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Integer Sets with Distinct Sums and Differences and Carrier Frequency Assignments for Nonlinear Repeaters

M. D. ATKINSON, N. SANTORO, AND J. URRUTIA

Abstract—**The problem of assigning n carrier frequencies so as to avoid certain types (third and fifth order) of intermodulation interference is discussed. For the third-order case, close upper and lower bounds on the optimal solution are established; and close to optimal solutions are given for $n < 100$ (previously, suboptimal solutions were known only for $n \leqslant 23$). For the fifth-order case, it is shown that some existing results can be applied to this problem, and suboptimal solutions obtained by this construction are given for $n \leqslant 17$ (no solutions were known previously).**

I. INTRODUCTION

Some 30 years ago Babcock [2], in work on assigning radio frequencies so as to avoid certain types ("third order" and "fifth order," respectively) of intermodulation interference caused by a common nonlinear power amplifier, formulated the following two problems.

For any given n, find integers $0 \leqslant a_0 < a_1 < \cdots < a_n$ such that no nontrivial equality $a_r + a_s - a_t = a_u$ holds.

For any given n, find integers $0 \leqslant a_0 < a_1 < \cdots < a_n$ such that no nontrivial equality $a_r + a_s + a_t - a_u - a_v = a_w$ holds.

In these problems the integers are radio frequencies and, since it is desirable to have a small spectral range, solutions are sought in which a_n is as small as possible ("optimal" solutions) or at least provably close to optimal ("suboptimal" solutions).

The word "nontrivial" appears in the problem statements since, of course, trivial equalities such as $a_r + a_s - a_r = a_s$ and $a_r + a_s + a_t - a_t - a_r = a_s$ cannot be avoided. However, equalities like $a_r + a_r - a_t = a_u$ and $a_r + a_r + a_t - a_u - a_u = a_w$ are deemed to be nontrivial. The problems can be formulated as the special cases $k = 2$ and $k = 3$ of the following interesting question.

If k and n are fixed nonnegative integers, find an optimal or suboptimal set of $n + 1$ integers

$$0 \leqslant a_0 < a_1 < \cdots < a_n$$

such that all the k-term sums

$$\sum_{t=1}^{k} a_{i_t}$$

are distinct.

Notice that, in an optimal set, $a_0 = 0$.

Babcock himself gave some suboptimal solutions to the first problem for $n \leqslant 9$. 25 years later these solutions were improved and extended to $n \leqslant 23$ by Fang and Sandrin [5], who formulated the problem as a distinct difference problem and applied some results from graph theory and coding theory. Rather less is known about the second problem; in [5] it was studied in terms of a "difference pyramid" but no actual solutions were given.

In this paper we consider both problems. For the case $k = 2$, we establish close lower and upper bounds on the optimal solution, give close to optimal values for $n < 100$, and make some remarks about a related problem. For the case of $k = 3$, we point out that some results given in [4] can be used to construct solutions (even for a general k) and give the results of a computer search based on these constructions for $n \leqslant 17$.

II. THE CASE $k = 2$

A. Distinct Difference Sets

If a set $a_0, \cdots, a_n$ has all its sums $a_i + a_j$, $i \leqslant j$, distinct, then also the differences $a_i - a_j$, $i \neq j$, will be distinct, and vice versa. Accordingly, to study the case $k = 2$ of our general problem we shall say that a set $a_0, \cdots, a_n$ (in increasing order) whose nonzero differences are distinct is a distinct difference set (DDS), and that the DDS is optimal if a_n is as small as possible.

The problem of finding optimal DDS's for each integer n has been investigated by engineers working on radio frequency assignments [5] and coding theory [8], and by mathematicians [6] who have studied the problem for its own sake. Trial-and-error search has yielded an optimal solution for all $n \leqslant 10$. For larger n it has been observed that the theory of perfect difference sets [9] can yield DDS's which seem to be close to optimal. For subsequent use we briefly recall the salient facts.

A perfect difference set (PDS) with parameters (v, k, e) is a set of k integers whose differences modulo v represent every nonzero residue from 1 to $v - 1$ the same number e of times. Obviously, $e(v - 1) = k(k - 1)$. The PDS is planar if $e = 1$; in this case we put $k = n + 1$ so that $v = n^2 + n + 1$. It is known [9] that planar PDS's exist for every prime power n but, despite much research, no planar PDS has been found for other integers n. Clearly, a planar PDS gives rise *a fortiori* to a DDS with $a_n \leqslant n^2 + n$. Moreover, by omitting elements from a planar PDS we obtain DDS's of smaller size.

However, there is another, less well-known construction due to Bose [3] which hitherto has not been applied to the problem. Let q be any prime power, let t be a primitive element in the finite field GF(q^2), and let $t^2 = ut + v$ be its minimal equation over GF(q). Using this equation, each power of t can be expressed as $t^i = u_i t + v_i$ with u_i, v_i in GF(q). Bose proved that $\{i : u_i = 1\}$ is a set of q integers whose differences modulo $v = q^2_{-1}$ are distinct.

In both cases we can obtain, for certain values of n and moduli v, a set $\{a_0, \cdots, a_n\}$ of integers whose differences modulo v are distinct; then also the set $\{b_0, \cdots, b_n\}$ defined by

$$b_i = ca_i + d, \qquad (c, v) = 1$$

has this same property. By a suitable choice of c and d we can often improve an initial set. In this way we have found good DDS's for all $n < 100$.

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B. Bounds

The precise value of a_n in an optimal DDS is difficult to calculate in general. An easy lower bound $a_n \geqslant n(n+1)/2$ follows by observing that each of the $n(n+1)/2$ positive differences are distinct. Our next results improve on this:

Lemma 1: $a_n \geqslant n(t+1)(n-t)/(t+2)$ for any t in $0, \cdots, n$.

Proof: It is obvious that, for any s in $0, \cdots, n$,

$$(s+1)a_n = A_s + B_s$$

where

$$A_s = (a_n - a_{n-s-1}) + (a_{n-1} - a_{n-s-2}) + \cdots + (a_{s+1} - a_0)$$

and

$$B_s = (a_0 + a_1 + \cdots + a_s) + (a_n - a_{n-1}) + (a_n - a_{n-2}) + \cdots + (a_n - a_{n-s}).$$

Hence,

$$\sum_{s=0}^{t} (s+1)a_n \geqslant \sum_{s=0}^{t} A_s.$$

Now observe that the right-hand side is a sum of $n + (n-1) + \cdots + (n-t) = (n-t/2)(t+1)$ differences, all of which are distinct, and therefore

$$\sum (s+1)a_n \geqslant 1+2+3+\cdots+(n-t/2)(t+1).$$

In other words,

$$(t+1)(t+2)a_n \geqslant (n-t/2)(t+1)[(n-t/2)(t+1)+1]$$

and the result follows.

Note: The bound in the lemma can be marginally improved but the one given suffices for the following proposition.

Proposition: In any DDS, $a_n \geqslant n^2 - 2n\sqrt{n}$.

Proof: In the above lemma put $t = [\sqrt{n}]$.

Remark: In view of the numerical results presented in the next subsection, it would be of significant interest to improve the lower bound to $n^2 - n\sqrt{n}$.

We have already observed that, when n is a prime power, the optimal a_n is bounded above by $n^2 + n$. For a general n we have the following upper bound.

Lemma 2: Let $q = n + t$ be a prime power greater than or equal to n. Then there exists a DDS in which $a_n \leqslant n^2 + nt$.

Proof: Let $a_0, a_1, \cdots, a_q$ be a planar PDS for the modulus $v = 1 + q + q^2$. Consider the following $t+1$ identities:

$$(a_1 - a_0) + (a_2 - a_1) + \cdots + (a_q - a_{q-1}) + (a_0 - a_q + v) = v$$

$$(a_2 - a_0) + (a_3 - a_1) + \cdots + (a_q - a_{q-2}) + (a_0 - a_{q-1} + v) + (a_1 - a_q + v) = 2v$$

$$\vdots$$

$$(a_{t+1} - a_0) + \cdots + (a_q - a_{q-t-1}) + (a_0 - a_{q-t} + v) + \cdots + (a_t - a_q + v) = (t+1)v.$$

The bracketed expressions are all positive, less than v, and consequently are distinct by the planar PDS property. Let M be the maximum of these $(q+1)(t+1)$ expressions. It follows that

$$M + (M-1) + \cdots + [M - (q+1)(t+1) + 1] \geqslant v + 2v + \cdots (t+1)v$$

and therefore

$$(q+1)(t+1)M - (q+1)(t+1)[(q+1)(t+1) - 1]/2 \geqslant (t+1)(t+2)v/2$$

from which, using $v = 1 + q + q^2$, we can deduce

$$M \geqslant qt + 3q/2 + t/2.$$

It is now convenient to regard the residues $a_0, a_1, \cdots, a_q$ as being arranged in circular order with a_0 following a_q. The inequality on M states that there are two residues a_i, a_j with subscripts $t+1$ apart (that is, $j - i = t + 1 \bmod q + 1$) with the separation from a_i circularly around to a_j being at least $qt + 3q/2 + t/2$. Then $i - j = (q+1) - (t+1) = n \bmod q + 1$, and the separation from a_j circularly around to a_i is less than $v - (qt + 3q/2 + t/2) < n^2 + nt$ (using $q = n + t$ and $v = 1 + q + q^2$).

But now subtracting the residue a_j from each of $a_0, \cdots, a_q$ gives a new planar PDS $b_0, \cdots, b_q$ with

$$0 = b_0 < b_1 \cdots < b_n = b_n - b_0 = a_i - a_j < n^2 + nt.$$

Remarks:

1) This lemma can be slightly improved by taking more care with the inequalities; for example, it can be shown that, if n is a prime power, there is a PDS with $a_n \leqslant n^2 - n/2$.

2) The numerical evidence in the next section strongly suggests that a DDS with $a_n \leqslant n^2$ always exists. However, to establish this, the proof technique of Lemma 2 will probably need to be enhanced by exploiting that PDS's retain their property when multiplied by a residue coprime to v.

3) It is proved in [7] that, for any sufficiently large n, there exists $t \leqslant n^{7/12}$ with $n + t$ prime. Consequently we have the following.

Theorem: For optimal DDS's, the numbers a_n are asymptotic to n^2 as $n \to \infty$.

Proof: The proposition and Lemma 1 give

$$1 - 2/\sqrt{n} < a_n/n^2 < 1 + n^{19/12}/n^2.$$

C. Numerical Results

The method given earlier of transforming planar PDS's or the sets given by Bose's construction by mappings $x \to cx + d$, and truncating them if n is not a prime power, has been carried out for all $n < 100$. It gave the values for a_n shown in Table I. (The entry in row i, column j is the value a_n for $n = 10i + j$.)

The associated DDS's are listed in full in [1]. Our belief that these sets are close to optimal is based on two observations:

1) For $n \leqslant 10$, where optimal DDS's have been found by exhaustive search, the only cases where the above a_n's are not minimal are a_7 and a_8, which can each be reduced by 1, and a_6 which can be reduced by 2.

2) The values a_n above agree closely with the expression $n^2 - n\sqrt{n}$, whereas the previous subsection demonstrated that $a_n \geqslant n^2 - 2n\sqrt{n}$.

A table of the best sets then known for $n \leqslant 23$ was given in [5]. Our results give the following improvements for n in this range:

$n=12$ {0, 3, 11, 38, 40, 47, 62, 72, 88, 92, 93, 105, 111}

$n=16$ {0, 5, 7, 17, 52, 56, 67, 80, 81, 100, 122, 138, 159, 165, 168, 191, 199}

TABLE I

	0	1	2	3	4	5	6	7	8	9
0	0	1	3	6	11	17	28	35	45	55
1	72	85	111	127	155	179	199	216	246	283
2	333	356	372	425	480	492	553	585	623	680
3	747	784	859	938	987	1005	1099	1146	1252	1282
4	1305	1397	1507	1596	1687	1703	1804	1915	1958	2094
5	2190	2270	2347	2373	2598	2725	2773	2851	2911	3019
6	3134	3215	3391	3527	3593	3757	3819	3956	4145	4217
7	4330	4478	4513	4753	4982	5089	5204	5299	5408	5563
8	5717	5814	6020	6159	6410	6537	6708	6745	6778	6967
9	7542	7617	7726	7884	7967	8121	8357	8509	8540	8831

TABLE II

n	e=2	e=3
1	0, 1	0, 1
2	0, 1, 2	0, 1, 2
3	0, 1, 2, 4	0, 1, 2, 3
4	0, 1, 3, 5, 6	0, 1, 2, 3, 5
5	0, 1, 4, 6, 8, 9	0, 1, 2, 4, 5, 7
6	0, 1, 4, 6, 10, 11, 13	0, 1, 2, 4, 5, 8, 10
7	0, 1, 4, 6, 10, 15, 17, 18	0, 1, 3, 4, 6, 8, 12, 13
8	0, 1, 4, 6, 11, 13, 19, 22, 23	0, 1, 3, 5, 9, 10, 12, 15, 16
9	0, 1, 3, 8, 10, 14, 20, 25, 28, 29	0, 1, 2, 5, 8, 11, 13, 15, 19, 20

TABLE III

n	a_n	$\{a_0, \ldots, a_n\}$
1	1	0, 1
2	4	0, 1, 4
3	11	0, 1, 8, 11
4	27	0, 1, 7, 16, 27
5	56	0, 14, 18, 45, 51, 56
6	120	0, 52, 73, 97, 106, 114, 120
7	156	0, 7, 15, 18, 72, 116, 154, 156
8	257	0, 18, 20, 49, 86, 128, 204, 229, 257
9	405	0, 80, 81, 85, 103, 223, 317, 337, 368, 405
10	591	0, 21, 30, 80, 91, 107, 224, 405, 563, 567, 591
11	737	0, 3, 35, 71, 88, 101, 228, 378, 582, 591, 628, 737
12	1013	0, 45, 72, 149, 309, 355, 379, 518, 538, 786, 983, 1000, 1013
13	1235	0, 129, 152, 212, 335, 349, 362, 733, 854, 971, 982, 1030, 1184, 1235
14	1873	0, 43, 152, 221, 434, 601, 656, 748, 1385, 1433, 1581, 1615, 1766, 1823, 1873
15	2154	0, 104, 137, 293, 339, 441, 874, 1037, 1157, 1161, 1447, 1772, 1849, 1962, 1967, 2154
16	2491	0, 240, 507, 581, 915, 968, 1143, 1212, 1997, 2015, 2078, 2193, 2252, 2266, 2383, 2467, 2491
17	2959	0, 29, 124, 136, 496, 611, 717, 751, 893, 1365, 1931, 2023, 2051, 2248, 2346, 2471, 2952, 2959

$n=21$ $\{0, 3, 15, 26, 65, 86, 93, 103, 133, 152, 177, 197, 228, 232, 234, 250, 286, 313, 342, 347, 355, 356\}$.

D. A Related Problem

A natural generalization of the distinct difference property is to allow the differences to represent each value at most e times for some small constant e. The proof of Lemma 1 generalizes easily to accommodate this hypothesis, and consequently one can obtain the lower bound

$$a_n \geqslant n^2/e - \text{lower order terms.}$$

However, comparably good upper bounds are not so easy to obtain, since (v, k, e) PDS's with fixed $e > 1$ are not so abundant. By trial-and-error search we have found the optimal sets given in Table II.

III. The Case $k = 3$

A. Construction

For the case $k = 3$ (and, in fact, for all k) there are two constructions in [4] which produce good solutions. These constructions generalize the planar difference set construction and the construction of Bose and depend on primitive elements in $GF(q^4)$ and $GF(q^3)$. Such a primitive element satisfies a quartic or a cubic equation with coefficients in $GF(q)$ and this equation is used to define certain powers of the element; these powers are the required integers. Both methods are well suited to computation and give, respectively:

i) a set of $q + 1$ integers whose triple sums modulo $v = q^3 + q^2 + q + 1$ are distinct,

ii) a set of q integers whose triple sums modulo $v = q^3 - 1$ are distinct.

B. Bounds

It follows almost immediately from the remarks above that, for all n, there exists a set of integers $a_0 < a_1 < \cdots < a_n$ whose triple sums are distinct with $a_n < n^3 + o(n^3)$. We have also the following lower bound.

Lemma 3: If $a_0 < a_1 < \cdots < a_n$ is a set of nonnegative integers with all sums $a_i + a_j + a_k$ distinct, then

$$a_n > \frac{10}{57} n^3.$$

Proof (Sketch): The expressions $a_r + a_s - a_t$, $r > s$, $r > t$, $s \neq t$ are easily seen to be distinct and positive. There are $\sum r(r-1) = (n^3 - n)/3$ such expressions, and hence

$$1+2+3+\cdots+(n^3-n)/3 \leq \sum (a_r+a_s-a_t) = \sum r(r-1)a_r (1).$$

The set $a_n - a_n$, $a_n - a_{n-1}$, $\cdots$, $a_n - a_0$ is also a set of increasing integers with distinct triple sums; and so, by induction $a_n - a_r \geqslant (n-r)^3 10/57$, or $a_r \leqslant a_n - (n-r)^3 10/57$ if $r > 0$. Using this last inequality in (1) now gives the result.

C. Numerical Results

Using the constructions mentioned above, we have found, as shown in Table III, the suboptimal sets of integers whose triple sums are distinct. The corresponding frequency assignments are therefore free of both third-order and fifth-order intermodulation interference.

It is of interest to know whether the limit $\lim_{n\to\infty} a_n/n^3$ exists and, if so, its value. By analogy with the case $k = 2$, one might conjecture that (the optimal) a_n is asymptotic to n^3.

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An Asymptotically Optimal Receiver for Heterodyne Optical Communication

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Abstract—**An incoherent receiver is derived for the heterodyne optical channel under the assumption of phase coherence at the start of each transmitted sequence. The receiver is optimal in the limit of small bit intervals with respect to the coherence time of the laser oscillators, and reduces to that proposed by Jeromin and Chan [1] when no initial phase coherence is present. Computer simulations indicate that the small amount of performance improvement obtained by resolving an initial phase uncertainty may not justify the extra complexity needed, for the data rates currently considered.**

I. Introduction

Recent developments in semiconductor lasers and other technology-related issues have focused considerable attention on heterodyne optical systems, both for fiber-optic as well as intersatellite link (ISL) applications.

One of the main problems that faces heterodyne system designers is the large phase noise associated with semiconductor laser oscillators. Even with the advent of semiconductor laser technology, the laser phase noise results in a spread of the intermediate-frequency (IF) spectrum of the laser, with typical linewidths in excess of 10 MHz. Although these linewidths are small compared to the optical carrier frequency, they are sometimes too large to be neglected compared to the data rates of a few megabits per second envisioned for some ISL applications. It is clear then that to avoid excessive performance degradation, the receiver design must account for laser phase noise. In a recent publication [1], Jeromin and Chan proposed that an incoherent receiver be used in heterodyne optical systems with unstable local oscillators. We show next that according to their channel model, the asymptotically optimal statistic consists of *two* terms. The first term corresponds to the one used in [1]. The second term vanishes only if an initial phase uncertainty is included at the start of each transmitted sequence.

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II. Channel Model

For a more complete description of the channel model assumed in this paper, the reader is referred to [2] and [3]. A summary of the model is given by the following equations and the explanation of the various quantities that follows.

$$r(t)=S(x,\ t,\ I_k(t))+\sqrt{\frac{N_0}{2}}\,\dot{u}(t) \tag{1}$$

$$S(x,\ t,\ I_k(t))=\sqrt{2P_{IF}}\ \sin\ [2\pi f_{IF}t+x(t)+I_k(t)] \tag{2}$$

$$dx(t)=\frac{1}{\sqrt{\tau_c}}\,dw(t),\ \ x(0)=\theta. \tag{3}$$

In (1), $r(t)$ is the observation process, and it is the sum of the signal process S and white Gaussian noise of spectral density $N_0/2$; $I_k(t)$ is the kth modulation sequence of length N symbols, with each symbol of duration T seconds derived in general from a Q-ary alphabet. The signal process S, described explicitly in (2), is a sinusoid at an electrical intermediate frequency f_{IF}, whose phase is corrupted by the random process $x(t)$, described by the Ito equation in (3). In integral form, (3) implies that $x(t)$ is the sum of a Wiener process $W(t) = (1/\sqrt{\tau_c})w(t)$ modeling the phase instability and a random variable θ, uniform in $(-\pi, \pi)$, modeling the absence of an absolute phase reference between transmitter and receiver at the start of each transmitted sequence of length N symbols. Notice that if the receiver has provisions for resolving the initial phase uncertainty θ, then (2) applies with $x(t) = W(t)$. The parameter τ_c will be referred to as the coherence time of the transmitter and receiver oscillators, and it is related to the transmitter and receiver linewidths, f_{Lt} and f_{Lr}, respectively, by [3]

$$\tau_c=\frac{1}{2\pi}\,\frac{1}{f_{Lt}+f_{Lr}}. \tag{4}$$

Thus, τ_c is a parameter that can be obtained relatively easily from experimental measurements and describes the quality of the transmitter and receiver laser oscillators. The motivation for modeling the laser phase noise as a Wiener process is that, for this model, the power spectrum of S can be shown to be Lorentzian, which matches the experimentally observed Lorentzian spectrum [4].

III. Derivation of the Incoherent Receiver

From the observation equation (1), it is obvious that given the phase noise $x(t)$, the maximum likelihood functional is well known (see, for example, Van Trees [5, ch. 4]) and is given by

$$l^n(I_j/x_{0,t})$$

$$=\exp\left\{\frac{2\sqrt{2P_{IF}}}{N_0}\int_{(n-1)T}^{nT} r(t)\cdot\sin\ [2\pi f_{IF}t+x(t)+I_{jn}(t)]\ dt\right.$$

$$\left.-\frac{2P_{IF}}{N_0}\int_{(n-1)T}^{nT}\sin^2\ [2\pi f_{IF}t+x(t)+I_{jn}(t)]\ dt\right\}. \tag{5}$$