A Constructive Method for Finding β-Invariant Measures for Transition Matrices of $M/G/1$ Type

Quan-Lin Li,[1,2] Yiqiang Q. Zhao[2]

Abstract: In this paper, we study the transition matrix of $M/G/1$ type. The radius of convergence is discussed, conditions on classification of the states are obtained, and expressions of the β-invariant measure are constructed. The censoring technique is generalized to deal with nonnegative matrices, which may not be either stochastic or substochastic. This allows us to prove a factorizaton result for the discounted transition matrix. This factorizaton provides a unified algorithmic approach for expressing the β-invariant measure for transition matrices with a block structure, including the matrix of $M/G/1$ type.

Keywords: β-invariant measures, duality, factorizations, $M/G/1$ type, quasi-stationary distributions, radius of convergence.

1 Introduction

We consider an irreducible aperiodic Markov chain $\{X_n, n = 1, 2, \ldots\}$ of $M/G/1$ type, whose transition matrix P is partitioned into block form:

$$P = \begin{bmatrix} D_1 & D_2 & D_3 & D_4 & \cdots \\ D_0 & C_1 & C_2 & C_3 & \cdots \\ & C_0 & C_1 & C_2 & \cdots \\ & & C_0 & C_1 & \cdots \\ & & & \ddots & \ddots \end{bmatrix}, \qquad (1)$$

where D_1 is a matrix of size $m_0 \times m_0$, all C_i are square matrices of finite size m, the sizes of the other block-entries are determined accordingly and all empty

[1]National Laboratory of Pattern Recognition, Institute of Automation, Chinese Academy of Sciences, P.O. Box 2728, Beijing 100080, P.R. China.

[2]School of Mathematics and Statistics, Carleton University, Ottawa, Ontario, Canada K1S 5B6. *qli@math.carleton.ca*, *zhao@math.carleton.ca*

entries are zero. P is assumed to be stochastic or strictly substochastic. By strictly substochastic, we mean that $P \geq 0$, $Pe \leq e$ and $Pe \neq e$, where e is a column vector of ones.

The state space of the above block-partitioned Markov chain can be expressed as $S = \cup_{i=0}^{\infty} L_i$, where $L_0 = \{(0, j); j = 0, 1, 2, \ldots, m_0\}$ and $L_i = \{(i, j); j = 0, 1, 2, \ldots, m\}$ for $i \geq 1$. In state (i, j), variable i is called the *level* and variable j, the *phase*. Therefore, L_i is the set of all states at level i. For convenience, we write $L_{\leq i} = \cup_{k=0}^{i} L_k$ and $L_{\geq i}$ for the complement of $L_{\leq (i-1)}$.

Let α be the radius of convergence for the transition matrix $P = (p_{(i,r);(j,s)})$. We know that $\alpha = \sup\{z > 0; \sum_{n=0}^{\infty} z^n p_{(i,r);(j,s)}^{(n)} < \infty\} \geq 1$, where $p_{(i,r);(j,s)}^{(n)}$ is the n-step transition probability and α is independent of states (i, r) and (j, s).

A nonnegative nonzero row vector π is said to be an invariant measure of P if $\pi = \pi P$. For $0 < \beta \leq \alpha$, a nonnegative nonzero row vector π is said to be a β-invariant measure of P if $\pi = \pi\beta P$. Call βP the discounted transition matrix at rate β. Then, a β-invariant measure is simply a invariant measure of the discounted matrix. It follows from the definition that a 1-invariant measure is simply an invariant measure.

For the transition matrix P of $M/G/1$ type, we are interested in

a) determination of the radius of convergence α;

b) conditions on further classification of the states when P is transient; and

c) expressions for the β-invariant measure for $0 < \beta \leq \alpha$.

There are a number of reasons why the above items are of interest.

1) It is well-known that $\pi = (\pi_i)$ is a quasistationary distribution if and only if, for some $\beta > 1$, π is a β-invariant measure satisfying $\sum_i \pi_i < \infty$. The study of quasistationary behaviour of a Markov chain is not only theoretically important, but also finds interesting and important applications in many areas, including, for example, biology (Scheffer 1951, Holling 1973, Pakes 1987 and Pollett 1987), chemistry (Oppenheim, Shuler and Weiss 1977, Parsons and Pollett 1987 and Pollett 1988), and telecommunications (Schrijner 1995).

2) When the entries π_i in π cannot be summed, the concept of the β-invariant measure is a generalization of invariant measures for a nonergodic chain (Derman 1955, Harris 1957, Latouche, Pearce and Taylor 1998, Gail, Hantler and Taylor 1998, Zhao, Li and Braun 1998). In this case, π can still be interpreted probabilistically in terms of the movement of particles whose initial states are governed by Poisson distributions (Derman 1955 and

Kelly 1983). Also, π can be used to define a time-reversed matrix or dual matrix, which has important applications (Kelly 1979, Ramaswami 1990, Asmussen and Ramaswami 1990, Bright 1996 and Zhao, Li and Alfa 1999).

3) It is well known how important the Perron-Frobenius Theorem is in the theory of finite nonnegative matrices. The radius of convergence α of P can be considered the Perron-Frobenius eigenvalue of the nonnegative matrix P and an α-invariant measure of P a Perron-Frobenius eigenvector of P.

It is believed that the study on quasistationary behaviour was originated by Yaglom (1947). Since then, significant advances in the theory of quasistationarity have been made through the efforts of many researchers. A detailed review on the literature can be found in the Ph.D. dissertation of Schrijiner (1995). This study has also successfully advanced to considering transition matrices with block structure since Kijima (1993) made a break through on the determination of the radius of convergence for Markov chains of $GI/M/1$ type and $M/G/1$ type without boundaries. For block-structured transition matrices, studies have been centered on obtaining probabilistic measures to express the radius of convergence and quasistationary distributions, including classifications of the states in terms of these measures. People are searching for expressions which are numerically preferable. Results on quasi-birth-and-death (QBD) processes can be found in Kijima (1993), Makimoto (1993), Bean *et al.* (1997), and Bean, Pollett and Taylor (1998, 2001). Some preliminary results on the expressions for the matrices of $GI/M/1$ type and $M/G/1$ type were obtained in Li (1997). A survey on quasistationary distributions of Markov chains arising from queueing processes was conducted by Kijima and Makimoto (1998).

In this paper, we will study the matrix of $M/G/1$ type with boundary blocks as defined in (1). The issue on the radius of convergence will be addressed by combining characteristic results obtained by Kijima (1993) and the boundary treatment based on censoring. For the case without boundaries, the matrix is always α-transient. With the presence of the boundary, the matrix can be either α-transient or α-recurrent. Conditions on classifications of the transient states will also be discussed in this paper. For the matrix of $M/G/1$ type, we have not noticed the existence of an expression for the β-invariant measure in the literature. We will provide a constructive way of expressing such a measure.

The technique used in this paper to study the radius of convergence and conditions on classifications of the transient states is based on censoring. This technique has been successfully used in studying many other aspects of block-structured stochastic or strictly substochastic matrices (for example, see Grassmann and Heyman 1990, Latouche 1993, Zhao, Li and Braun 1998, 2001, Zhao, Li and Alfa 1999, Latouche and Ramaswami 1999, and Zhao 2000). In order to use the censoring technique to deal with the issue of the β-invariant measure, we need to generalize results of stochastic or strictly substochastic matrices to that of nonnegative matrices.

What we will use to obtain expressions for the β-invariant measure is the method of factorization, where $I - \beta P$ is factorized into the product of an upper triangular matrix and a lower triangular matrix. We shall call it the RG-factorization, since the factors in factorization involve the R- and G-measures, which are two key probabilistic measures in our study and defined later. This factorization may be viewed as a UL-factorization for the infinite matrix $I - \beta P$. The procedure of obtaining a solution for the β-invariant measure can be considered a generalization of using a UL-factorization to solve a finite system of linear equations. Expressions for the β-invariant measure are different according to the classification of the states and the value of β. When we use the factorization technique, the key is how to associate the middle factor or the diagonal matrix with either the upper triangular or the lower triangular matrix. Our study will provide a way to successively identify two different sets of solutions for the β-invariant measure. When $\beta = 1$, an equivalent form of this factorization was obtained and studied by Heyman (1995), Zhao, Li and Braun (1997, 2000) and Zhao (2000). In Li (1997), the matrix $I - \beta P$ was factored into an equivalent form of the RG-factorization without using the R-measure. There are three possible difficulties when using the RG-factorization on infinite matrices. Firstly, the associativity of matrix multiplications cannot be taken for granted, secondly, the existence of a non-trivial solution to a linear system of infinitely many equations cannot be taken for granted, and thirdly, the method of dealing with a recurrent matrix and a transient matrix should be distinguished. When the Markov chain is positive recurrent, these issues have been successfully addressed in the literature, for example, see Heyman (1995). Ramaswami (1988) presented a stable recursion, equivalent to the factorization of Heyman, for the steady state vector for Markov chains of $M/G/1$ type. Also, Meini (1997) studied the matrix of $M/G/1$ type in terms of a method of factorization. For quasistationary distributions, the method employed by Bean, Pollett and Taylor (2001) to the quasi-birth-and-death process is essentially equivalent the factorization method used in this paper. However, they did not indicate how the expressions for the β-invariant measure are constructed.

It is our belief that the idea presented here can also be used to study other types of block-structured matrices, for example, matrices of $GI/M/1$ type and, more generally, $GI/G/1$ type.

The rest of the paper is organized as follows.

In Section 2, some basic properties on matrix βP are provided, including properties on the existence of an inverse of $I - \beta P$, the minimal nonnegative inverse and the fundamental matrix. These properties are needed in later sections.

When P is transient, the states of P can be further classified as α-recurrent or α-transient according to $\alpha\mathbf{P} = \sum_{k=0}^{\infty} \alpha^k P^k = \infty$ or $< \infty$, respectively. The matrix $\alpha\mathbf{P}$ is referred to as the fundamental matrix of αP. If P is α-recurrent, either $\lim_{n\to\infty} \alpha^n p^{(n)}_{(i,r),(j,s)} > 0$ for all states (i,r) and (j,s), or $\lim_{n\to\infty} \alpha^n p^{(n)}_{(i,r),(j,s)} = 0$ for all states (i,r) and (j,s). In the former case, P is called α-positive and in the latter case, α-null. In Section 3, we provide a determination of the radius α of convergence and conditions on classifications of the transient states, based on the combination of the result of determining the radius α of convergence for the matrix of $M/G/1$ type without boundaries and a new treatment for the boundary.

In Section 4, the RG-factorization for matrix $I - \beta P$ is proven. We show that

$$I - \beta P = [I - R_U(\beta)][I - U_D(\beta)][I - G_L(\beta)],$$

where $R_U(\beta)$ is a block-form upper triangular matrix involving only the R-measure, $G_L(\beta)$ is a block-form lower triangular matrix involving only the G-measure, and $U_D(\beta)$ is a block-form diagonal matrix. The R-measure is a sequence of matrices defined by (14) and (15) and the G-measure for the matrix $M/G/1$ type consists of two matrices defined by (4) and (16). Probabilistic interpretations for both R- and G-measures are provided after the definition formulas. In this section, we also show that the RG-factorization exists for the matrix of level-dependent $M/G/1$ type.

In Section 5, based on the RG-factorization, expressions for the β-invariant measure are obtained. There are two different sets of expressions. One is for the α-invariant measure when P is α-recurrent. In this case, the α-invariant measure is unique up to multiplication by a positive constant. For all other cases, we provide a common expression for the β-invariant measure. When the β-invariant measure cannot be summed, this uniqueness is no longer guaranteed.

The final section, Section 6, consists of concluding remarks.

2 Preliminaries

In this section, we provide some properties on negative matrix βP, which will be used in later sections. Most of these results can be viewed as generalizations of the counterparts for a stochastic or strictly substochastic matrix. Proofs to these properties may not be obvious. However, since they can be proved either in the same way as that for a stochastic or strictly substochastic matrix or in a similar fashion, we omit most of the proofs. Relevant references are Seneta (1980), Kemeny, Snell and Knap (1976), Cinlar (1975) among possible others.

A general statement on the existence and uniqueness of an α-invariant measure can be found in the literature, for example Seneta (1980) which is stated in the following lemma. In order to do so, we need the concept of superregularity. A row vector x is called a superregular measure of P if $x \geq xP$. A row vector x is called a β-superregular measure of P if $x \geq \beta xP$. A 1-superregular measure is simply superregular.

Lemma 1 *For irreducible aperiodic matrix P, there always exists a positive α-superregular measure x. If P is α-recurrent, then the unique α-superregular measure x, up to multiplication by a positive constant, of P is α-regular and positive.*

The following are some basic properties about the existence of an inverse, minimal nonnegative inverse and the fundamental matrix.

Lemma 2 *(i) For $0 < \beta < \alpha$ if P is α-recurrent, or for $0 < \beta \leq \alpha$ if P is α-transient, $(I - \beta P)$ is invertible. (ii) If $(I - \beta P)$ is invertible, then*

$$\beta\hat{P} = \sum_{k=0}^{\infty} \beta^k P^k \tag{2}$$

is the minimal nonnegative inverse of $(I - \beta P)$, which is often referred to as the fundamental matrix of βP. (iii) Let P be partitioned into

$$P = \begin{bmatrix} T & H \\ L & Q \end{bmatrix}. \tag{3}$$

Then, both $(I - \beta T)$ and $(I - \beta Q)$ are invertible for $0 < \beta \leq \alpha$.

The following lemma plays an important role in later sections, which will be used to establish a relationship between block-entries of the fundamental matrix $\beta\hat{P}$.

Lemma 3 *Let P be partitioned as in (3) and let βP be partitioned accordingly as*

$$\beta P = \begin{bmatrix} \beta T & \beta H \\ \beta L & \beta Q \end{bmatrix}, \quad 0 < \beta \leq \alpha.$$

Assume that $I - \beta P$ is invertible. Then, the minimal nonnegative inverse Φ of $(I - \beta P)$ is given by

$$\Phi = \begin{bmatrix} (I - \beta T - \beta H \Phi_Q \beta L)^{-1}_{\min} & (I - \beta T - \beta H \Phi_Q \beta L)^{-1}_{\min} \beta H \Phi_Q \\ \Phi_Q \beta L (I - \beta T - \beta H \Phi_Q \beta L)^{-1}_{\min} & \Phi_Q + \Phi_Q \beta L (I - \beta T - \beta H \Phi_Q \beta L)^{-1}_{\min} \beta H \Phi_Q \end{bmatrix}$$

or equivalently,

$$\Phi = \begin{bmatrix} \Phi_T + \Phi_T \beta H (I - \beta Q - \beta L \Phi_T \beta H)^{-1}_{\min} \beta L \Phi_T & \Phi_T \beta H (I - \beta Q - \beta L \Phi_T \beta H)^{-1}_{\min} \\ (I - \beta Q - \beta L \Phi_T \beta H)^{-1}_{\min} \beta L \Phi_T & (I - \beta Q - \beta L \Phi_T \beta H)^{-1}_{\min} \end{bmatrix},$$

where $(I - X)^{-1}_{\min} = \sum_{i=0}^{\infty} X^i$ is the minimal nonnegative inverse of $I - X$.

Remark 1 By sample path argument or the above lemma, we can show that the fundamental matrix is invariant under censoring. Let E be a subset of the state space. Let βP be partitioned according to E and its complement E^c as in (3). And let the fundamental matrix Φ of βP be expressed as in (4). Then, the fundamental matrix of the censored matrix $(\beta P)^E$ is equal to the block-entry corresponding to the states in E in the fundamental matrix Φ.

3 Radius of convergence and classification of states

Let α be the radius of convergence for P. If $\alpha = 1$, the classification of states is conventional. So, we are only interested in the classification of states when $\alpha > 1$. This corresponds to a further classification of the transient states. The main purpose of this section is to determine the radius of convergence α and to provide conditions on classification of the states. To pursue that, we first define the matrix $G(\beta)$ which, together with matrix $G_{1,0}(\beta)$ defined in Section 4, is referred to as the G-measure for the transition matrix P of $M/G/1$ type. The main results in this section will be expressed in terms of the G-measure through the analysis of the fundamental matrix and censored matrices $N(\beta)$ and $N_0(\beta)$. By introducing the G-measure, not only can the theoretical analysis be carried out, but also it is computable.

Partition the discounted transition matrix βP of $M/G/1$ type as in (3) with $\beta T = \beta D_1$, and βH, βL and βQ being determined accordingly. Notice that, in the partition, Q is the transition matrix of $M/G/1$ type without boundaries. Therefore, βQ is the discounted transition matrix from Q. Let $\hat{Q} = (\hat{Q}_{i,j}(\beta))_{i,j=1,2,\ldots}$ be the fundamental matrix for $Q(\beta)$ and write $N(\beta) = \hat{Q}_{1,1}(\beta)$.

The matrix $G(\beta)$ is defined by

$$G(\beta) = N(\beta)\beta C_0. \tag{4}$$

$G(\beta)$ is a matrix of size m. The (r,s)th entry of $G(\beta)$ can be interpreted as the total expected discounted reward with rate β induced by hitting state (i,s) upon the process entering $L_{\le i}$ for the first time, given that the process starts in state $(i+1, r)$.

Remark 2 Though the matrix $G(\beta)$ is defined as the product of $N(\beta)$ and βC_0, we usually first compute $G(\beta)$ and then determine $N(\beta)$ in terms of $G(\beta)$. To do so, we need the following lemma, that says that all the other block-entries in the first block-column in $\hat{Q}$ can be explicitly expressed in terms of $N(\beta)$, the $(1,1)$st block-entry in $\hat{Q}$.

Lemma 4 *For the fundamental matrix* $\hat{Q} = (\hat{Q}_{i,j}(\beta))_{i,j=1,2,\ldots}$,

$$\hat{Q}_{j,1}(\beta) = G(\beta)^{j-1} N(\beta); \quad j \ge 1. \tag{5}$$

Proof: It follows from (4) in Lemma 3 that

$$(\hat{Q}_{2,1}(\beta)^T, \hat{Q}_{3,1}(\beta)^T, \ldots)^T = \hat{Q}\beta L N(\beta).$$

The repeating structure and the property of skip-free-to-left of the transition matrix $Q(\beta)$ leads to

$$(\hat{Q}_{2,1}(\beta)^T, \hat{Q}_{3,1}(\beta)^T, \ldots)^T = (N(\beta)^T, \hat{Q}_{2,1}(\beta)^T, \ldots)^T \beta C_0 N(\beta).$$

The proof is completed by the above recursive expression and repeatedly using $N(\beta)\beta C_0 = G(\beta)$. ∎

For the discounted transition matrix βP of $M/G/1$ type, we partition the fundamental matrix $\hat{P}$ of βP according to levels. The block-entries of $\hat{P}$ are denoted by $\hat{P}_{i,j}(\beta)$. It is clear that to study the radius of convergence and to classify the states, it is sufficient to only consider an arbitrary block-entry in $\hat{P}$. For the block-structured transition matrix P in (1), partition P according to (3) with $T = D_1$. It suffices to consider the $(1,1)$st block-entry, denoted by $N_0(\beta)$, in $\hat{P}$. We express $N(\beta)$ in terms of $G(\beta)$ and $N_0(\beta)$ in terms of $N(\beta)$. This will enable us to determine the radius of convergence α and provide conditions on classification of the states.

Theorem 1 *For the transition matrix of $M/G/1$ type, the $(1,1)$st block entry $N(\beta)$ in $\widehat{\beta Q}$ can be expressed as*

$$N(\beta) = [I - \beta \sum_{k=1}^{\infty} C_k G(\beta)^{k-1}]^{-1}, \tag{6}$$

or $N(\beta)$ is the fundamental matrix for $U(\beta) = \beta \sum_{k=1}^{\infty} C_k G(\beta)^{k-1}$. The $(1,1)$st block entry $N_0(\beta)$ in $\widehat{\beta P}$ can be expressed as

$$N_0(\beta) = [I - U_0(\beta)]^{-1}, \tag{7}$$

where

$$U_0(\beta) = \beta D_1 + \sum_{k=1}^{\infty} \beta D_{k+1} G(\beta)^{k-1} N(\beta) \beta D_0, \tag{8}$$

or $N_0(\beta)$ is the fundamental matrix for $U_0(\beta)$.

Proof: Apply Lemma 3 to the discounted transition matrix βQ. It follows from (4) that $N(\beta) = \hat{\Phi}_{1,1}(\beta)$ is the fundamental matrix for $\beta T + \beta H \widehat{\beta Q} \beta L$. Then,

$$\begin{aligned}\beta T + \beta H \widehat{\beta Q} \beta L &= \beta C_1 + \beta H (\hat{\Phi}_{1,1}(\beta), \hat{\Phi}_{2,1}(\beta), \ldots)^T \beta C_0 \\ &= \beta C_1 + \sum_{k=2}^{\infty} \beta C_k \hat{\Phi}_{k-1,1}(\beta) \beta C_0.\end{aligned}$$

Noticing that $N(\beta) \beta C_0 = G(\beta)$ and using Lemma 4 will complete the proof to the first assertion.

To prove the second, apply Lemma 3 to the discounted transition matrix βP. Then, $N_0(\beta)$ is the fundamental matrix for $\beta T + \beta H \widehat{\beta Q} \beta L$, where $\beta T = \beta D_1$, $\beta H = \beta(D_2, D_3, \ldots)$, $\widehat{\beta Q}$ is the fundamental matrix of βQ and $\beta L = \beta(D_0, 0, \ldots)$. Therefore,

$$U_0(\beta) = \beta D_1 + \sum_{k=2}^{\infty} \beta D_k \hat{\Phi}_{k-1,1}(\beta) \beta D_0.$$

The proof is complete by using Lemma 4. ∎

Remark 3 It follows from the definition equation (4) and equation (6) that $G(\beta)$ satisfies the following equation:

$$G(\beta) = \sum_{k=0}^{\infty} \beta C_k G(\beta)^k. \tag{9}$$

We can further prove that $G(\beta)$ is the minimal nonnegative solution to equation (9).

The determination of the radius of convergence α and the conditions on classification of the states given below are based on the combination of the classification result for the matrix without boundaries given by Kijima (1993) and the treatment of the boundary. For convenience, we state two results by Kijima here.

For the transition matrix P of $M/G/1$ type in (1) without boundaries, or all $D_k = C_k$ for $k = 0, 1, \ldots$, Kijima (1993) provided a method for determining the radius of convergence α and showed that P is always α-transient.

Lemma 5 (Kijima) *Let $C^*(z)$ be defined by*

$$C^*(z) = \sum_{k=0}^{\infty} C_k z^k; \quad 0 \le z < z_0. \tag{10}$$

Let $\chi(z)$ be the Perron-Frobenius eigenvalue of $C^*(z)$. If $z_0 > 1$, then there always exists a unique γ such that $\chi(z) \ge \gamma z$ for all $0 < z < z_0$, and there exists some θ with $0 < \theta \le z_0$ such that $\chi(\theta) = \theta\gamma$. If $\theta = z_0$, then $\gamma = \chi(z_0)/z_0$. Otherwise, γ and θ can be determined by solving the simultaneous equations

$$\chi(\theta) = \gamma\theta \quad \text{and} \quad \chi'(\theta) = \gamma. \tag{11}$$

By using this lemma, Kijima was able to show the following result.

Theorem 2 (Kijima) *For the transition matrix P of $M/G/1$ type without boundaries ($D_k = C_k$ for all $k \ge 0$), if γ is the quantity determined in Lemma 5, then the radius of convergence α of P satisfies $\alpha = 1/\gamma$ and P is α-transient.*

Remark 4 In fact, θ given in the above lemma is the maximal eigenvalue of the $G(\alpha)$. Makimoto (1993) obtained two types of expressions for the quasistationary distributions of the $PH/PH/c$ queue in terms of θ and γ, and Kijima (1998) generalized those results to the matrix of $GI/M/1$ type without boundaries.

Remark 5 Kijima (1993) also related μ and γ to the mean drift. The fact that the matrix of $M/G/1$ type without boundaries is always α-transient is independent of the mean drift. However, the matrix with boundaries can be α-transient, α-positive recurrent or α-null recurrent.

For P of $M/G/1$ type in (1) with boundaries, we can perform the spectral analysis on the censored matrix to level 0, $U_0(\beta)$, to obtain conditions on classifications of the transient states and a determination of the radius of convergence. However, it seems more convenient to reach this goal by considering the relationship between the censored matrix $U_0(\beta)$ and its fundamental matrix $N_0(\beta)$.

Let $u_0(\beta)$ and $n_0(\beta)$ be the maximal eigenvalues of the censored matrix $U_0(\beta)$ and its fundamental matrix $N_0(\beta)$, respectively. It follows from results of linear algebra that the first two statements are true, for example, Seneta (1980), and the third one follows from the definitions of the radius of convergence and $N_0(\beta)$.

Lemma 6 *Let $\bar{\alpha}$ and α be the radii of convergence of Q and P respectively. In i) and ii), assume $0 < \beta \le \bar{\alpha}$.*

i) *Both $u_0(\beta)$ and $n_0(\beta)$ are strictly increasing in β, and*

ii) $u_0(\beta) < 1$ *if and only if* $N_0(\beta) < \infty$.

iii) $N_0(\beta) < \alpha$ *if* $\beta < \alpha$ *and* $N_0(\beta) = \infty$ *if* $\beta > \alpha$.

iv) $\alpha \le \bar{\alpha}$.

The classification of the states is characterized by the following conditions.

Theorem 3

i) *If for all* $0 < \beta \le \bar{\alpha}$, $u_0(\beta) < 1$, *then* $N_0(\bar{\alpha}) < \infty$ *and* $\alpha = \bar{\alpha}$. *Therefore, P is α-transient;*

ii) *If there exists a β^* with* $0 < \beta^* \le \bar{\alpha}$ *such that* $u_0(\beta^*) = 1$, *then* $\alpha = \beta^*$ *and* $N_0(\alpha) = \infty$. *Therefore, P is α-recurrent.*

Proof: Since $P(\beta)$ is assumed irreducible, the censored matrix $U_0(\beta)$, and therefore the fundamental matrix $N_0(\beta)$, are irreducible. Let $s_k(\beta)$ be the sum of the kth row in $N_0(\beta)$. Then, the Perron-Frobenius Theorem implies (for example, Corollary 1 to Theorem 1.1 of Seneta (1981)):

$$\min_k s_k(\beta) \le n_0(\beta) \le \max_k s_k(\beta); \tag{12}$$

or

$$\min_k s_k(\beta) \le \frac{1}{1 - u_0(\beta)} \le \max_k s_k(\beta) \tag{13}$$

according to Lemma 6. Since the size of $N_0(\beta)$ is finite, the radius of convergence α for P equals

$$\begin{aligned} \alpha &= \sup\{\beta : N_0(\beta) < \infty\} \\ &= \sup\{\beta : \min_k s_k(\beta) < \infty\} = \sup\{\beta : \max_k s_k(\beta) < \infty\} \end{aligned}$$

according to Theorem 1. There are two cases: i) there exists no solution to $1 - u_0(\beta) = 0$ for $0 < \beta \le \bar{\alpha}$. In this case, $n_0(\bar{\alpha}) < \infty$. Therefore, $\alpha \ge \bar{\alpha}$. This, together with iv) of Lemma 6, implies $\alpha = \bar{\alpha}$. Hence, P is α-transient. ii) There exists a solution β^* to $1 - u_0(\beta) = 0$ for $0 < \beta^* \le \bar{\alpha}$. In this case $n_0(\beta^*) = \infty$ and $\alpha = \beta^* \le \bar{\alpha}$. Therefore, P is α-recurrent. This completes the proof. ■

Remark 6 The above result provides a way to classify the transient states and to determine the radius of convergence of P. For an α-recurrent P, it is possible for us to find a condition to further determine when it is α-null or α-positive. For example, if both $\sum_{k=1}^{\infty} kD_k G(\alpha)^{k-1} < \infty$ and $\sum_{k=1}^{\infty} kC_k G(\alpha)^{k-1} < \infty$, then the α-recurrent Markov chain is α-positive; otherwise, it is α-null.

Remark 7 Theorem 3 is also a generalization of classifying an irreducible stochastic matrix into either a recurrent or transient matrix based on censoring. For example, P is recurrent if and only if every censored matrix of P is stochastic. Therefore, the maximal eigenvalue of the censored matrix is one, or $u_0(1) = 1$. P is transient if and only if every censored finite matrix of P is strictly substochastic. Therefore, $u_0(1) < 1$ and $\alpha \ge 1$. If we replace $u_0(1)$ mentioned above by $u_0(\alpha)$, we then have the conditions for α-recurrence and α-transience.

4 *RG*-factorization

The RG-factorization of $(I - P)$, where P is stochastic or strictly substochastic, is a version of UL-factorization having probabilistic interpretations. This factorization was discussed by Heyman (1995), Zhao, Li and Braun (1997, 2000), and Zhao (2000). Heyman showed how to use this factorization to determine the stationary probability vector of a positive recurrent Markov chain. When studying the quasi behaviour of transition matrix P of $M/G/1$ type without boundaries, Li (1997) obtained a UL-factorization for $(I - \beta P)$ without using the R-measure defined in this paper.

The RG-factorization of $(I - \beta P)$ can be proved for an arbitrary transition matrix P, with or without a structure. However, in this paper, we only concentrate on the transition matrix of $M/G/1$ type defined in (1). We first need to define the R-measure and the matrix $G_{1,0}(\beta)$.

Consider the fundamental matrix $\widehat{\beta Q}$ of βQ. Let the first block-column of $\widehat{\beta Q}$ be $(\widehat{\Phi}_{1,1}(\beta)^T, \widehat{\Phi}_{2,1}(\beta)^T, \ldots)^T$. The R-measure for the matrix βP in (1) consists of two sequences of matrices $R_{0,k}(\beta)$ and $R_k(\beta)$, $k = 1, 2, \ldots$, defined by

$$R_{0,k}(\beta) = \sum_{i=1}^{\infty} \beta D_{k+i} \widehat{\Phi}_{i,1}(\beta) \tag{14}$$

and

$$R_k(\beta) = \sum_{i=1}^{\infty} \beta C_{k+i} \widehat{\Phi}_{i,1}(\beta). \tag{15}$$

The (r,s)th entry of $R_{0,k}(\beta)$ can be interpreted as the total expected discounted reward with rate β induced by all visits to state (k,s) before hitting any state in $L_{\le k-1}$, given that the process starts in state $(0,r)$. Similarly, the (r,s)th entry of $R_k(\beta)$ can be interpreted as the total expected discounted reward with rate β induced by all visits to state $(i+k,s)$ before hitting any state in $L_{\le i+k-1}$, given that the process starts in state (i,r), where $i \ge 1$.

The G-measure for βP of $M/G/1$ type consists of two matrices, $G(\beta)$ as defined in (4) and $G_{1,0}(\beta)$ defined by

$$G_{1,0}(\beta) = \widehat{\Phi}_{1,1}(\beta)\beta D_0 = N(\beta)\beta D_0. \tag{16}$$

The (r,s)th entry of $G_{1,0}(\beta)$ can be interpreted as the total expected discounted reward with rate β induced by hitting state $(0,s)$ upon the process entering level 0 for the first time, given that the process starts in state $(1,r)$.

Using Lemma 4 in (14) and (15), the R-measure can then be expressed as

$$R_{0,k}(\beta) = \sum_{i=1}^{\infty} \beta D_{k+i} G(\beta)^{i-1} N(\beta) \tag{17}$$

and

$$R_k(\beta) = \sum_{i=1}^{\infty} \beta C_{k+i} G(\beta)^{i-1} N(\beta) \qquad (18)$$

for $k = 1, 2, \ldots$.

Remark 8 Up to now, we have obtained all components needed in the factorization equation and expressed in terms of $G(\beta)$ only.

For the matrix P of $M/G/1$ type with boundaries, the RG-factorization can be stated in the following theorem.

Theorem 4 *For the matrix P of $M/G/1$ type in (1), $I - \beta P$ can be factorized as*

$$I - \beta P = [I - R_U(\beta)][I - U_D(\beta)][I - G_L(\beta)], \qquad (19)$$

where

$$[I - R_U(\beta)] = \begin{bmatrix} I & -R_{0,1}(\beta) & -R_{0,2}(\beta) & -R_{0,3}(\beta) & \cdots \\ & I & -R_1(\beta) & -R_2(\beta) & \cdots \\ & & I & -R_1(\beta) & \cdots \\ & & & I & \cdots \\ & & & & \ddots \end{bmatrix}, \qquad (20)$$

$U_D(\beta)$ is the diagonal matrix in block form with the first block-entry on the diagonal equal to $U_0(\beta)$ and all the other diagonal block-entries equal to $U(\beta)$, or $U_D(\beta) = diag(U_0(\beta), U(\beta), U(\beta), \ldots)$, and

$$[I - G_L(\beta)] = \begin{bmatrix} I & & & & \\ -G_{1,0}(\beta) & I & & & \\ & -G(\beta) & I & & \\ & & -G(\beta) & I & \\ & & & \ddots & \ddots \end{bmatrix}. \qquad (21)$$

Proof: We only prove the factorization equation for the first block row and first block column entries. The remainder can be similarly proved.

The entry on $(1, 0)$ on the right-hand side is $-[I - U(\beta)]$, which is equal to $-\beta D_0$ from the definition of $G_{1,0}(\beta)$.

The entry on $(0,k)$ with $k \geq 1$ on the right-hand side is

$$
\begin{aligned}
&-R_{0,k}(\beta)[I - U(\beta)] + R_{0,k+1}(\beta)[I - U(\beta)]G(\beta) \\
&= -\sum_{i=1}^{\infty} \beta D_{i+k} G(\beta)^{i-1} + \sum_{i=1}^{\infty} \beta D_{i+k+1} G(\beta)^{i-1} G(\beta) \\
&= -\beta D_{k+1},
\end{aligned}
$$

where the first equality is due to Lemma 4.

Finally, to see the entry on $(1,1)$ on the right-hand side is equal to the entry on the left-hand side, we have

$$
\begin{aligned}
&[I - U_0(\beta)] + R_{0,k}(\beta)[I - U(\beta)]G_{1,0}(\beta) \\
=&[I - U_0(\beta)] + \sum_{i=1}^{\infty} \beta D_{i+1} G(\beta)^{i-1} N(\beta) \beta D_0 \\
=&[I - \beta D_1 - \sum_{k=1}^{\infty} \beta D_{k+1} G(\beta)^{k-1} N(\beta) \beta D_0] + \sum_{i=1}^{\infty} \beta D_{i+1} G(\beta)^{i-1} N(\beta) \beta D_0 \\
=&I - \beta D_1.
\end{aligned}
$$

where the first equality is due to Lemma 4 and the second one due to (8). ■

We will use the RG-factorization to obtain expressions for the β-invariant measure. As we mentioned earlier, the RG-factorization is valid for an arbitrary transition matrix P. As a special case, it is valid for the transition matrix of level-dependent $M/G/1$ type. Therefore, for the transition matrix of level-dependent $M/G/1$ type, expressions for the β-invariant measure can be obtained using the same approach as given in the next section. We only provide the RG-factorization here and leave other details for the reader.

For the transition matrix P of level-dependent $M/G/1$ type given by

$$
P = \begin{bmatrix}
C_1^{(0)} & C_2^{(0)} & C_3^{(0)} & C_4^{(0)} & \cdots \\
C_0^{(1)} & C_1^{(1)} & C_2^{(1)} & C_3^{(1)} & \cdots \\
 & C_0^{(2)} & C_1^{(2)} & C_2^{(2)} & \cdots \\
 & & C_0^{(3)} & C_1^{(3)} & \cdots \\
 & & & \ddots & \ddots
\end{bmatrix},
$$

let

$$Q_k = \begin{bmatrix} C_1^{(k)} & C_2^{(k)} & C_3^{(k)} & \cdots \\ C_0^{(k+1)} & C_1^{(k+1)} & C_2^{(k+1)} & \cdots \\ & C_0^{(k+2)} & C_1^{(k+2)} & \cdots \\ & & \ddots & \ddots \end{bmatrix}, \quad k \geq 1.$$

Let the first block-column of the fundamental matrix $\hat{\Phi}_k$ of βQ_k be $(\hat{\Phi}_{1,1}^{(k)}(\beta)^T, \hat{\Phi}_{2,1}^{(k)}(\beta)^T, \ldots)^T$. Define $R_j^{(k)}(\beta)$ by

$$R_j^{(k)}(\beta) = \sum_{i=1}^{\infty} \beta C_{i+j}^{(k)} \hat{\Phi}_{i,1}^{(k+1)}(\beta), \quad k \geq 0, j \geq 1,$$

and define $G^{(k)}(\beta)$ by

$$G^{(k)}(\beta) = \hat{\Phi}_{1,1}^{(k)}(\beta) \beta C_0^{(k)}, \quad k \geq 1.$$

We also define

$$U_k(\beta) = \beta C_1^{(k)} + R_1^{(k)}(\beta) \beta C_0^{(k+1)}, \quad k \geq 0.$$

Theorem 5 *For the matrix P of $M/G/1$ type in (4), $I - \beta P$ can be factorized as*

$$I - \beta P = [I - R_U(\beta)][I - U_D(\beta)][I - G_L(\beta)], \tag{22}$$

where

$$[I - R_U(\beta)] = \begin{bmatrix} I & -R_1^{(0)}(\beta) & -R_2^{(0)}(\beta) & -R_3^{(0)}(\beta) & \cdots \\ & I & -R_1^{(1)}(\beta) & -R_2^{(1)}(\beta) & \cdots \\ & & I & -R_1^{(2)}(\beta) & \cdots \\ & & & I & \cdots \\ & & & & \ddots \end{bmatrix}, \tag{23}$$

$U_D(\beta)$ is the diagonal matrix in block form with the diagonal entries equal to $U_k(\beta)$, $k = 0, 1, 2, \ldots$, or $U_D(\beta) = diag(U_0(\beta), U_1(\beta), U_2(\beta), \ldots)$, and

$$[I - G_L(\beta)] = \begin{bmatrix} I & & & & \\ -G^{(1)}(\beta) & I & & & \\ & -G^{(2)}(\beta) & I & & \\ & & -G^{(3)}(\beta) & I & \\ & & & \ddots & \ddots \end{bmatrix}. \tag{24}$$

5 β-invariant measures

In this section, we use the RG-factorization to obtain β-invariant measures for the transition matrix P of $M/G/1$ type with boundaries, where $0 < \beta \le \alpha$. Since the RG-factorization is a version of the UL-factorization for a matrix of infinite size, the procedure of obtaining an expression for the β-invariant measure is similar to the Gaussian elimination for solving a finite linear system. For $0 < \beta \le \alpha$, let π be a β-invariant measure of P. We present two sets of expressions: One for an α-recurrent matrix with $\beta = \alpha$ and the other for all the other cases. Since for an α-recurrent matrix, its α-invariant measure is unique up to multiplication by a positive constant, the solution given here is a unique solution up to multiplication by a positive constant. When P is α-transient, the β-invariant measure may not be unique. Examples and remarks will be given.

In the RG-factorization in (19), the three matrices, $[I - R_U(\beta)]$, $[I - U_D(\beta)]$ and $[I - G_L(\beta)]$, are associative. We can also prove that they are associative with any nonnegative vector π, which will lead to solutions for the β-invariant measure.

Lemma 7 *Let P be the transition matrix of $M/G/1$ type and let π be any nonnegative row vector. Then,*

$$\begin{aligned}\pi[I - \beta P] &= \{\pi[I - R_U(\beta)]\}\{[I - U_D(\beta)][I - G_L(\beta)]\} \\ &= \{\pi[I - R_U(\beta)][I - U_D(\beta)]\}[I - G_L(\beta)].\end{aligned}$$

Proof: This is clear, for example, from the sufficient conditions provided in Corollary 1-9 of Kemeny *et al.*. ∎

5.1 α-recurrent with $\beta = \alpha$

In this case, we solve $\pi(I - \alpha P) = 0$ by two steps. In the first step, we let

$$x = \pi[I - R_U(\alpha)]. \tag{25}$$

If $x = (x_0, x_1, \ldots)$ and $\pi = (\pi_0, \pi_1, \ldots)$ are partitioned according to levels, then (25) is equivalent to

$$x_0 = \pi_0,$$

$$x_k = -\pi_0 R_{0,k}(\alpha) - \sum_{i=1}^{k-1} \pi_i R_{k-i}(\alpha) + \pi_k, \quad k \ge 1.$$

Expressing π_k in terms of x_k, we have

$$\pi_0 = x_0, \tag{26}$$

$$\pi_k = \pi_0 R_{0,k}(\alpha) + \sum_{i=1}^{k-1} \pi_i R_{k-i}(\alpha) + x_k, \quad k \geq 1. \tag{27}$$

In the second step, we solve

$$x[I - U_D(\alpha)][I - G_L(\alpha)] = 0 \tag{28}$$

for a nontrivial nonnegative x. If such a solution exists, then π given in (26) and (27) will be nonnegative and nonzero. According to Lemma 7, the above π is an α-invariant measure of P and it is unique up to multiplication by a positive constant.

Equation (28) is equivalent to

$$x_0[I - U_0(\alpha)] - x_1[I - U(\alpha)]G_{1,0}(\alpha) = 0,$$
$$x_k[I - U(\alpha)] - x_{k+1}[I - U(\alpha)]G(\alpha) = 0, \quad k \geq 1.$$

Since P is α-recurrent, it follows from Theorem 3 that the maximal eigenvalue of $U_0(\alpha)$ is $u_0(\alpha) = 1$. Therefore, for nonnegative and irreducible $U_0(\beta)$, there exists a positive x_0 such that

$$x_0[I - U_0(\alpha)] = 0.$$

Hence, $(x_0, 0, 0, \ldots)$ is a solution to (28).

Theorem 6 *If P is α-recurrent, then the unique, up to multiplication by a positive constant, α-invariant measure is given by*

$$\pi_0 = x_0, \tag{29}$$

$$\pi_k = \pi_0 R_{0,k}(\alpha) + \sum_{i=1}^{k-1} \pi_i R_{k-i}(\alpha), \tag{30}$$

where x_0 is the unique, up to multiplication by a positive constant, solution to $x_0[I - U_0(\alpha)] = 0$.

We may notice that this form of solution is the same as that of the invariant measure for a recurrent Markov chain as obtained using the same procedure in Heyman (1995) or an equivalent method in Ramaswami (1988).

5.2 α-recurrent with $\beta < \alpha$ or α-transient with $\beta \le \alpha$

In this case, we also proceed in two steps, but the matrices are associated differently. In the first step, let

$$y = \pi[I - R_U(\beta)][I - U_D(\beta)]. \tag{31}$$

This is equivalent to

$$\begin{aligned} y_0 &= \pi_0[I - U_0(\beta)]; \\ y_1 &= [-\pi_0 R_{0,1}(\beta) + \pi_1][I - U(\beta)]; \\ y_k &= \left[-\pi_0 R_{0,k}(\beta) - \sum_{i=1}^{k-1} \pi_i R_{k-i}(\beta) + \pi_k\right][I - U(\beta)]; \quad k \ge 2. \end{aligned}$$

Since both $[I - U_0(\beta)]$ and $[I - U(\beta)]$ are invertible in this case, we can express π_k in terms of y_k:

$$\pi_0 = y_0[I - U_0(\beta)]^{-1}; \tag{32}$$

$$\pi_1 = \pi_0 R_{0,1}(\beta) + y_1[I - U(\beta)]^{-1}; \tag{33}$$

$$\pi_k = \pi_0 R_{0,k}(\beta) + \sum_{i=1}^{k-1} \pi_i R_{k-i}(\beta) + \pi_k[I - U(\beta)]^{-1}; \quad k \ge 2. \tag{34}$$

In the second step, solve

$$y[I - G_L(\beta)] = 0 \tag{35}$$

for nonnegative nonzero y. If such a solution exists, then π calculated by (32), (33) and (34) is nonnegative and nonzero. According to Lemma 7, the above π is a β-invariant measure of P. Though in many cases such a β-invariant measure is unique up to multiplication by a positive constant, in some other cases, it is simply not unique.

Equation (35) is equivalent to

$$\begin{aligned} y_0 - y_1 G_{1,0}(\beta) &= 0; \\ y_k - y_{k+1} G(\beta) &= 0; \quad k \ge 1. \end{aligned}$$

In the following, we construct a nonnegative nonzero solution y to (35). First, we need the following lemma.

Lemma 8 *For every $0 < \beta \le \alpha$, there exist a $\theta_\beta > 0$ and a nonnegative nonzero vector z such that*

$$\theta_\beta z = zG(\beta). \tag{36}$$

Proof: Since $G(\beta) \geq 0$, the maximal eigenvalue μ_β of $G(\beta)$ is nonnegative. If $\mu_\beta > 0$, then the lemma is proved by choosing z to be the left eigenvector of $G(\beta)$ associated with μ_β.

It follows from Neuts (1989), by using irreducibility of P, that $\mu_1 > 0$. Therefore, $\mu_\beta > 0$ for all $\beta \geq 1$ since $G(\beta)$ is increasing in β.

For $0 < \beta < 1$, the proof also relies on the irreducibility of P. Suppose that there was an s with $0 < s < 1$ such that $\mu_s = 0$. Then, $\mu_\beta = 0$ for all $0 < \beta \leq s$. Therefore, all the eigenvalues of $G(\beta)$ when $0 < \beta \leq s$ are zero according to the Perron-Frobenius theorem for nonnegative matrices. It follows from the Cayley-Hamilton theorem that

$$G^m(\beta) = 0; \quad \text{for all } 0 < \beta \leq s; \tag{37}$$

where m is the size of matrix $G(\beta)$. On the other hand, according to the probabilistic interpretation of $G^m(\beta)$ and the assumption of irreducibility on P, $G^m(\beta) \neq 0$, which contradicts (37). ■

Remark 9 For a finite stochastic matrix, the maximal eigenvalue is one, and for a finite strictly substochastic matrix, the maximal eigenvalue is smaller than one. This property is no longer true if the size of the matrix P is infinite. For example, the maximal eigenvalue is α, which can be strictly greater than one if P is transient. When $\beta < 1$, βP is strictly substochastic, but 1 can still be an eigenvalue of βP. This is true when there exists a β-invariant measure for P.

By using Lemma 8 and letting $y_0 = zG_{1,0}(\beta)$, we can easily check that $y = (y_0, z, z/\mu_\beta, z/\mu_\beta^2, \ldots)$ is a nonnegative nonzero solution to (35). Substituting y into (32), (33) and (34), a β-invariant measure is found.

Theorem 7 *For the following values of β: $\beta < \alpha$ if P is α-recurrent or $\beta \leq \alpha$ if P is α-transient, a nonnegative nonzero β-invariant measure of P is given by*

$$\pi_0 = y_0 N_0(\beta); \tag{38}$$

$$\pi_1 = z[N(\beta) + G_{1,0}(\beta)N_0(\beta)R_{0,1}(\beta)]; \tag{39}$$

$$\pi_2 = \frac{z}{\mu_\beta}\{N(\beta) + G(\beta)N(\beta)R_1(\beta) \tag{40}$$

$$+G(\beta)G_{1,0}(\beta)N_0(\beta)[R_{0,1}(\beta)R_1(\beta) + R_{0,2}(\beta)]\}; \tag{41}$$

$$\pi_3 = \frac{z}{\mu_\beta^2}\{N(\beta) + G(\beta)N(\beta)R_1(\beta) + G(\beta)^2 N(\beta)[R_1(\beta)^2 + R_2(\beta)] \tag{42}$$

$$+G(\beta)^2 G_{1,0}(\beta)N_0(\beta)[R_{0,1}(\beta)R_2(\beta) + R_{0,1}(\beta)R_1(\beta)^2 \tag{43}$$

$$+R_{0,2}(\beta)R_1(\beta) + R_{0,3}(\beta)]\}$$

$$\cdots\cdots$$

or it can be written as one common expression for $k \ge 1$*:*

$$\pi_k = \frac{z}{\theta^{k-1}} \left\{ N(\beta) + \sum_{i=1}^{k-1} G(\beta)^i N(\beta) \sum_{\substack{0 \le j_1 \le \cdots \le j_i \le i \\ \sum_t j_t = i}} R_{j_1}(\beta) R_{j_2}(\beta) \cdots R_{j_i}(\beta) \right.$$

$$\left. + G(\beta)^{k-1} G_{1,0}(\beta) N_0(\beta) \sum_{i=1}^{k} R_{0,i}(\beta) \sum_{\substack{0 \le j_1 \le \cdots \le j_{k-i} \le k-i \\ \sum_t j_t = k-i}} R_{j_1}(\beta) R_{j_2}(\beta) \cdots R_{j_{k-i}}(\beta) \right\}; \tag{44}$$

where $R_0(\beta) = I$.

Example 1 *Consider the quasi-birth-and-death process, or consider the transition matrix* P *in (1) with* $D_i = C_i = 0$ *for* $i \ge 3$*. In this case,* $R_i = R_{0,i} = 0$ *for* $i \ge 2$ *and*

$$U_0(\beta) = \beta D_1 + R_{0,1}(\beta) \beta D_0 = \beta D_1 + \beta D_2 G_{1,0}(\beta);$$

where

$$R_{0,1}(\beta) = \beta D_2 N(\beta); \quad G_{1,0}(\beta) = N(\beta) \beta D_0.$$

The β*-invariant measure is given as*

$$\pi_0 = y_0 N_0,$$

$$\pi_k = \frac{z}{\theta^{k-1}} \left[N(\beta) + \sum_{i=1}^{k-1} G(\beta)^i N(\beta) R_1(\beta)^i + G(\beta)^{k-1} G_{1,0}(\beta) N_0(\beta) R_{0,1}(\beta) R_1(\beta)^{k-1} \right].$$

Remark 10 For a fixed value of β, $G(\beta)$ can be effectively computed by a similar computational scheme for the case of $\beta = 1$, for example, Ramaswami (1988), Latouche (1994) and Meini (1997). When $G(\beta)$ becomes available, other matrices, including $U(\beta)$, $N(\beta)$, $U_0(\beta)$, $N_0(\beta)$, and the R-measure can be computed. Finally, the β-invariant measure π_k can be computed up to a desired index value. A detailed analysis of computational scheme has been carried out and computational complexity has been counted out. We omit all the details here.

Remark 11 To see why we need two different sets of expressions for the β-invariant measure, let us consider the scalar case. If P is α-transient, one is not an eigenvalue of $U_0(\alpha)$. Therefore, $x_0[I - U_0(\alpha)] = 0$ only provides the trivial solution. This means that the method used for the case in 5.1 is not valid. If P is α-recurrent, y given in Section 5.2 is zero. In fact, this y cannot satisfy (31) unless $y_0 = 0$. For example, $I - U_0(\alpha) = 1 - 1 = 0$ for the scalar case, which gives $y_0 = 0$.

While in many cases there exists a unique β-invariant measure up to multiplication by a positive constant, in some other cases, the β-invariant measure is simply not unique. One such example was provided by Gail, Hantler and Taylor (1998), which is given as below.

Example 2 *In this example,* $C_1 = 0$*,* $D_k = C_k = 0$ *for* $k \geq 3$*, and*

$$D_1 = \begin{bmatrix} 1 - p - q_1 & q_1 \\ q_2 & 1 - p - q_2 \end{bmatrix}; \quad D_2 = C_2 = \begin{bmatrix} 0 & p \\ p & 0 \end{bmatrix};$$

$$D_0 = C_0 = \begin{bmatrix} 0 & 1 - p \\ 1 - p & 0 \end{bmatrix};$$

where, $0 < p < 1$, $0 < q_1 < 1$, $0 < q_2 < 1$, $0 < p + q_1 < 1$ *and* $0 < p + q_2 < 1$. *Assume* $p > 1/2$. *In this case, the transition matrix is transient. Two* 1*-invariant measures* $\hat{\pi}$ *and* π *have been found for this example, which are given by* $\hat{\pi}_k = \pi_0 \hat{R}^k$*,* $k = 1, 2, \ldots$*, with* $\hat{\pi}_0 = [1, (p + q_1 - 1/2)/(p + q_2 - 1/2)]$ *and*

$$\hat{R} = \frac{1}{2}\begin{bmatrix} \sigma - 1 & \sigma + 1 \\ \sigma + 1 & \sigma - 1 \end{bmatrix};$$

where $\sigma = p/(1 - p)$*, and by* $\pi_k = \pi_0 R^k$*,* $k = 1, 2, \ldots$*, with* $\pi_0 = (1, q_1/q_2)$ *and*

$$R = \begin{bmatrix} 0 & \sigma \\ \sigma & 0 \end{bmatrix};$$

where $\sigma = p/(1 - p)$*.*

6 Concluding remarks

In this paper, we considered the matrix of $M/G/1$ type with boundaries. We generalized the censoring technique such that it can be used to deal with the nonnegative matrix $\bar{P}$. Based on the generalized censoring technique, we proposed a method for determining the radius of convergence, we obtained conditions on classifying transient states, and proved a factorization theorem for the matrix $I - \bar{P}$. This factorization was then used to obtain expressions for the β-invariant measure.

The method developed here can also be used to study the radius of convergence and β-invariant measures for transition matrices with other type of block-structure, such as, for the matrix of $GI/M/1$ type and even for the matrix of $GI/G/1$ type.

Acknowledgements

The authors acknowledge that this work was supported by a research grant from the Natural Sciences and Engineering Research Council of Canada (NSERC). Dr. Li also acknowledges the support from Carleton University.

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