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THE OUTER CAPACITY OF AN INTERNAL SET FUNCTION

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1. Introduction

In [5], LOEB showed how to obtain a classical probability space via a standard part mapping starting from an internal probability space in a nonstandard universe. In this work we follow LOEB's approach in order to obtain a classical capacity by taking the standard part of a very simple set function defined on an internal paving. We give here the necessary background on capacities, but only a few facts on nonstandard extensions of superstructures. Good expositions of the later subject can be found in [1], [4], [6].

Let $\mathcal{F}$ be a family of subsets of a set F. If $\mathcal{F}$ is closed with respect to finite unions and intersections, and, furthermore, contains the empty set $\emptyset$, then we say that $\mathcal{F}$ is a *paving over F*. A function $C\colon \mathcal{P}(F) \to \overline{\mathbb{R}}$, where $\mathcal{P}(F)$ is the power set of F, is an *$\mathcal{F}$-capacity* if

(i) C is increasing, i.e. $A \subseteq B$ implies $C(A) \leqq C(B)$,

(ii) for every increasing sequence $(A_n)_{\mathrm{N}}$ of subsets of F,

$$C(\textstyle\bigcup_{\mathrm{N}} A_n) = \sup_{\mathrm{N}} C(A_n),$$

(iii) for every decreasing sequence $(A_n)_{\mathrm{N}}$ of elements of $\mathcal{F}$,

$$C(\textstyle\bigcap_{\mathrm{N}} A_n) = \inf_{\mathrm{N}} C(A_n).$$

Under certain conditions it is possible to extend a function C defined originally only on a paving $\mathcal{F}$ to an $\mathcal{F}$-capacity defined on $\mathcal{P}(F)$: Let $C\colon \mathcal{F} \to \overline{\mathbb{R}}_+$ be increasing and strongly subadditive (i.e. for every $A, B \in \mathcal{F}$, $C(A \cup B) + C(A \cap B) \leqq C(A) + C(B)$). Let us suppose in addition that C is upper-continuous, that is, for every increasing sequence $(A_n)_{\mathrm{N}}$ of elements of $\mathcal{F}$ whose union $\bigcup_{\mathrm{N}} A_n$ also belongs to $\mathcal{F}$, one has $C(\bigcup_{\mathrm{N}} A_n) = \sup_{\mathrm{N}} C(A_n)$. For every $A \in \mathcal{F}_\sigma$ (the closure of $\mathcal{F}$ with respect to enumerable unions), we define

$$C^\star(A) := \sup_{B \in \mathcal{F},\, B \subseteq A} C(B),$$

and, for every $D \in \mathcal{P}(F)$, we define

$$C^\star(D) := \inf_{D \subseteq A \in \mathcal{F}_\sigma} C^\star(A),$$

where $\inf \emptyset := +\infty$.

The theorem of construction of capacities [2] says that $C^\star$ is an $\mathcal{F}$-capacity (the *outer capacity associated with* C) if and only if, for every decreasing sequence $(A_n)_{\mathrm{N}}$ of elements of $\mathcal{F}$,

$$C^\star(\textstyle\bigcap_{\mathrm{N}} A_n) = \inf_{\mathrm{N}} C(A_n). \tag{1}$$

In what follows, we will work in an $\aleph_1$-saturated, nonstandard extension $V(^*S)$ of the superstructure $V(S)$ of a set S which contains the real numbers. $V(S)$ consists of all relations over S, of any finite arity and any finite level, e.g. $\mathcal{P}(S)$ and $\mathcal{P}(\mathcal{P}(S))$ belong to $V(S)$. Let us recall that an extension $V(^*S)$ contains $^*\mathbb{R}$ (the set of hyperreal numbers) and $^*\mathbb{N}$ (the set of hyperintegers) as elements. $^*\mathbb{R}$ is a proper extension of $\mathbb{R}$ and $\mathbb{N}$ is a proper initial segment of $^*\mathbb{N}$. $^*\mathbb{N}$ and $^*\mathbb{N}-\mathbb{N}$ are external elements of $V(^*S)$. The mapping $^\circ\colon {}^*\mathbb{R}\to\bar{\mathbb{R}}$, which associates to every finite hyperreal d the unique real c that is infinitely close to it ($c={}^\circ d$), is called the *standard part.* Finally, we recall that $V(^*S)$ is $\aleph_1$*-saturated* iff, for every sequence $(a_n)_{n\in\mathbb{N}}$ of internal objects of $V(^*S)$, there is an internal sequence $(b_n)_{n\in{}^*\mathbb{N}}$ which extends it ($b_n=a_n$, for every $n\in\mathbb{N}$). The existence of nonstandard extensions with these properties is proved in [3].

2. Non-standard capacities with non-standard pavings

Let F be an internal set and $\mathcal{F}$, an internal paving over F. Using the $\aleph_1$-saturation of the extension $V(^*S)$, Loeb [5] proved the following proposition for the case where $\mathcal{F}$ is an internal algebra of sets. However, the same proof works when $\mathcal{F}$ is an internal paving.

2.1. Proposition. *For every* $n\geqq 0$, $n\in\mathbb{N}$, *let* $A_n\in\mathcal{F}$. *If* $A_0\subseteqq\bigcup_{1\leqq n\in\mathbb{N}}A_n$, *then, for some* $m\in\mathbb{N}$,

$$A_0\subseteqq\bigcup_{n=1}^m A_n. \ \square$$

2.2. Lemma. *If* $(A_n)_{1\leqq n\in\mathbb{N}}$ *is a strictly increasing sequence of elements of* $\mathcal{F}$, *then*

$$\bigcup_{1\leqq n\in\mathbb{N}}A_n\notin\mathcal{F}.$$

Proof. If $A_0:=\bigcup_{1\leqq n\in\mathbb{N}}A_n\in\mathcal{F}$, then, by proposition 2.1, $A_0\subseteqq\bigcup_{n=1}^m A_n$ for some $n\in\mathbb{N}$. This contradicts the hypothesis that the original sequence is strictly increasing. $\square$

Now, let $J\colon\mathcal{F}\to{}^*\mathbb{R}_+$ be internal, increasing and strongly subadditive. Then, we have in our nonstandard extension an internal triple $(F,\mathcal{F},J)$ that satisfies very simple hypothesis. We point out that it is easy to verify, using $\aleph_1$-saturation, that the paving $\mathcal{F}$ is semi-compact. We will not use this property of $\mathcal{F}$. In a second step, starting from this triple, we can obtain a classical capacity based on paving $\mathcal{F}$ in a very natural way. In order to do this, let us define $I(A):={}^\circ(J(A))$, for every $A\in\mathcal{F}$.

2.3. Lemma. $I\colon\mathcal{F}\to\bar{\mathbb{R}}_+$ *is increasing, strongly subadditive, and upper-continuous.*

Proof. The upper-continuity follows trivially from lemma 2.2. $\square$

2.4. Proposition. *The mapping* $I^\star\colon\mathcal{P}(F)\to\bar{\mathbb{R}}_+$ *associated with* I *is an* $\mathcal{F}$*-capacity.*

Proof. By lemma 2.3, it suffices to verify condition (1), that is to say, for every decreasing sequence $(A_n)_\mathbb{N}$ of elements of $\mathcal{F}$,

$$I^\star(\bigcap_\mathbb{N}A_n)=\inf_\mathbb{N}I(A_n).$$

This condition can be reformulated [2] directly in terms of I, the original mapping: if $(A_n)_\mathbb{N}$, $(B_n)_\mathbb{N}$ are respectively decreasing and increasing sequences of elements of $\mathcal{F}$, such that $\bigcap_\mathbb{N}A_n\subseteqq\bigcup_\mathbb{N}B_n$, then $\inf_\mathbb{N}I(A_n)\leqq\sup_\mathbb{N}I(B_n)$. We will show that this condition is satisfied by I.

Let $A:=\bigcap_\mathbb{N}A_n$, $B:=\bigcup_\mathbb{N}B_n$. As A_n, $B_n\in\mathcal{F}$ and $\mathcal{F}$ is internal, we know that A_n, B_n are internal. By the saturation condition on the nonstandard extension, there are internal extensions $(A_n)_{*\mathbb{N}}$, $(B_n)_{*\mathbb{N}}$ of $(A_n)_{\mathbb{N}}$, $(B_n)_{\mathbb{N}}$, respectively. Let us define

$$K_1 := \{i \in {}^*\mathbb{N} \mid \forall j \leqq i \colon A_j \in \mathcal{F} \wedge A_{j-1} \supseteqq A_j\},$$

$$K_2 := \{i \in {}^*\mathbb{N} \mid \forall j \leqq i \colon B_j \in \mathcal{F} \wedge B_{j-1} \subseteqq B_j\},$$

K_1 and K_2 are internal sets because they are defined by conditions expressed in terms of internal parameters and bounded quantifiers. Furthermore, $K_1, K_2 \supseteqq \mathbb{N}$. Since $\mathbb{N}$ is external, there are $\gamma, \omega \in {}^*\mathbb{N} - \mathbb{N}$, such that $\gamma \in K_1$, $\omega \in K_2$. Thus,

$$A_1 \supseteqq A_2 \supseteqq \ldots \supseteqq A_{\gamma-1} \supseteqq A_\gamma \in \mathcal{F} \qquad \text{and} \qquad B_1 \subseteqq B_2 \subseteqq \ldots \subseteqq B_{\omega-1} \subseteqq B_\omega \in \mathcal{F}.$$

For $\eta := \min\{\gamma, \omega\}$ $(\in {}^*\mathbb{N} - \mathbb{N})$,

$$A_1 \supseteqq A_2 \supseteqq \ldots \supseteqq A_{\eta-1} \supseteqq A_\eta \qquad \text{and} \qquad B_1 \subseteqq B_2 \subseteqq \ldots \subseteqq B_{\eta-1} \subseteqq B_\eta.$$

In addition, $\forall n \in \mathbb{N}$, $\forall j \leqq \eta, j \in {}^*\mathbb{N} - \mathbb{N}$: $A_n \supseteqq A_j$ and $B_n \subseteqq B_j$. Since $\bigcup_{\mathbb{N}} B_n \supseteqq \bigcap_{\mathbb{N}} A_n$, we have $\forall j \leqq \eta, j \in {}^*\mathbb{N} - \mathbb{N}$: $B_j \supseteqq A_j$. Thus, $\forall j \leqq \eta, j \in {}^*\mathbb{N} - \mathbb{N}$: $J(B_j) \geqq J(A_j)$. The set

$$K := \{j \in {}^*\mathbb{N} \mid j \leqq \eta \wedge J(A_j) \leqq J(B_j)\}$$

is internal and satisfies $\{1, \ldots, \eta\} \supseteqq K \supseteqq \{1, \ldots, \eta\} \cap ({}^*\mathbb{N} - \mathbb{N})$. This last set is external, so that there is an $m \in \mathbb{N}$, such that $m \in K$. For this m we have $J(A_m) \leqq J(B_m)$. Thus, $I(A_m) \leqq I(B_m)$. Therefore, we finally obtain

$$\inf_{\mathbb{N}} I(A_n) \leqq I(A_m) \leqq I(B_m) \leqq \sup_{\mathbb{N}} I(B_n). \ \square$$

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