad v_7 \quad v_8]^T \quad Av = 0$$

$$\begin{array}{ll}
 v_1 = r_{21} & v_5 = \alpha r_{11} \\
 v_2 = r_{22} & v_6 = \alpha r_{12} \\
 v_3 = r_{23} & v_7 = \alpha r_{13} \\
 v_4 = T_Y & v_8 = \alpha T_X
 \end{array}
 \quad
 A = \begin{bmatrix}
 x_1 X_1^w & x_1 Y_1^w & x_1 Z_1^w & x_1 & -y_1 X_1^w & -y_1 Y_1^w & -y_1 Z_1^w & -y_1 \\
 x_2 X_2^w & x_2 Y_2^w & x_2 Z_2^w & x_2 & -y_2 X_2^w & -y_2 Y_2^w & -y_2 Z_2^w & -y_2 \\
 \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
 x_N X_N^w & x_N Y_N^w & x_N Z_N^w & x_N & -y_N X_N^w & -y_N Y_N^w & -y_N Z_N^w & -y_N
 \end{bmatrix}$$

- If there are N points, A is an N by 8 matrix
 - Need to solve $A v = 0$ but require an extra constraints
 - This is a homogenous system with constraints

Estimating Camera Parameters

Let \bar{v} be the obtained solution vector, then

$$\bar{v} = \gamma (r_{21} \quad r_{22} \quad r_{23} \quad T_Y \quad \alpha r_{11} \quad \alpha r_{12} \quad \alpha r_{13} \quad \alpha T_X)^T$$

Since $r_{21}^2 + r_{22}^2 + r_{23}^2 = 1$, we have

$$\sqrt{v_1^2 + v_2^2 + v_3^2} = \sqrt{\gamma^2 (r_{21}^2 + r_{22}^2 + r_{23}^2)} = |\gamma|$$

- Using magnitude of γ we can determine α

Since $r_{21}^2 + r_{22}^2 + r_{23}^2 = 1$, we have

$$\sqrt{v_5^2 + v_6^2 + v_7^2} = \sqrt{\gamma^2 \alpha^2 (r_{21}^2 + r_{22}^2 + r_{23}^2)} = \alpha |\gamma|$$

Estimating Camera Parameters

Determine r_{31} , r_{32} , r_{33}

- Can be estimated as the cross product of R_1 and R_2 :

$$R_3 = R_1 \times R_2$$

- The sign of R_3 is already fixed (the entries of R_3 remain unchanged if the signs of all the entries of R_1 and R_2 are reversed).

Estimating Camera Parameters

Determine the sign of γ

- Consider the following equations again:

$$x = -f/s_x \frac{r_{11}X_w + r_{12}Y_w + r_{13}Z_w + T_x}{r_{31}X_w + r_{32}Y_w + r_{33}Z_w + T_z} = -f/s_x \frac{X_c}{Z_c}$$

$$y = -f/s_y \frac{r_{21}X_w + r_{22}Y_w + r_{23}Z_w + T_y}{r_{31}X_w + r_{32}Y_w + r_{33}Z_w + T_z} = -f/s_y \frac{Y_c}{Z_c}$$

- If $Z_c > 0$, then x and $r_{11}X_w + r_{12}Y_w + r_{13}Z_w + T_x$ must have opposite signs (it is sufficient to check the sign for one of the points).

if $x(r_{11}X_w + r_{12}Y_w + r_{13}Z_w + T_x) > 0$, then
reverse the signs of r_{11} , r_{12} , r_{13} , and T_x
else
no further action is required

Estimating Camera Parameters

- Similarly, if $Z_c > 0$, then y and $r_{21}X_w + r_{22}Y_w + r_{23}Z_w + T_x$ must have opposite signs (it is sufficient to check the sign for one of the points).

if $y(r_{21}X_w + r_{22}Y_w + r_{23}Z_w + T_x) > 0$, then
 reverse the signs of r_{21} , r_{22} , r_{23} , and T_x
else
 no further action is required

Must do more work to compute T_z and f_x , and
also to find O_x , O_y

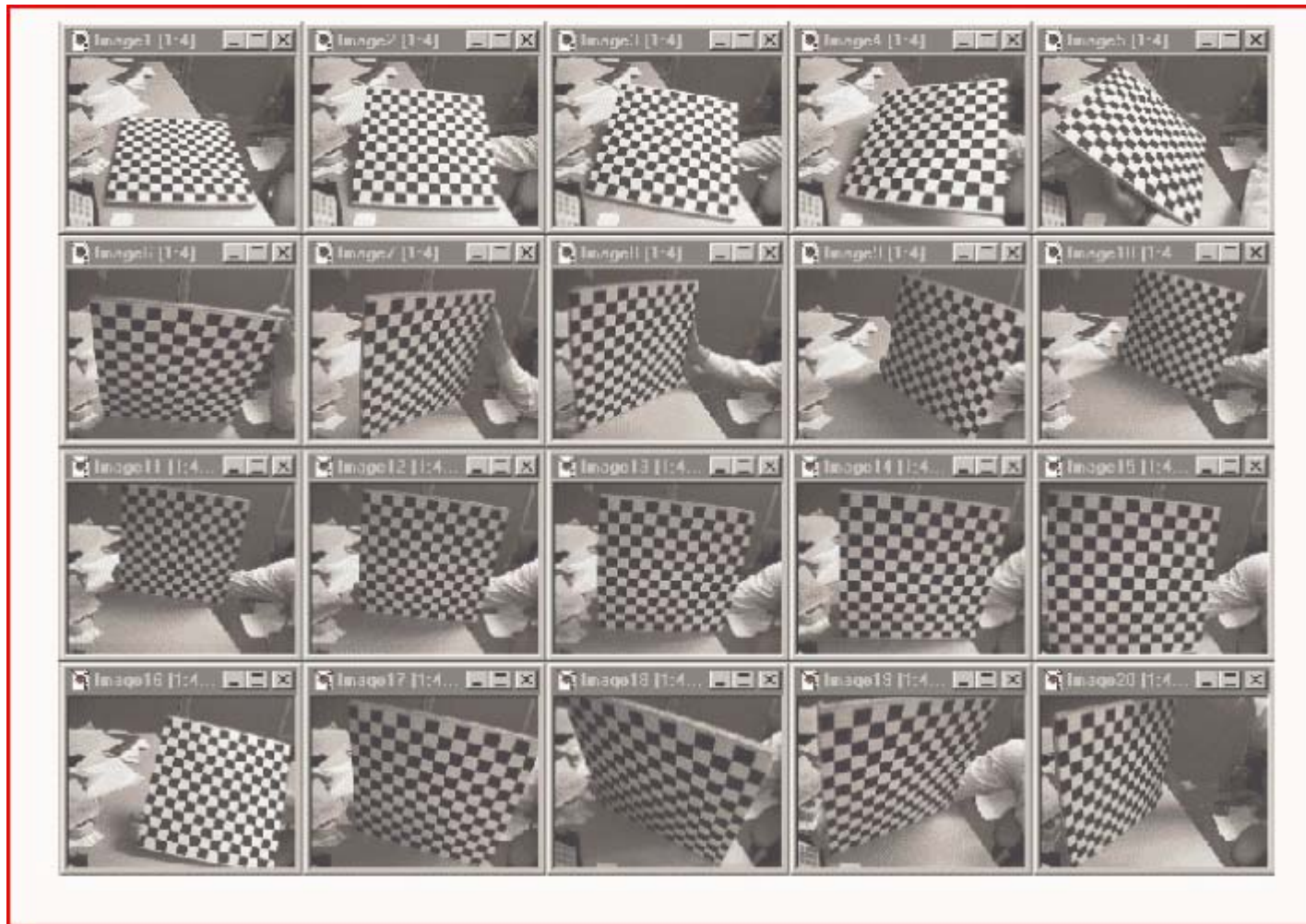
Calibration Summary

- **Comparison of methods**
 - Direct approach requires extra step to find O_x , O_y
 - Projection approach finds O_x , and O_y at same time
 - Is simpler mathematically than the direct approach
 - Both methods require a refit to find a “valid” R matrix
- **There are other calibration methods**
 - Zhang approach uses flat plane (implemented in OpenCV)
 - Plane must be flat, but do not need 3D co-ordinates
- **But all calibration methods**
 - Have some known targets with known 3D geometry or shape
 - Take a number of images of these targets
 - From these measurements calculate the camera parameters
 - Are essential for further processing like reconstruction

Multiple View/Camera Calibration

- Previous math describes the calibration process for a single image
 - We usually take multiple images of the same calibration target (from a variety of different views)
 - Simultaneously find all extrinsic parameters and all the intrinsic parameters of the single camera
- Also calibrate radial distortion using fact that there are straight lines in the pattern
- OpenCV code can do this using a checkerboard pattern
- Zhang's algorithm is used most in practice

Input set of 2d Calibration Patterns



Final Camera positions and the pattern

