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CHUAQUI'S DEFINITION OF PROBABILITY IN SOME STOCHASTIC PROCESSES

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ABSTRACT. Models for Markov Dependent Bernoulli Trials, Markov Chains, Random Walks and Brownian Motion are constructed in the framework of Chuaqui's Definition of probability.

Chuaqui 1980 and 1981 explains how a semantical definition of probability can be applied to random experiments that give rise to compound outcomes. In order to do this, he introduces what he calls "compound probability structures" (CPS). These CPS are based on causal trees of the form (T,R) where T is a nonempty set and R is a partial order in T which reflects the causal dependence relation between the simple outcomes which make up the compound outcome.

In the applications we are interested in, the elements of T are time moments and R is a the natural linear order $\leqslant$.

A compound outcome is a function f with domain T for which $f(t)$ is an outcome in a simple probability structure (SPS) (see Chuaqui 1977 and 1981). Starting with known probability measures on these SPS, he defines a probability measure on the set of compound outcomes (see Chuaqui 1980).

In what follows we show how this definition works for some known stochastic processes.

1. MARKOV DEPENDENT BERNOULLI TRIALS (MDBT)

We repeat n times an experiment which has only two possible outcomes, s and f (for success and failure). We assume that $p_{s,f}$ is the probability of f on the $(k+1)$-st trial, given that the outcome was s on the k-th trial, and that the analogously defined probabilities $p_{s,s}$, $p_{f,s}$, $p_{f,f}$ are known and independent of k. We also assume the initial probabilities p_s, p_f to be known.

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Here $T = \{t_0,\ldots,t_{n-1}\}$, and we consider in T the natural order relation: $t_i \leqslant t_j$ if and only if $i \leqslant j$. We associate with each $t_i \in T$, $i > 0$, two simple probability structures $\mathbf{K}^s$ and $\mathbf{K}^f$. The models in $\mathbf{K}^s$ are of the form $\langle A^s, S^s, F^s, \{a\}\rangle$, with A^s a non-empty set, $\{S^s, F^s\}$ a fixed partition of A^s where the proportion of elements of A^s which are in S^s is $p_{s,s}$ and that of the elements which are in F^s is $p_{s,f}$, and a an element of A^s.

The appearance of an element of S^s gives outcome s and the appearance of an element of F^s gives outcome f.

We write $\mathfrak{A}^s_s$ for $\{\langle A^s, S^s, F^s, \{a\}\rangle\colon a \in S^s\}$ and $\mathfrak{A}^s_f$ for $\{\langle A^s, S^s, F^s, \{a\}\rangle\colon a \in F^s\}$ and simplifying, we write $\mathbf{K}^s = \{\mathfrak{A}^s_f, \mathfrak{A}^s_s\}$. Analogously, we define $\mathbf{K}^f$ and write $\mathbf{K}^f = \{\mathfrak{A}^f_f, \mathfrak{A}^f_s\}$; $\mathfrak{A}^f_f$ and $\mathfrak{A}^s_f$ give outcome f; $\mathfrak{A}^f_s$ and $\mathfrak{A}^s_s$ give outcome s.

To complete our formulation, we associate with t_0 the simple probability structure $\mathbf{K}_0 = \{\mathfrak{A}^0_s, \mathfrak{A}^0_f\}$, where the definition of $\mathfrak{A}^0_s$ and $\mathfrak{A}^0_f$ is analogous to that of $\mathfrak{A}^s_s$ and $\mathfrak{A}^s_f$, respectively.

The following probabilities follow immediately from the above definition:

$$
\begin{array}{ll}
\mu(\mathfrak{A}^s_f) = p_{s,f} & \mu(\mathfrak{A}^s_s) = p_{s,s} \\
\mu(\mathfrak{A}^f_f) = p_{f,f} & \mu(\mathfrak{A}^f_s) = p_{f,s} \\
\mu(\mathfrak{A}^0_s) = p_s & \mu(\mathfrak{A}^0_f) = p_f .
\end{array}
$$

A *compound outcome* is a function f which satisfies the following conditions:

a) Domain of $\mathrm{f} = T$,

b) $\mathrm{f}(t_0) \in \mathbf{K}_0$,

c) $\mathrm{f}(t_k) \in \mathfrak{A}^*_f$ implies $\mathrm{f}(t_{k+1}) \in \mathbf{K}^f$ and $\mathrm{f}(t_k) \in \mathfrak{A}^*_s$ implies $\mathrm{f}(t_{k+1}) \in \mathbf{K}^s$,

with $* \in \{s,f\}$ for $k > 0$ and $* = 0$ for $k = 0$.

The compound probability structure corresponding to these MDBT is $\boldsymbol{H} = \langle T, \leqslant, \mathbf{H}\rangle$ where $\mathbf{H}$ is the set of all functions that satisfy (a)-(c). On the basis of the probabilities assigned above and the relation $\leqslant$ in T, we define a probability measure μ on $\mathbf{H}$. In the case of MDBT, it is interesting to calculate the probabilities:

$$
\begin{aligned}
p^k_s &= \text{probability of } s \text{ on the } k\text{-th trial} \\
p^k_f &= \text{probability of } f \text{ on the } k\text{-th trial.}
\end{aligned}
$$

To do this, it is enough to solve a difference equation whose derivation is based on the "total probability theorem" which can be formulated and proved in this context in the usual fashion.

Clearly, the situation corresponding to Markov Chains can be formulated in a form completely analogous to that of MDBT. In considering Markov Chains, it is merely necessary to choose a greater number of simple probability structures associated with each moment of time and a greater number of transition probabilities.

2. RANDOM WALKS.

Let us consider a one dimensional random walk which, starting from the origin, is controlled by a coin thrown n times, where the step size is constant and equal to 1.

Here $T = \{t_0, \ldots, t_n\}$, T ordered as in §1. We associate with t_0 the simple probability structure $\mathbf{K} = \{\mathfrak{A}_0\}$ with $\mathfrak{A}_0 = \langle\{0\},\{0\}\rangle$. For each t_k, $k > 0$, define

$$\boldsymbol{u}_{t_k} := \{-k, -k+2, \ldots, k-2, k\} \text{ and } \boldsymbol{u}_{t_0} := \{0\}.$$

A *random walk* (a compound outcome) is a function f which satisfies the following conditions:

a) Domain of $f = T$,

b) $f(t_0) = \mathfrak{A}_0$,

c) $f(t_k)$ is a model of the form $\langle\boldsymbol{u}_{t_k}, \{\lambda\}\rangle$ with $\lambda \in \boldsymbol{u}_{t_k}$,

d) $f(t_k) = \langle\boldsymbol{u}_{t_k}, \{\lambda\}\rangle$ implies $f(t_{k+1}) \in \mathbf{K}_{k+1,\lambda} := \{\langle\boldsymbol{u}_{t_{k+1}}, \{\lambda+1\}\rangle, \langle\boldsymbol{u}_{t_{k+1}}, \{\lambda-1\}\rangle\}$.

With each $t_k \in T$, $k > 0$, we associate a family of SPS with the same similarity type and a common universe, namely the family $\{\mathbf{K}_{k,\lambda} : \lambda \in \boldsymbol{u}_{t_{k-1}}\}$.

The probability in $\mathbf{K}_{k,\lambda}$ is uniformely distributed if the coin which controls the walk is unbiased, but, in general,

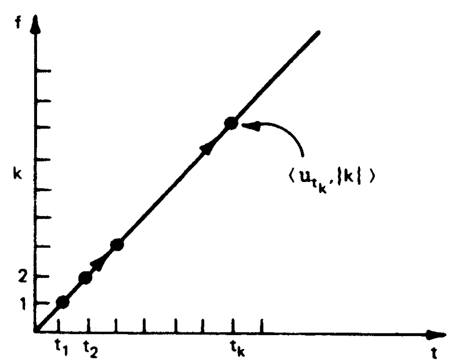
$$\mu(\langle\boldsymbol{u}_{t_k}, \{\lambda+1\}\rangle) = p \quad \text{and}$$

$$\mu(\langle\boldsymbol{u}_{t_k}, \{\lambda-1\}\rangle) = 1-p \quad \text{for each } k \text{ and for each } \lambda \in \boldsymbol{u}_{t_{k-1}}.$$

we assign to $\mathfrak{A}_0$ the probability 1.

The CPS corresponding to this kind of random walk is $\mathcal{H} = \langle T, <, \mathbf{H}\rangle$, where $\mathbf{H}$ is the set of all functions which satisfy (a)-(d). On $\mathbf{H}$ one obtains a probability measure μ determined by $\mathcal{H}$ and the probabilities assigned above to the SPS $\mathbf{K}_{k,\lambda}$.

We can calculate probabilities according to Chuaqui 1980, such as, for example, the probability of the path $f \in \mathbf{H}$ shown in the given figure.



We show that $\mu(f) = p^n$, as expected. In Chuaqui 1980, the measure μ on $\mathbf{H}$ is defined by induction on ordinals.

Let $g \in \mathbf{H}$, $t \in T$ and $T_t := \{s \in T : s \leqslant t, s \neq t\}$, then $\mathbf{H}(g,t) := \{h(t) : h \in \mathbf{H}, g \upharpoonright T_t = h \upharpoonright T_t\}$ is an SPS where a probability measure with values p and $1\text{-}p$ is defined. Denote this measure by $\mu_{g,t}$. We need some definitions from Chuaqui 1980:

T'_α is the set of all minimal elements of $T\text{-}\cup\{T'_\beta : \beta \subset \alpha\}$

$T_\alpha := \cup\{T'_\beta : \beta \subset \alpha\}$; $\bar{T}_\alpha := \cup\{T'_\beta : \beta \subseteq \alpha\}$, α ordinal.

$\bar{T}_t := T_t \cup \{t\}$, $t \in T$.

$\mathbf{H}(S) := \{f \upharpoonright S : f \in \mathbf{H}\}$, $S \subseteq T$.

$\mathbf{A}(S) := \{f \upharpoonright S : f \in \mathbf{A}\}$, $S \subseteq T$, $\mathbf{A} \subseteq \mathbf{H}$.

Then we have $T'_i = \{t_i\}$, $T_i = \{t_0,\ldots,t_{i-1}\}$, $\bar{T}_i = \{t_0,\ldots,t_i\}$.

We have to find the measure μ on $\mathbf{H}$. Clearly $\mathbf{H} = \mathbf{H}(\bar{T}_{t_n})$. Then the measure on $\mathbf{H}$ is $\bar{\mu}_{t_n}$ which is defined for $\mathbf{A} \subseteq \mathbf{H}$ by

$$\bar{\mu}_{t_n}(\mathbf{A}(\bar{T}_{t_n})) = \int_{\mathbf{A}(T_{t_n})} \mu_{g,t_n}(\mathbf{A}(g,t_n))\,d\mu_{t_n}$$

where $\mathbf{A}(g,t_n) := \{h(t_n) : h \in \mathbf{A}$ and $h \upharpoonright T_{t_n} = g \upharpoonright T_{t_n}\}$. In our case $\mathbf{A} = \{f\}$, so that

$$\begin{aligned}\mu(f) = \bar{\mu}_{t_n}(\{f\}) &= \int_{\{f \upharpoonright T_{t_n}\}} \mu_{g,t_n}(\mathbf{A}(g,t_n))\,d\mu_{t_n} \\ &= \mu_{f,t_n}(f(t_n)) \cdot \mu_{t_n}(f \upharpoonright T_{t_n}) \\ &= p \cdot \mu_{t_n}(f \upharpoonright T_{t_n}).\end{aligned}$$

The measure μ_{t_n} is defined by

$$\mu_{t_n} = \pi\langle \bar{\mu}_s : s \in T'_{n-1} \text{ and } s < t_n \rangle = \bar{\mu}_{t_{n-1}}.$$

Thus, $\mu(f) = p \cdot \bar{\mu}_{t_{n-1}}(f \upharpoonright T_{t_n})$. If we calculate $\bar{\mu}_{t_{n-1}}$ as we calculated $\bar{\mu}_{t_n}$, we have, upon iteration, $\mu(f) = p^n$.

Within this formulation we can prove all the results of Probability Calculus involving random walks.

3. BROWNIAN MOTION.

Our formulation is motivated by the known fact thay by speeding up a random walk it is possible to obtain a good model of Brownian Motion. We avoid this explicit acceleration process using non-standard techniques. Let us consider a Brownian Motion during a unit of time and a ω_1-saturated non-standard extension $V(^*\mathbf{R})$ of the superstructure $V(\mathbf{R})$ of the real numbers.

Let $\eta \in {}^*\mathbf{N} \setminus \mathbf{N}$ be an infinite natural number and $T = \{0, 1/\eta, 2/\eta, \ldots, 1\}$ ordered in the natural way. We associate to each $t_\lambda = \lambda/\eta \in T$ the set

$$\boldsymbol{u}_{t_\lambda} := \{-\frac{\lambda}{\sqrt{\eta}}, \frac{-\lambda+2}{\sqrt{\eta}}, \ldots, \frac{\lambda-2}{\sqrt{\eta}}, \frac{\lambda}{\sqrt{\eta}}\}, \quad \lambda \geqslant 1$$

$$\boldsymbol{u}_0 := \{0\}.$$

Now, if $\alpha \in \boldsymbol{u}_{t_{\lambda-1}}$, $\lambda \geqslant 1$, then $\mathbf{K}_{\lambda,\alpha} := \{\mathcal{a}_{\lambda,\alpha+}, \mathcal{a}_{\lambda,\alpha-}\}$, with $\mathcal{a}_{\lambda,\alpha+} := \langle u_{t_\lambda}, \{\alpha + \frac{1}{\sqrt{\eta}}\}\rangle$ and $\mathcal{a}_{\lambda,\alpha-} := \langle \boldsymbol{u}_{t_\lambda}, \{\alpha - \frac{1}{\sqrt{\eta}}\}\rangle$ is a simple probability structure with $\mu_{\lambda,\alpha}(\mathcal{a}_{\lambda,\alpha+}) = \mu_{\lambda,\alpha}(\mathcal{a}_{\lambda,\alpha-}) = 1/2$.

A possible path of Brownian Motion is a function f such that:

a) Domain of f = T,

b) $f(0) = \langle \boldsymbol{u}_0, \{0\}\rangle$,

c) $f(t_{\lambda-1}) = \langle \boldsymbol{u}_{t_{\lambda-1}}, \{\alpha\}\rangle$ implies $f(t_\lambda) \in \mathbf{K}_{\lambda,\alpha}$, $\lambda \geqslant 1$.

Let **H** be the set of all possible trajectories. On **H** one obtains a probability measure μ induced by the $\mu_{\lambda,\alpha}$'s. As indicated in Chuaqui 1980, μ is defined by induction on ordinals which in this situation may be hyperfinite.

We define random variables $(X_{t_\lambda})_{\lambda=0}^{\eta}$ on **H** by $X_{t_\lambda}(f) := \mathrm{Var.}(f(t_\lambda))$, where $\mathrm{Var.}(f(t_\lambda)) \in {}^*\mathbf{R}$ is the real number that belongs to the variable part of $f(t_\lambda)$. For example, if $f(t_\lambda) = \langle \boldsymbol{u}_{t_\lambda}, \{\alpha\}\rangle$, then $\mathrm{Var.}(f(t_\lambda)) = \alpha$.

Using some results of Anderson 1976, it may be shown that this is a good model for Brownian Motion. Indeed, if $f \in \mathbf{H}$, we define $X_s(f)$ for each $s \in {}^*[0,1]$ by

$$X_s(f) := X_{t_{[\eta s]}}(f) + (\eta s - [\eta s]) \cdot (X_{t_{[\eta s]+1}}(f) - X_{t_{[\eta s]}}(f)).$$

In this way we have a set **H** that contains all possible trajectories, a measure μ defined on **H**, or more precisely, on a family $\boldsymbol{A}$ of subsets of **H** and a family $(X_s)_{s \in {}^*[0,1]}$ of random variables. Furthermore, all these objects (T, the functions f, **H**, $\boldsymbol{A}$, the X_s's) are internal. This is also the case for the measure μ, because it is defined in terms of standard measures and internal ordinals. $(\mathbf{H}, \boldsymbol{A}, \mu)$ is an internal probability space.

Now, we consider Loeb's standard probability space $(\mathbf{H}, L(\boldsymbol{A}), P)$ associated with $(\mathbf{H}, \boldsymbol{A}, \mu)$ (see Loeb 1975). $L(\boldsymbol{A})$ is the σ-algebra generated by $\boldsymbol{A}$, and P is the probability measure defined on $L(\boldsymbol{A})$ and generated by the standard part ${}^o\mu$ of μ. If we now define

$$Y_s(f) := {}^oX_s(f), \quad s \in [0,1],$$

then

$$P(Y_s \leqslant \alpha) = \frac{1}{\sqrt{2\pi s}} \int_{-\infty}^{\alpha} \exp(-y^2/2s)\,dy, \quad \alpha \in \mathbf{R}.$$

In fact,

$$\begin{aligned} P(Y_s \leqslant \alpha) &= P({}^oX_s \leqslant \alpha) \\ &= P({}^oX_{t_{[\eta s]}} \leqslant \alpha) \\ &= \left({}^o\sum_{k=0}^{[\eta s]-1} (X_{t_{k+1}} - X_{t_k}) \leqslant \alpha\right) \end{aligned}$$

$$= P\left(\frac{\sum\limits_{k=0}^{[\eta s]-1} (X_{t_{k+1}} - X_{t_k})}{\sqrt{[\eta s]}\frac{1}{\sqrt{\eta}}} \leqslant \frac{\alpha}{\sqrt{\frac{[\eta s]}{\eta}}}\right).$$

Because the random variables $X_{t_{k+1}} - X_{t_k}$ are independent with mean 0 and variance $1/\eta$, by a non-standard version of central limit theorem (Anderson 1976) one has that the last expression equals

$$\lim_{m\to\infty} {}^{\circ}({}^{*}\psi)\left(\sqrt{\frac{\eta}{[\eta s]}}\,(\alpha + \tfrac{1}{m})\right) = \lim_{m\to\infty} \psi\left(\frac{\alpha + \frac{1}{m}}{\sqrt{s}}\right) = \psi\left(\frac{\alpha}{\sqrt{s}}\right),$$

where ψ is the distribution function of the normal probability law with mean 0 and variance 1. Thus, Y_s has normal distribution $N(0,s)$, with mean 0 and variance s.

It is known (Anderson 1976) that P is an extension of the Wiener measure on $C[0,1]$.

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