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THE FORMAL LANGUAGE L_t AND TOPOLOGICAL PRODUCTS

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1. Preliminaries

We will consider topological structures of the form $\langle \mathfrak{A}, \sigma \rangle$, where $\mathfrak{A}$ is a classical algebraic structure for first order logic and σ is a topology or a basis for a topology on the universe A of $\mathfrak{A}$. Given a set of symbols S compatible with the algebraic parts of the structures under consideration, we construct the formal language L_t^S for expressing properties of those topological structures. L_t^S is constructed as the classical first-order language L^S adding to it the following rules for building formulas:

if t is a (first-order) term of L^S, then $t \in X$ (where X is a second-order variable for open sets) is an atomic formula;

if φ is a formula of L_t^S positive in X and t is a term of L^S, then $(\forall X \ni t)\, \varphi$ is a formula of L_t^S;

if φ is a formula of L_t^S negative in X and t is a term of L^S, then $(\exists X \ni t)\, \varphi$ is a formula of L_t^S.

It is easy to show that the formulas of L_t^S are basis-invariant, i.e. for each $\varphi \in L_t^S$ and structure $\langle \mathfrak{A}, \sigma \rangle$:

$$\langle \mathfrak{A}, \sigma \rangle \models \varphi \quad \text{iff} \quad \langle \mathfrak{A}, \tilde{\sigma} \rangle \models \varphi .$$

Here $\tilde{\sigma}$ is the topology generated by σ: $\tilde{\sigma} = \{\bigcup s \mid s \subseteq \sigma\}$. For historical remarks, precise definitions and other interesting properties of L_t, see [4], [7].

2. The topological product

Let $(\langle \mathfrak{A}_i, \sigma_i \rangle)_{i \in I}$ be topological structures and $D \subseteq P(I)$ a filter on I. We define the *D-product of the* $\langle \mathfrak{A}_i, \sigma_i \rangle$ as the topological structure $\langle \prod_I \mathfrak{A}_i, \tilde{\sigma}_D \rangle$, where $\prod_I \mathfrak{A}_i$ is the direct product of the $\mathfrak{A}_i$ and $\tilde{\sigma}_D$ is the topology generated by the basis

$$\sigma_D = \{\prod_I U_i \mid U_i \in \sigma_i \text{ and } \{i \in I \mid U_i = A_i\} \in D\}.$$

Some facts that are easy to verify are the following:

a) The projection maps p_i are continuous iff $D \supseteq Co(I)$, the filter of cofinite sets.

b) If $D = Co(I)$, then $\tilde{\sigma}_D$ is the Tychonov topology (the usual product topology).

c) If $D = \{I\}$ (the trivial filter), then $\tilde{\sigma}_D$ is the trivial topology.

d) If $D = P(I)$ (the improper filter), then $\tilde{\sigma}_D$ is the box topology.

e) If D, E are filters on I and $D \subseteq E$, then $\tilde{\sigma}_E$ is finer than $\tilde{\sigma}_D$.

f) $\langle \prod_I A_i, \tilde{\sigma}_D \rangle$ is Hausdorff iff $D \supseteq Co(I)$ and all $\langle A_i, \sigma_i \rangle$ are Hausdorff.

If $\mathfrak{P}(I)$ is the Boolean algebra $\langle P(I), \cap, \cup, {}^c, \emptyset, I\rangle$, then the structure $\langle \mathfrak{P}(I), D\rangle$ is a Boolean algebra with distinguished filter D. Let $\mathrm{L}^{\mathrm{BAF}}$ be the first-order language for such algebras.

It is also possible to extend the Feferman-Vaught-type theorem [3] given in [4] for usual topological products to our products:

Theorem. *To each sentence $\varphi \in \mathrm{L}_t^S$ there can be effectively associated a tuple $(\Phi; \theta_1, \ldots, \theta_n)$, where $\Phi(z_1, \ldots, z_n) \in \mathrm{L}^{\mathrm{BAF}}$ and $\theta_1, \ldots, \theta_n \in \mathrm{L}_t^S$ are sentences, such that for all I, $(\langle \mathfrak{A}_i, \sigma_i\rangle)_{i\in I}$, D:*

$$\langle \prod_i \mathfrak{A}_i, \tilde{\sigma}_D\rangle \models \varphi \quad \textit{iff} \quad \langle \mathfrak{P}(I), D\rangle \models \Phi[S(\theta_1), \ldots, S(\theta_n)],$$

where $S(\theta_j) = \{i \in I \mid \langle \mathfrak{A}_i, \sigma_i\rangle \models \theta_j\}$. □

As in [3], we can everywhere assume that $(\Phi; \theta_1, \ldots, \theta_n)$ is a partitioning tuple.

Combining this result with Ershov's theorem [2] on the decidability of the first-order theory of all algebras of the form $\langle \mathfrak{P}(I), D\rangle$, we obtain

Theorem. *If K is a class of topological structures with decidable L_t^S-theory, then the L_t^S-theory of the class of all products (in our sense) of structures in K is also decidable. The same is true if we restrict ourselves to the case where the index sets are infinite and/or the filters extend the corresponding filter of cofinite sets.* □

3. Examples

a) Let $K = \{\langle 2, \sigma\rangle\}$, where 2 is the Boolean algebra with two elements and σ is the discrete topology. Let L^{BA} be the first-order language for Boolean algebras and $\mathrm{L}_t^{\mathrm{BA}}$ the topological language for Boolean algebras with topology. The $\mathrm{L}_t^{\mathrm{BA}}$-theory of K is decidable since the unique structure contained in it is finite. Thus, the class $\{\langle 2^I, \tilde{\sigma}_D\rangle \mid I$ is a set and D is a filter on $I\}$ has a decidable $\mathrm{L}_t^{\mathrm{BA}}$-theory. Furthermore, we shall see that it is possible to classify all its structures according to elementary equivalence.

Theorem. *For all filters D and E on I and J, respectively,*

$$\langle 2^I, \tilde{\sigma}_D\rangle \equiv_{\mathrm{L}_t^{\mathrm{BA}}} \langle 2^J, \tilde{\sigma}_E\rangle \quad \textit{iff} \quad \langle \mathfrak{P}(I), D\rangle \equiv_{\mathrm{L}^{\mathrm{BAF}}} \langle \mathfrak{P}(J), E\rangle.$$

Proof. One direction follows from the Feferman-Vaught-type theorem. The other one is obtained interpreting the $\mathrm{L}^{\mathrm{BAF}}$-theory of $\langle \mathfrak{P}(I), D\rangle$ in the $\mathrm{L}_t^{\mathrm{BA}}$-theory of $\langle 2^I, \tilde{\sigma}_D\rangle$. More precisely, to each formula $\varphi(\boldsymbol{x}) \in \mathrm{L}^{\mathrm{BAF}}$ we associate a formula $\varphi^*(\boldsymbol{x}) \in \mathrm{L}_t^{\mathrm{BA}}$, such that, for each $I, D, \boldsymbol{a} \in 2^I$:

$$\langle \mathfrak{P}(I), D\rangle \models \varphi[\boldsymbol{a}] \quad \text{iff} \quad \langle 2^I, \tilde{\sigma}_D\rangle \models \varphi^*[\boldsymbol{a}],$$

where $\boldsymbol{x} = (x_1, \ldots, x_n)$, $\boldsymbol{a} = (a_1, \ldots, a_n)$. φ^* is constructed replacing each subformula of φ of the form $t \in D$ by the $\mathrm{L}_t^{\mathrm{BA}}$-formula $(\exists X \ni t)\, \forall x(x \in X \to x \cap t = x)$. □

Therefore, using the classification in elementary-equivalence types of all algebras of the form $\langle \mathfrak{P}(I), D\rangle$ made by Ershov [2], it is possible to classify the algebras of the form $\langle 2^I, \tilde{\sigma}_D\rangle$. In particular, we can see that there are many pairs of them which are not elementary equivalent.

b) Let $(\langle A_i, \sigma_i\rangle)_{i\in I}$, $(\langle B_j, \tau_j\rangle)_{j\in J}$, be T_3-spaces (regular and Hausdorff) which are not singletons. In contrast with that obtained in a), we have

Theorem. *If I, J are infinite and $D \supseteqq Co(I)$, $E \supseteqq Co(J)$, then*

$$\langle \prod_I A_i, \tilde{\sigma}_D \rangle \equiv_{L_t} \langle \prod_I B_j, \tilde{\tau}_E \rangle.$$

Proof. $\langle \prod_I A_i, \tilde{\sigma}_D \rangle$ and $\langle \prod_J B_j, \tilde{\tau}_E \rangle$ are T_3-spaces without isolated points. It is known [4] that all T_3-spaces without isolated points are L_t-equivalent. □

c) Let us consider the non-regular T_2 topological space $\langle \mathsf{R}, \sigma \rangle$, where the basic neighbourhoods of points different from 0 are as usual, but the basic neighbourhoods of 0 are of the form $(c, d) \setminus \{\pm 1/k \mid k \in \mathsf{N}\}$. It is possible to show that the L_t-theory of $\langle \mathsf{R}, \sigma \rangle$ is decidable. Consequently, the theory of all topological powers $\langle \mathsf{R}^I, \tilde{\sigma}_D \rangle$ is also decidable. Not all these powers are elementary equivalent; for example, $\langle \mathsf{R}, \sigma \rangle$ and $\langle \mathsf{R}^I, \tilde{\sigma}_D \rangle$, with $|I| > 2$, are not elementary equivalent since "there exists a unique non-regular point" is expressible in L_t. It is an open problem to classify all these powers according to elementary equivalence; in particular, for infinite index sets and filters that extend the filter of cofinite sets. In this context, it may be of interest to recall that the L_t-theory of all Hausdorff spaces is undecidable [4].

4. Properties of formulas which are preserved under topological products

A sentence $\varphi \in L_t^S$ is *preserved* (under topological products) if for all I, $(\langle \mathfrak{A}_i, \sigma_i \rangle)_{i \in I}$, filter D on I: for all $i \in I$, $\langle \mathfrak{A}_i, \sigma_i \rangle \models \varphi$ implies $\langle \prod_I \mathfrak{A}_i, \tilde{\sigma}_D \rangle \models \varphi$.

From now on, the set of symbols S will be finite.

Theorem. *The class of all preserved sentences of L_t^S is recursively enumerable.*

Proof. By the Feferman-Vaught-type theorem, we can associate effectively to each sentence $\varphi \in L_t^S$ a partitioning tuple $(\Phi; \theta_1, \ldots, \theta_n)$ with $\Phi(x_0, \ldots, x_{n-1}) \in L^{BAF}$ and $\theta_1, \ldots, \theta_n$ L_t^S-sentences.

If $\hat{\varphi}$ is the L^{BAF}-sentence

$$\forall x_0 \ldots \forall x_{n-1} \Big(\bigwedge_{i<j<n} x_i \cap x_j = 0 \wedge x_0 \cup \ldots \cup x_{n-1} = 1 \wedge \bigwedge_{j \in C(\varphi)} x_j = 0 \rightarrow \Phi(x_0, \ldots, x_{n-1}) \Big),$$

where $C(\varphi) = \{ j \mid 0 \leqq j < n \text{ and } \models_{L_t^S} \varphi \rightarrow \neg \theta_j \}$, then it is easy to verify that φ is preserved iff $\hat{\varphi}$ belongs to the L^{BAF}-theory of all algebras of the form $\langle \mathfrak{P}(I), D \rangle$. As we have seen, this theory is decidable; furthermore, $\{\theta \in L_t^S \mid \models_{L_t^S} \varphi \rightarrow \neg \theta\}$ is recursively enumerable and $\hat{\varphi}$ can be obtained effectively from φ and $C(\varphi)$. From this we can conclude that the class of all preserved sentences is recursively enumerable. □

For the usual first-order logic there is a theorem by Vaught [3] that establishes that a formula is preserved under direct products iff it is preserved under products of two factors. In [4] it is shown that this fact, which simplifies the study of preserved first-order formulas, does not hold for L_t and Tychonov topological products. In our case, we have the following lemma as a substitute for Vaught's theorem. This lemma can be proved using Skolem's decision method for the L^{BA}-theory of all algebras of the form $\mathfrak{P}(I)$ (see e.g. [6]).

Lemma. *To each sentence $\varphi \in \mathrm{L}_t^S$ there can be effectively associated a natural number n_0, such that for all $\langle \mathfrak{A}, \sigma\rangle$ if $\langle \mathfrak{A}, \sigma\rangle^{n_0} \models \varphi$, then for all $m \geqq n_0$ $\langle \mathfrak{A}, \sigma\rangle^m \models \varphi$.* ($\langle \mathfrak{A}, \sigma\rangle^m$ is the usual topological power). □

From this lemma and a series of technical results with classical analogues, it follows, as in [5] for first-order logic and reduced products, the following theorem:

Theorem. *Each sentence of L_t^S is logically equivalent to a Boolean combination of sentences which are preserved under topological products with respect to non-trivial filters.* □

In what follows, we assume that the filters extend the corresponding filter of cofinite sets. The next two theorems give us a large syntactical class of formulas of L_t which are preserved. The first one can be proved by induction.

Theorem.

a) *Atomic formulas and negations of atomic formulas are preserved.*

b) *If φ is preserved, then $\exists x\varphi$ and $\forall x\varphi$ are preserved.*

c) *If φ and ψ are preserved, then $\varphi \wedge \psi$ is preserved.*

d) *If φ is preserved and positive in X, then $(\forall X \ni t)\,\varphi$ is preserved.*

Theorem. *If $\varphi \in \mathrm{L}_t^S$ is positive, without second-order quantifiers and Q is a block of second-order existential quantifiers, then the following formulas of L_t^S are preserved:*

a) $\varphi \to \theta$ *if φ is preserved.*

b) $\mathsf{Q}(\varphi \to \theta)$ *if θ is preserved and positive.*

c) $\mathsf{Q}\forall y(\varphi \to \theta)$ *if θ is a second-order atomic formula, i.e. of the form $t \in X$.*

Proof.

a) It is possible to show that the formulas of L_t^S which are preserved under continuous surjective homomorphisms are those that are equivalent to positive formulas without second-order existential quantifiers. On the other hand, by the hypothesis on the filters, the projection maps are continuous. The combination of these facts gives the proof.

b) As $\mathsf{Q}(\varphi \to \theta) \in \mathrm{L}_t^S$, the second-order variables in Q do not appear in θ; hence $\mathsf{Q}(\varphi \to \theta) \dashv\vdash \mathsf{Q}'\varphi \to \theta$, where Q' is a block of second-order universal quantifiers. Therefore we can apply a).

c) For simplicity, we restrict ourselves to the case $\mathsf{Q} = (\exists Y \ni t)$, $\mathbf{y} = y$, that is, we have to prove that

$$\psi := (\exists Y \ni t(x, \ldots))\,\forall y(\varphi(y, x, \ldots, Y, X, \ldots) \to \theta(y, x, \ldots, X))$$

is preserved.

Let us suppose that for all $i \in I$

$$\langle \mathfrak{A}_i, \sigma_i\rangle \models \psi(x, \ldots, X, \ldots)\,[b_i, \ldots, U_i, \ldots]$$

(here $b, \ldots \in \prod_I A_i, \prod_I U_i, \ldots \in \sigma_D$). Then, for each $i \in I$, let $Y \in \sigma_i$ be such that $t^{\mathfrak{A}_i}[b_i, \ldots] \in Y_i$ and

(1) $$\langle \mathfrak{A}_i, \sigma_i\rangle \models \forall y(\varphi(y, x, \ldots, Y, X, \ldots) \to \theta(y, x, \ldots, X))\,[b_i, \ldots, Y_i, U_i, \ldots].$$

If, for all $i \in I$, one has $\langle \mathfrak{A}_i, \sigma_i \rangle \models \forall y \theta(y, x, \ldots, X)\,[b_i, \ldots, U_i]$, then, by the preceding theorem, $\langle \prod_I \mathfrak{A}_i, \tilde{\sigma}_D \rangle \models \forall y \theta(y, x, \ldots, X)\,[b, \ldots, \prod_I U_i]$. Taking $Y = \prod_I A_i$, it follows that $\langle \prod_I \mathfrak{A}_i, \tilde{\sigma}_D \rangle \models \psi[b, \ldots, \prod_I U_i, \ldots]$.

Otherwise, there exist sets $I_0 \subseteqq I_1 \in D$ such that

$$i_0 \in I_0 \quad \text{iff} \quad (\mathfrak{A}_{i_0}, \sigma_{i_0}) \models \exists y \neg \theta[b_{i_0}, \ldots, U_{i_0}].$$

Then, for all $i_0 \in I_0$ and all $a_{i_0} \in A_{i_0}$ such that $\langle \mathfrak{A}_{i_0}, \sigma_{i_0} \rangle \models \neg \theta[a_{i_0}, b_{i_0}, \ldots, U_{i_0}]$, one has by (1) that $\langle \mathfrak{A}_{i_0}, \sigma_{i_0} \rangle \models \neg \varphi(y, x, \ldots, Y, X)\,[a_{i_0}, b_{i_0}, \ldots, Y_{i_0}, U_{i_0}, \ldots]$.

As $\neg \varphi$ is negative without second-order universal quantifiers, $\neg \varphi$ is preserved under preimages of continuous surjective homomorphisms, hence

$$\langle \prod_I \mathfrak{A}_i, \tilde{\sigma}_D \rangle \models \neg \varphi(y, x, \ldots, Y, X, \ldots)\,[\tilde{a}_{i_0}, \tilde{b}_{i_0}, \ldots, p_{i_0}^{-1}(Y_{i_0}), p_{i_0}^{-1}(U_{i_0}), \ldots],$$

where $\tilde{a}_{i_0}, \tilde{b}_{i_0}, \ldots \in \prod_I A_i$ are preimages of $a_{i_0}, b_{i_0}, \ldots$ under $p_{i_0}^{-1}$ (in particular, we can take $\tilde{b}_{i_0}, \ldots = b, \ldots$). In consequence, letting $V := \bigcap_{i \in I_1} p_i^{-1}(Y_i) \subseteqq \bigcap_{i \in I_0} p_i^{-1}(Y_i) \subseteqq p_{i_0}^{-1}(Y_{i_0})$, we obtain

$$\langle \prod_I \mathfrak{A}_i, \tilde{\sigma}_D \rangle \models \neg \varphi[\tilde{a}_{i_0}, b, \ldots, V, \prod_I U_i, \ldots]. \tag{2}$$

Notice that $V \in \sigma_D$ and therefore we can replace $p_{i_0}^{-1}(Y_{i_0})$ by V because $\neg \varphi$ is negative.

Now let us suppose that $\langle \prod_I \mathfrak{A}_i, \tilde{\sigma}_D \rangle \not\models \psi[b, \ldots, \prod_I U_i, \ldots]$; then

$$\langle \prod_I \mathfrak{A}_i, \tilde{\sigma}_D \rangle \models (\forall Y \ni t(x, \ldots))\, \exists y(\varphi(y, x, \ldots, Y, X, \ldots) \wedge \\ \wedge \neg \theta(y, x, \ldots, X))\,[b, \ldots, \prod_I U_i, \ldots].$$

In particular, taking the above defined V as assignment for Y, we obtain an $a \in \prod_I A_i$ such that

$$\langle \prod_I \mathfrak{A}_i, \tilde{\sigma}_D \rangle \models \varphi(y, x, \ldots, Y, X, \ldots)\,[a, b, \ldots, V, \prod_I U_i, \ldots] \quad \text{and} \\ \langle \prod_I \mathfrak{A}_i, \tilde{\sigma}_D \rangle \models \neg \theta(y, x, \ldots, X)\,[a, b, \ldots, \prod_I U_i]. \tag{3}$$

From this last fact there follows the existence of $i_0 \in I_0$ such that $\langle \mathfrak{A}_{i_0}, \sigma_{i_0} \rangle \models \neg \theta(y, x, \ldots, X)\,[a_{i_0}, b_{i_0}, \ldots, U_{i_0}]$. As we now are in the situation that led us to (2), we conclude that

$$\langle \prod_I \mathfrak{A}_i, \tilde{\sigma}_D \rangle \models \neg \varphi(y, x, \ldots, Y, X, \ldots)\,[\tilde{a}_{i_0}, b, \ldots, V, \prod_I U_i, \ldots].$$

In particular, with $\tilde{a}_{i_0} = a$, it follows

$$\langle \prod_I \mathfrak{A}_i, \tilde{\sigma}_D \rangle \models \neg \varphi(y, x, \ldots, Y, X, \ldots)\,[a, b, \ldots, V, \prod_I U_i, \ldots],$$

which contradicts (3). $\square$

Applying systematically the two preceding theorems it is possible to verify syntactically that the following topological properties expressable in L_t are preserved:

$$\forall x \forall y (((\forall X \ni x)\, y \in X \wedge (\forall Y \ni y)\, x \in Y) \rightarrow x = y) \qquad (\text{“}T_0\text{”}),$$

$$\forall x \forall y ((\forall X \ni x)\, y \in X \rightarrow x = y) \qquad (\text{“}T_1\text{”}),$$

$$\forall x \forall y (\exists X \ni x)\, (\exists Y \ni y)\, (\exists z (z \in X \wedge z \in Y) \rightarrow x = y) \qquad (\text{“Hausdorff”}),$$

$$\forall x(\forall X \ni x)\,(\exists Y \ni x)\,\forall y((\forall W \ni y)\,\exists z(z \in W \wedge z \in Y) \to y \in X) \qquad (\text{"regular"}),$$

$$\forall x(\forall X \ni f(x))\,(\exists Y \ni x)\,\forall z(z \in Y \to f(z) \in X) \qquad (\text{"}f\text{ is continuous"}),$$

$$\forall x((\forall X \ni x)\,\exists y(Py \wedge y \in X) \to Px) \qquad (\text{"}P\text{ is closed"}).$$

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