Review of Linear Algebra

Dr. Chang Shu

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Linear Equations

A system of linear equations, e.g.

$$2x_1 + 4x_2 = 2$$
$$4x_1 + 11x_2 = 1$$

can be written in matrix form:

$$\begin{bmatrix} 2 & 4 \\ 4 & 11 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

or in general:

$$Ax = b$$

Vectors

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad \text{e.g.} \quad x = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$$

The **length** or the **norm** of a vector is

$$\|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

e.g. $\|x\| = \sqrt{2^2 + 3^2 + 5^2} = \sqrt{38}$

Vector Arithmetic

Vector addition

$$u + v = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}$$

Vector subtraction

$$u - v = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} - \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} u_1 - v_1 \\ u_2 - v_2 \end{bmatrix}$$

Multiplication by scalar

$$\alpha u = \alpha \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \alpha u_1 \\ \alpha u_2 \end{bmatrix}$$

Dot Product (inner product)

$$a = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} \qquad b = \begin{bmatrix} 4 \\ -3 \\ 2 \end{bmatrix}$$

$$a \cdot b = a^T b = \begin{bmatrix} 2 & 3 & 5 \end{bmatrix} \begin{bmatrix} 4 \\ -3 \\ 2 \end{bmatrix} = 2 \cdot 4 + 3 \cdot (-3) + 5 \cdot 2 = 9$$

$$a \cdot b = a^T b = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

Linear Independence

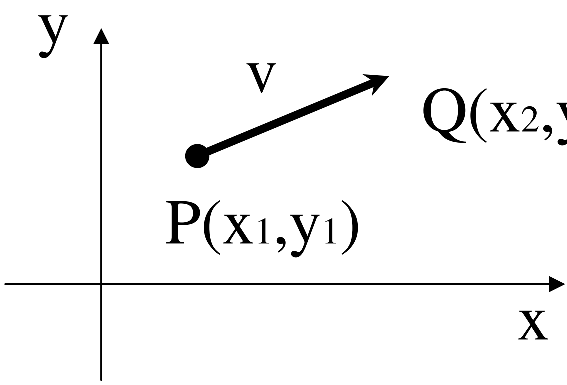
- A set of vectors is linear dependant if one of the vectors can be expressed as a linear combination of the other vectors.

$$v_k = \alpha_1 v_1 + \cdots + \alpha_{k-1} v_{k-1} + \alpha_{k+1} v_{k+1} + \cdots + \alpha_n v_n$$

- A set of vectors is linearly independent if none of the vectors can be expressed as a linear combination of the other vectors.

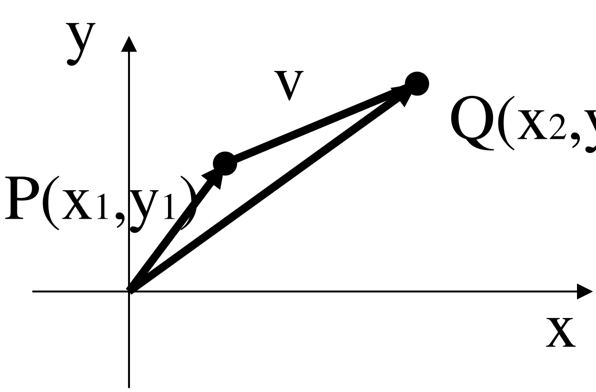
Vectors and Points

Two points in a Cartesian coordinate system define a vector



$$v = \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix}$$

A point can also be represented as a vector, defined by the point and the origin (0,0).



$$P = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \qquad Q = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$$

$v = Q - P$ or $Q = P + v$

Note: point and vector are different; vectors do not have positions

Matrix

A matrix is an $m \times n$ array of numbers.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \left[a_{ij}\right]$$

Example:

$$A = \begin{bmatrix} 2 & 3 & 5 & 4 \\ -4 & 1 & 3 & 9 \\ 0 & 7 & 10 & 11 \end{bmatrix}$$

Matrix Arithmetic

Matrix addition

$$A_{m\times n} + B_{m\times n} = \left[a_{ij} + b_{ij}\right]_{m\times n}$$

Matrix multiplication

$$A_{m\times n} B_{n\times p} = C_{m\times p} \qquad c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

Matrix transpose

$$A^T = \left[a_{ji}\right]$$

$$(A+B)^T = A^T + B^T \qquad (AB)^T = B^T A^T$$

Matrix multiplication is not commutative

$$AB \neq BA$$

Example:

$$\begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 17 & 29 \\ 19 & 11 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 18 & 32 \\ 17 & 10 \end{bmatrix}$$

Symmetric Matrix

We say matrix A is symmetric if

$$A^T = A$$

Example:

$$A = \begin{bmatrix} 2 & 4 \\ 4 & 5 \end{bmatrix}$$

A symmetric matrix has to be a square matrix

Inverse of matrix

If A is a square matrix, the inverse of A, written A^{-1} satisfies:

$$AA^{-1} = I \qquad A^{-1}A = I$$

Where *I*, the identity matrix, is a diagonal matrix with all 1's on the diagonal.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Trace of Matrix

The trace of a matrix:

$$Tr(A) = \sum_{i=1}^{n} a_{ii}$$

Orthogonal Matrix

A matrix A is orthogonal if

$$A^T A = I \quad \text{or} \quad A^T = A^{-1}$$

Example:

$$A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Matrix Transformation

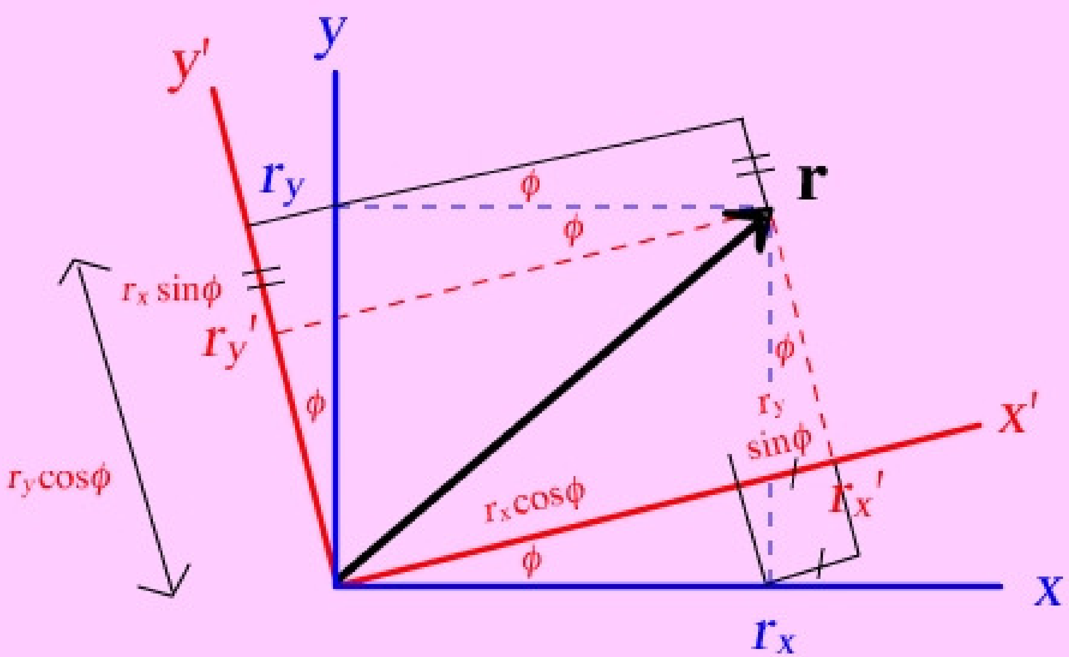
A matrix-vector multiplication transforms one vector to another

$$A_{m\times n} x_{n\times 1} = b_{m\times 1}$$

Example:

$$\begin{bmatrix} 2 & 5 \\ 3 & 1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 7 \end{bmatrix} = \begin{bmatrix} 41 \\ 16 \\ 26 \end{bmatrix}$$

Coordinate Rotation



$$r_x' = \; r_x \cos\phi + r_y \sin\phi$$

$$r_y' = -r_x \sin\phi + r_y \cos\phi$$

$$\begin{bmatrix} r_x' \\ r_y' \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} r_x \\ r_y \end{bmatrix}$$

Eigenvalue and Eigenvector

We say that x is an eigenvector of a square matrix A if

$$Ax = \lambda x$$

λ is called eigenvalue and x is called eigenvector.

The transformation defined by A changes only the magnitude of the vector x

Example:

$$\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \end{bmatrix} = 2\begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

5 and 2 are eigenvalues, and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ are eigenvectors.

Properties of Eigen Vectors

- If $\lambda_1, \lambda_2, \ldots, \lambda_q$ are distinct eigenvalues of a matrix, then the corresponding eigenvectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_q$ are linearly independent.

- A real, symmetric matrix has real eigenvalues with eigenvectors that can be chosen to be orthonormal.

Least Square

When m>n for an m-by-n matrix A, $Ax = b$ has no solution.

In this case, we look for an approximate solution.
We look for vector x such that

$$\|Ax - b\|^2$$

is as small as possible.

This is the least square solution.

Least Square

Least square solution of linear system of equations

$$Ax = b$$

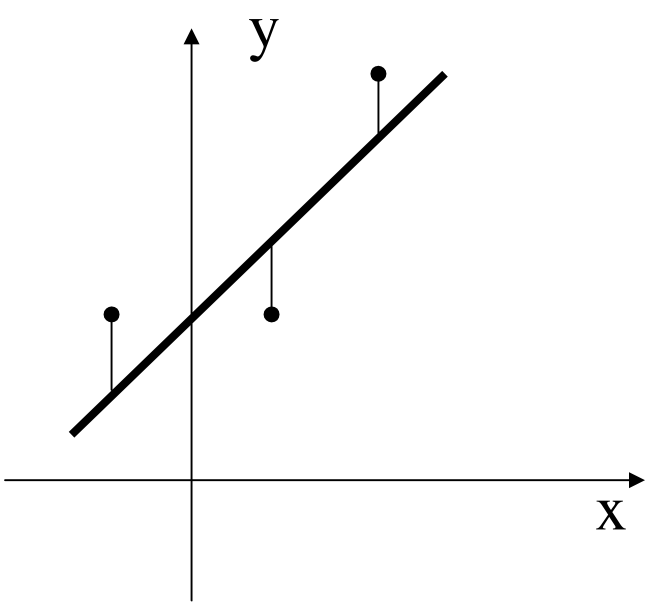
Normal equation: $A^T Ax = A^T b$

$A^T A$ is square and symmetric

The Least Square solution $\bar{x} = (A^T A)^{-1} A^T b$

makes $\left\| A\bar{x} - b \right\|^2$ minimal.

Least Square Fitting of a Line



Line equations:

$$\begin{aligned} c + dx_1 &= y_1 \\ c + dx_2 &= y_2 \\ &\vdots \\ c + dx_m &= y_m \end{aligned}$$

$$\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_m \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$$

$$Ax = y$$

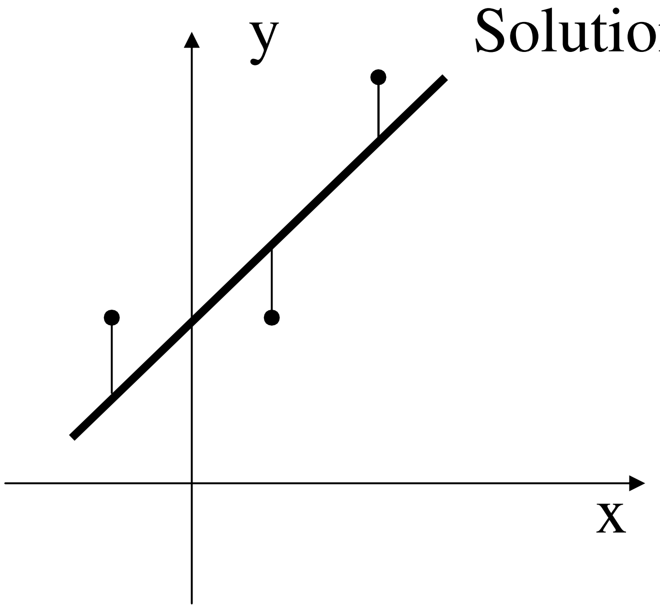
The best solution c, d is the one that minimizes:

$$E^2 = \|y - Ax\|^2 = (y_1 - c - dx_1)^2 + \cdots + (y_m - c - dx_m)^2.$$

Least Square Fitting - Example

Problem: find the line that best fit these three points:

P1=(-1,1), P2=(1,1), P3=(2,3)



Solution:

$$c - d = 1$$
$$c + d = 1$$
$$c + 2d = 3$$

or

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$$

$A^T A x = A^T b$ is $\begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$

The solution is $c = \frac{9}{7}, d = \frac{4}{7}$ and best line is $\frac{9}{7} + \frac{4}{7} x = y$

SVD: Singular Value Decomposition

An $m \times n$ matrix A can be decomposed into:

$$A = UDV^T$$

U is $m \times m$, V is $n \times n$, both of them have orthogonal columns:

$$U^T U = I \qquad V^T V = I$$

D is an $m \times n$ diagonal matrix.

Example:

$$\begin{bmatrix} 2 & 0 \\ 0 & -3 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$