Rayleigh Flat Fading Channels' Capacity

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Abstract

In this paper, we consider single-user transmission over a Rayleigh flat fading channel, in which the Channel State Information (CSI) is known by the receiver only. Subject to an average transmit power constraint, we study the capacity of an Additive White Gaussian Noise (AWGN) channel with Rayleigh fading. Under an independently identically distributed fading assumption, lower and upper bounds of the channel capacity are given and proved and they are compared to the capacity results numerically computed. Besides, an approximation result of such channel capacity is proposed, and by conducting numerical comparison it is shown that our suggested approximation result has a better performance in approximating Rayleigh fading channels capacity than the bounds given above. In addition, the channel capacity with outage probability is discussed and compared with different outage probabilities.

1. Introduction

The discrete-time channel with Additive White Gaussian Noise (AWGN) model is expressed by

$$V_t = U_t + N_t, \tag{1}$$

where U_t is the channel input, V_t is the channel output, and N_t is an AWGN random variable with mean zero and variance N_0 at time t. Assume that the channel total bandwidth is B and the transmit power is S. Then the noise power in the bandwidth B is then N_0B. The channel Signal-Noise Ratio (SNR) is

$$\gamma = \frac{S}{N_0 B}. \tag{2}$$

Then the channel capacity, which is defined to be an upper bound for the data rate that can be achieved with an arbitrarily small error probability in an AWGN channel, is given by [6]

$$C = Blog_a(1+\gamma) \tag{3}$$

where a is a positive real number, and C is the channel capacity expressed in bits per second (bps) when $a = 2$ or nats per second (nats/s) when $a = e$. In the following discussion, we use nats per second as the unit of channel capacity.

A radio channel exploits an extremely random characteristic, which does not allow us to use the simple AWGN channel model mentioned above to analyze the channel capacity. Radio signals propagate by means of reflection, diffraction, and scattering, which result in three effects a radio signal experiences: attenuation, large-scale shadowing, and small-scale fading. It was proved that these three effects are independent of each other. Signal attenuation is mainly introduced by the location of a receiver (distance between the transmitter and the receiver), which can be predicted by a deterministic model. Large-scale shadowing of a signal is mainly caused by multiple reflections and/or diffractions of the signal while propagating, whose characteristics can be captured with a log-normal distribution model. Small-scale fading of a signal is caused by multiple versions of a transmitted signal with different delay times such that it has both time and location varying property. One type of channels with the fading effect caused by the multi-path time delay spread is flat fading channels in which the period of the transmitted signal is larger than the multi-path delay spread. Since the received signal power varies significantly (in 20 or 30 dB) in a flat fading channel, it is critical to precisely capture the distribution of the channel gain for designing a radio communication system [5]. The most common used signal amplitude distribution in flat fading channels is the Rayleigh distribution, which is the focus of this paper.

The outline of this paper is as follows. Section 2 analyzes the Shannon capacity of a Rayleigh fading channel mathematically. Section 3 compare the numerically computed capacity results with the capacity bounds and the proposed approximation capacity result. Conclusions are made in Section 4.

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2 Rayleigh Fading Channels' Capacity

For a flat-fading channel, we consider the system model as shown in Figure 1, which is expressed by

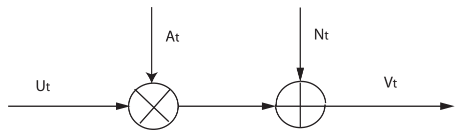

Figure 1. System Model of a Fading Channel

$$V_t = \sqrt{A_t} exp(j\varphi)U_t + N_t, \quad (4)$$

where V_t, U_t, and N_t are the same as defined in (1). $\sqrt{A_t} exp(j\varphi)$ is a complex channel gain with amplitude $\sqrt{A_t}$ and phase φ at time t. The phase φ is uniformly distributed in $[0, 2\pi)$, and the signal amplitude $\sqrt{A_t}$ is a random variable with a Rayleigh probability density function (pdf). A_t is referred to as the channel power gain with the time-varying property, which is independent of the channel input U_t, and could be either independent or correlated over time. The scope of this paper is limited to the channel power gain that is independent of each other over time such that a flat-fading channel has the discrete-time and memoryless properties, i.e., the signal amplitude $\sqrt{A_t}$ is constant over $T \geq 1$ symbol period time units, and after T time units, $\sqrt{A_t}$ is changed to an independent value according to some density function f (the Rayleigh pdf in this paper).

Let S be the average transmit power. The channel bandwidth and the power spectral density of the noise are B and N_0, respectively. The instantaneous SNR at time t is given by

$$\gamma_t = \frac{SA_t}{N_0 B}. \quad (5)$$

Theoretically, γ_t is greater than or equal to zero and does not have an upper bound since the range of the signal amplitude $\sqrt{A_t}$ is between zero and infinity with the density function f. The distributions of A_t and γ_t are determined by each other, and the mean $\bar{\gamma}$ of SNR is given by

$$\bar{\gamma} = \frac{S\bar{A}}{N_0 B} \quad (6)$$

where $\bar{A}$ is the mean of the channel power gain A_t.

We analyze the channel capacity for the case that channel distribution information (CDI) is known by the transmitter and the receiver, and meanwhile, the instantaneous signal amplitude $\sqrt{A_t}$ (the channel state information or CSI) is known at the receiver at time t. So is γ_t. There can be two types of channel capacity: the Shannon capacity and the capacity with outage. Shannon capacity is the maximum data rate that can be sent over the radio channel with asymptotically small error probability, so is also called the ergodic capacity. Capacity with outage is the maximum data rate that can be transmitted over a channel with some outage probability that is the percentage of data that can not be received correctly due to the deep fading. Apparently, a higher data rate (capacity with outage) can be achieved by allowing systems to lose some data in the event of deep fading compared with the data rate always received correctly regardless of the deep fading event (Shannon capacity). However, such capacity with outage only exists in the case where only the receiver knows CSI, since if the transmitter also knows the instantaneous signal amplitude value, then it knows the instantaneous SNR value, thus can decide not to transmit data while over a deep fading channel.

Since the instantaneous SNR γ_t is not known by the transmitter, the channel data rate is constant regardless of γ_t. The maximum data rate can be achieved after the channel has experienced all possible fading states during a sufficiently long transmitting time, which can be expressed as follows in the unit of nats per second [1].

$$C_s = \int_0^{+\infty} Blog_e(1+\gamma)p(\gamma)d\gamma \quad (7)$$

where $p(\gamma)$ is the pdf of the channel SNR γ_t, which is determined by the channel power gain A_t. We can ignore the time t since the channel is observed in steady state. The concavity of the logarithmic function curve gives rise to

$$C_s \leq Blog_e(1+\bar{\gamma}) \quad (8)$$

by applying the Jensen's Inequality [2]. (8) leads to the conclusion that the Shannon capacity of a fading channel with CSI known by the receiver is less than the Shannon capacity of an AWGN channel with the SNR $\bar{\gamma}$.

The following lemma gives an expression of Rayleigh fading channels capacity and its lower/upper bounds .

Lemma 2.1 *A Rayleigh flat-fading channel has the average transmitting power S, receiving bandwidth B and the mean channel gain $\sigma\sqrt{\pi/2}$. Assume the AWGN experienced by the channel has the power N_0. Let $d = \frac{S}{N_0 B}$ be a constant real number. When CDI is known by both the transmitter and the receiver and CSI is known by the receiver only, the Shannon capacity of the Rayleigh fading channel $C_{r,e}$ can be expressed by*

$$C_{r,e} = Bexp(\frac{1}{2d\sigma^2})E_1(\frac{1}{2d\sigma^2}) \quad (9)$$

where $E_n(x) = \int_1^{+\infty} \frac{e^{-xt}}{t^n}dt$, $(x > 0,\ n = 0, 1, \cdots)$ is the exponential integral, and

$$\frac{B}{2}log_e(1+4d\sigma^2) \leq C_{r,e} \leq Blog_e(1+2d\sigma^2). \quad (10)$$

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Proof: The channel gain $\sqrt{A_t}$ has a Rayleigh distribution with pdf

$$f(x) = \frac{x}{\sigma^2} exp(-\frac{x^2}{2\sigma^2}), \quad x \geq 0. \tag{11}$$

By the transformation theorem for single random variables, the channel power gain A_t has an exponential distribution with the mean $2\sigma^2$. The pdf of A_t is

$$p(y) = \frac{1}{2\sigma^2} exp(-\frac{y}{2\sigma^2}), \quad y \geq 0. \tag{12}$$

From (5), we have $\gamma_t = dA_t$, where $d = \frac{S}{N_0 B}$ is a real constant number. Therefore, γ_t has an exponential distribution with the mean $2d\sigma^2$. Now for a Rayleigh fading channel, by (7), we have

$$C_{r,e} = \int_0^{+\infty} Blog_e(1+y)\frac{1}{2d\sigma^2} exp(-\frac{y}{2d\sigma^2})dy. \tag{13}$$

Let $u = log_e(1+y)$, and $dv = exp(-\frac{y}{2d\sigma^2})dy$. Using integral by parts, we have

$$\begin{aligned} C_{r,e} &= \frac{B}{2d\sigma^2}\left[log_e(1+y)(-2d\sigma^2 exp(-\frac{y}{2d\sigma^2})\,|_0^{+\infty} \right. \\ &\quad \left. -\int_0^{+\infty} \frac{-2d\sigma^2}{1+y} exp(-\frac{y}{2d\sigma^2})dy\right] \\ &= B\int_0^{+\infty} \frac{1}{1+y} exp(-\frac{y}{2d\sigma^2})dy \\ &= B\int_1^{+\infty} \frac{1}{x} exp(\frac{1}{2d\sigma^2}) exp(-\frac{x}{2d\sigma^2})dx \\ &= Bexp(\frac{1}{2d\sigma^2})\int_1^{+\infty} \frac{1}{x} exp(-\frac{x}{2d\sigma^2})dx, \end{aligned}$$

which is the Shannon capacity expression given in the lemma. Next, we prove the capacity lower and upper bounds of a Rayleigh fading channel. Apparently, the the second inequality of (10) can be shown by applying the general result of (8) using the fact $E[\gamma_t] = 2d\sigma^2$. Now we give proof for the first inequality of (10) and an easier proof to the second one. For the exponential integral $E_1(x)$ $(x > 0)$, we have known that [3]

$$\frac{1}{2}log_e(1+\frac{2}{x}) \leq e^x E_1(x) \leq log_e(1+\frac{1}{x}), \quad \forall\, x > 0, \tag{14}$$

which is followed by (10) through replacing x by $1/(2d\sigma^2)$ and multiplying each expression by B. The lemma is now proved.

Now we discuss the capacity with outage probability for a flat-fading channel. With the discrete-time of a radio channel, we consider that the instantaneous SNR γ_t is constant over a bursty transmission of data (a time block T), and then γ_t changes to an independent value based on some distribution. Over such a time block T, the channel data rate can reach $Blog_e(1+\gamma_t)$ with a negligible error probability. However, since the transmitter does not know the instantaneous SNR value γ_t, a constant transmission data rate has to be used and it is independent of the instantaneous received SNR γ_t. p_{out} is the defined outage probability $p_{out} = Pr\left[\gamma_t \leq \gamma_{(p_{out})}\right]$, where $\gamma_{(p_{out})}$ is a threshold value of SNR. The data bits received over a transmit time block T are correctly received if the instantaneous SNR γ_t is greater than or equal to $\gamma_{(p_{out})}$ over that time block, and corrupted otherwise. Hence, it is the capacity of a fading channel with an error probability p_{out}. From [4], we can have

$$R \leq \frac{1}{1-p_{out}} \int_0^{+\infty} Blog_e(1+\gamma)p(\gamma)d\gamma, \tag{15}$$

where R is channel data rate. Then the channel capacity with outage probability p_{out} is

$$C_{out} = \frac{1}{1-p_{out}} \int_0^{+\infty} Blog_e(1+\gamma)p(\gamma)d\gamma, \tag{16}$$

from which we can treat Shannon capacity of an independent flat fading channel, where channel state information is known only by the receiver, as a special case of capacity with outage with the outage probability zero. Since deep fading happens rarely, capacity with outage could increase by allowing a small error probability. This will be verified in the next numerical analysis section. For a Rayleigh fading channel, the channel capacity with outage probability p_{out} is

$$C_{r,o} = \frac{1}{1-p_{out}} Bexp(\frac{1}{2d\sigma^2})E_1(\frac{1}{2d\sigma^2}), \tag{17}$$

where $d = S/(N_0 B)$.

3 Numerical Analysis of Channels' Capacity

In this section, we compare the results of (9) by numerical methods with its lower and upper bounds. We also provide an approximation formula.

It is easy to prove the following results

$$log_e(1+\frac{1}{2x}) - \frac{1}{2}log_e(1+\frac{2}{x}) \begin{cases} > 0 & when\ 0 < x < \frac{1}{4}, \\ = 0 & when\ x = \frac{1}{4}, \\ < 0 & when\ x > \frac{1}{4}. \end{cases} \tag{18}$$

Then, when $0 < x \leq \frac{1}{4}$, we have

$$\frac{1}{2}log_e(1+\frac{2}{x}) \leq log_e(1+\frac{1}{2x}) \leq log_e(1+\frac{1}{x}), \tag{19}$$

where the second inequality follows from the monotonicity of log function. Let $x = \frac{1}{2d\sigma^2}$ and multiply 19 by B, we obtain, when $d\sigma^2 \geq 2$,

$$\frac{B}{2} log_e(1+4d\sigma^2) \leq Blog_e(1+d\sigma^2) \leq Blog_e(1+2d\sigma^2). \quad (20)$$

Using 10, when the mean value of a Rayleigh fading channel power gain is no less than 4 (or 6 dB), we have an approximation expression of Shannon capacity as follows

$$C_{r,e} \approx Blog_e(1+d\sigma^2). \quad (21)$$

Figure 2 illustrates a comparison of the Shannon capacity to their lower and upper bounds, and to approximated capacity computed using (20) with different average SNR values subject to the average transmit power constraint. The top curve corresponds to the capacity upper bound, which is also the AWGN capacity with the same average SNR. It shows that, for this range of average SNR values, the fading channel capacity is located between the AWGN capacity and the capacity lower bound computed using (10). It verifies the result in Lemma 2.1. The curve of the approximated capacity is much closer to the curve of numerically computed results than the lower bound curve when the mean of SNR is greater than 6 dB, which shows that our approximation of Shannon capacity of a Rayleigh fading channel in (20) is valid with the range of the average SNR 6 dB to 50 dB.

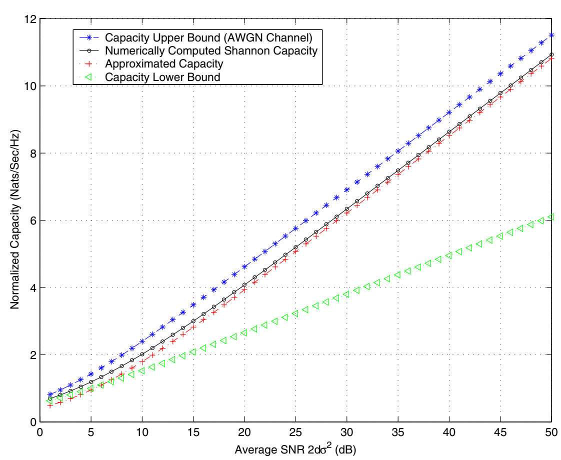

Figure 2. Rayleigh Fading Channels' Capacity

Figure 3 shows a comparison of the channel capacity with outage with different outage probabilities. The AWGN and Rayleigh fading channel Shannon capacity curves are added to the figure for comparison. It shows that, capacity with outage is at least the Shannon capacity. The larger is the outage probability, the greater increase is the capacity. For a fixed outage probability, the capacity may exceed the AWGN Shannon capacity after some average SNR value.

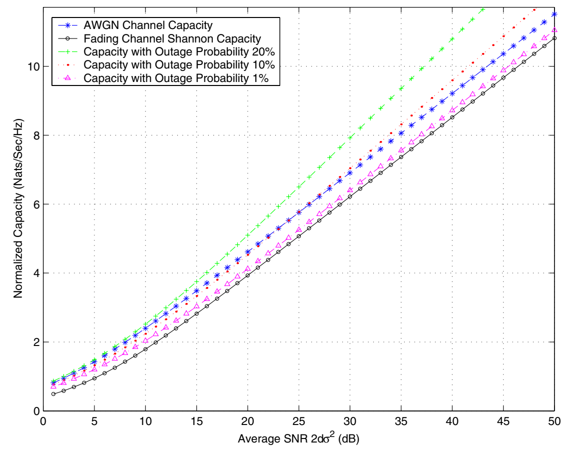

Figure 3. Channel Capacity with Outage

4 Conclusion

Given an average transmit power, we have studied the capacity of a Rayleigh flat fading channel, in which the Channel State Information (CSI) is known by the receiver only. With an independently identical distributed fading assumption, a lower bound and an upper bound of Shannon channel capacity for such a channel have been derived mathematically. An approximation result of a Rayleigh fading channel capacity was proposed. The comparisons of the Shannon capacity to its approximation result and to its lower/upper bounds were made, which verified our approximation result.

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