The Visual Computer

Optical clustering

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This paper presents a definition of 'optical clusters' which is derived from the concept of optical resolution. The clustering problem (induced by this definition) is transformed such that the application of well known Computational Geometry methods yields efficient solutions. One result (which can be extended to different classes of objects and metrices) is the following:
Given a set S of N disjoint line segments in E^2.
(a) The optical clusters with respect to a given separation parameter $r \in R$ can be computed in time $O(N \log^2 N)$.
(b) Given an interval $[a, b]$ for the number $m(S, r)$ of optical clusters which we want to compute, then time $O(N \log^2 N)$ $[O(N \log^2 N + CN)]$ suffices to compute the interval $[R(b), R(a)] = \{r \in R / m(S, r) \in [a, b]\}$ [all C optical clusterings with $R(b) \leq r \leq R(a)$].

Key words: Clustering methods – Computational geometry – Picture analysis

This paper considers the following problem: Given a set S of N objects (i.e. line segments in E^2, see Fig. 1) find a suitable clustering of S which supports picture recognition (Deday and Simon 1980).
There are several ways to specify such a clustering process. Most of the proposed strategies in clustering literature can be classified according to Fig. 2.

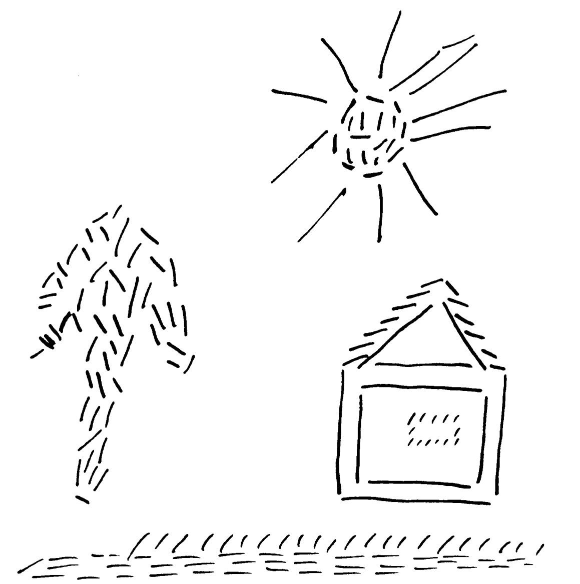

Fig. 1

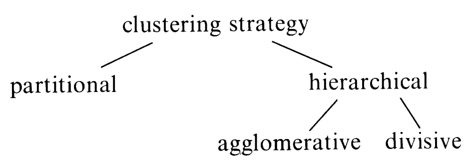

Fig. 2

Agglomerative hierarchical (divisive hierarchical) algorithms produce a sequence of nested partitions with decreasing (increasing) number of clusters. Partitional strategies divide S into C clusters at once (approximately) optimizing some given clustering measure (and mostly trying to improve this partitioning in some postprocessing steps) – refer to Day and Edelsbrunner 1983; Dubes and Jain 1980; Dehne and Noltemeier 1985a–c; Murtagh 1983; Page 1974; Rohlf 1973).
In this paper we propose the following clustering strategy:
If you look at Fig. 1 and assume that it be-

The Visual Computer (1986) 2:39–43

comes blurred then all objects seem to grow and those ones forming a 'natural' cluster start to unite. Thus, we consider optical selectivity as a model for our clustering strategy. With D denoting the minimum distance of two points (i.e. in E^2) such that both points are separable from each other, all points which lie inside some circle of radius $D/2$ are not separable and have to be considered identical. With this we define two line segments separable if no two points (one of each segment) exist which are not separable. This yields a definition of clusters which we will call optical clusters.
Part 2 of this paper gives a general framework including a precise definition of optical clusters and a transformation of this clustering problem such that well known Computational Geometry methods can be applied and yield efficient solutions. Part 3 and 4 show how these results can be applied to several classes of geometric objects.

A general framework for optical clustering

Let $S=\{s_1, \ldots, s_N\}$ be a set of N disjoint objects in R^n (compact subsets of R^n without holes), consider some convex distance function $d: R^n x R^n \rightarrow R\,[(\forall s, s' \in S)$: $d(s, s') := \inf\{d(x, y)/x \in s, y \in s'\}]$, and let $c(P, r) := \{x \in R^n / d(P, x) \leqq r\}$ denote the ball with center P and radius r.

Definition. $s_i, s_j \in S$ are *r-connected* $(s_i \sim_r s_j)$ iff

$$(\exists c(P, r'), r' \leqq r)\colon\ c(P, r') \cap s_i \neq \emptyset \quad \text{and} \quad c(P, r') \cap s_j \neq \emptyset.$$

Since the transitive closure $cl(\sim_r)$ of $\sim_r$ is an equivalence relation we define 'optical clusters' as follows:

Definition. The equivalence classes of $cl(\sim_r)$ are called *optical clusters with resp. to (separation parameter) r.*
Let $m(S, r)$ denote the number of optical clusters of S with resp. to r.
With this the following lemma is obvious:

Lemma 1. $r \leqq r' \Rightarrow m(S, r) \geqq m(S, r')$.

Given the task to construct the optical clusters of a set S of geometric objects with respect to some given resolution r a naive solution may be the following:
(1) Compute the graph $(S, \sim_r)$ [Let $(S, \sim_r)$ denote the graph with vertex set S and edges between all pairs of vertices $(s, s') \in \sim_r$.].
(2) Find the connected components of $(S, \sim_r)$.

But there is one major drawback to such a simple solution: Since $|\sim_r| \in \Omega(N^2)$ in the worst case we may have to compute a graph with $\Omega(N^2)$ edges.
However, we are only interested in the equivalence classes of $cl(\sim_r)$ which raises the following question:
Is there another relation $\diamond_r$ with $cl(\diamond_r) = cl(\sim_r)$ and $|\diamond_r| \in O(N)$?
We will answer this question affirmatively for $n=2$.

Definition
(a) $s_i, s_j \in S$ are *Delaunay connected* with resp. to r $(s_i \diamond_r s_j)$ iff
$(\exists r' \leqq r, P \in R^n)$: $d(P, s_i) = r' = d(P, s_j)$ and
$(\forall s_k \in S - \{s_i, s_j\})$: $d(P, s_k) > r'$.
(b) $s_i, s_j \in S$ are *directly connected* with resp. to r $(s_i \approx_r s_j)$ iff
$(\exists c(P, r'), r' \leqq r)$: $c(P, r') \cap s_i \neq \emptyset$ and $c(P, r') \cap s_j \neq \emptyset$ and
$(\forall s_k \in S - \{s_i, s_j\})$: $s_k \cap c(P, r') = \emptyset$.
With this, we prove the following

Lemma 2.
(a) $cl(\sim_r) = cl(\approx_r)$
(b) $\approx_r = \diamond_r$

Proof.
(a) (i) Since $\approx_r \subset \sim_r$, $cl(\approx_r) \subset cl(\sim_r)$.
(ii) Let $s_i \sim_r s_j$, then we have objects $t_0, \ldots, t_w \in S$, such that $s_i = t_0 \sim_r t_1 \sim_r t_2 \sim_r \ldots \sim_r t_{w-1} \sim_r t_w = s_j$.
Thus, we get $(s_i, s_j) \in cl(\approx_r)$ if the following lemma holds:

Lemma. $(\forall s, s' \in S)$: $(s, s') \in \sim_r \Rightarrow (s, s') \in cl(\approx_r)$.

Proof. The proof of this lemma is sketched by Fig. 3: If s, s' are connected by some ball B that is intersected by some other objects, then there is a path of balls $b_1, \ldots, b_L$ which transitively connect s and s' and do not intersect other objects.

(b) (i) $\oplus_r \subset \approx_r$ is obvious.
(ii) Let $s \approx_r s'$, and let $c(P, r)$ be a ball as described in the definition of relation $\approx_r$. Let $q[q']$ be the point of $s[s']$ closest to P, $1[1']$ be the line segment connecting P and $q[q']$, and L be the union of 1 and 1'. With $BS(s, s') := \{x \in R^n / d(x, s) = d(x, s')$ denoting the bisector of s and s' it is easy to see that the ball $c(P', r')$ with $P' := L \cap BS(s, s')$ and $r' := d(P', s)$ satisfies the conditions described in the definition of $\oplus_r$, thus $s \oplus_r s'$ (see Fig. 4 for an illustration in E^2).

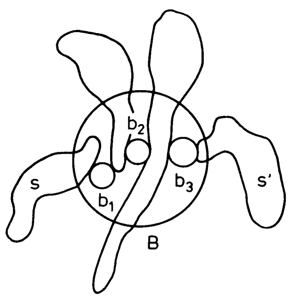

Fig. 3

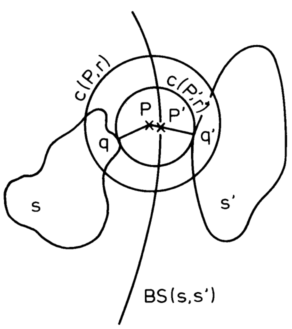

Fig. 4

This yields the following
Theorem 1. (a) $cl(\sim_r) = cl(\oplus_r)$
(b) $|\oplus_r| \in O(N)$ *for* $n = 2$.

Proof. (a) is a trivial consequence from Lemma 2. For $n = 2$ it is easy to see that the graph $(S, \oplus_r)$ is planar (see Fig. 5), thus (b) follows immediately.

Thus, the optical clusters of S with respect to separation parameter r are exactly the connected components of the graph $(S, \oplus_r)$, and $(S, \oplus_r)$ has $O(N)$ edges in the planar case.

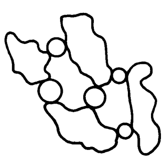
Fig. 5

On the other hand it is easy to see that if $(S, DT(S))$ denotes the *Delaunay Triangulation* (Shamos and Hoey 1975; Lee and Drysdale 1981) then we have $DT(S) = \bigcup_{r \geq 0} \oplus_r$.
With this we define $\min(s, s') := \min\{r \geqq 0 / (s, s') \in \oplus_r\}$ for all $(s, s') \in DT(S)$ and call the labeled graph $(S, DT(S))$ with labeling $(s, s') \rightarrow \min(s, s')$ clustering graph of S (denoted by $CG(S)$).

Hence, $\oplus_r = \bigcup_{[(s, s') \in DT(S), \min(s, s') \leqq r]} (s, s')$.

Clustering sets of line segments in E^2

Let $S = \{s_1, \ldots, s_N\}$ be a set of disjoint line segments in E^2. From Lee and Drysdale (1981) we know that the Voronoi diagram $V(S)$ and the Delaunay triangulation $DS(S)$ (which is the dual of $V(S)$, see Fig. 6) can be constructed in time $O(N \log^2 N)$. For each edge $e_i = (s, s') \in DT(S)$ we compute the value $\min(s, s')$ as follows:
Consider the Voronoi edge v which is an edge of the Voronoi polygons of both s and s'. It is

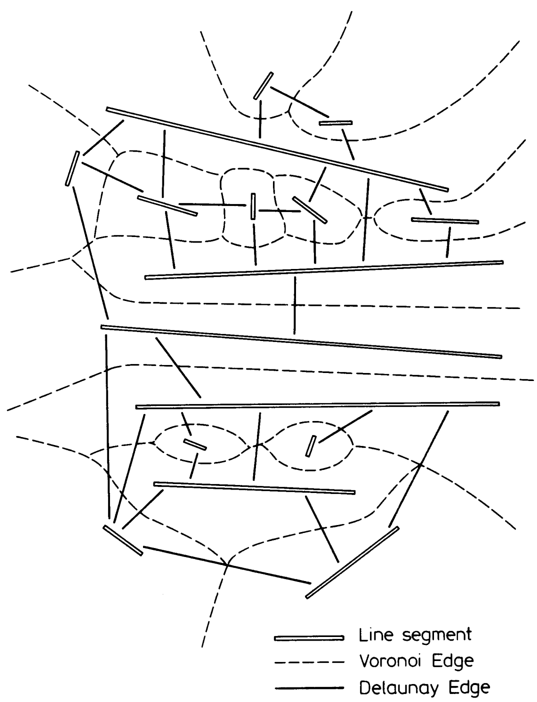

Fig. 6 (from Lee and Drysdale 1981)

easy to see that v is exactly the set of all centers of circles which "connect' s and s' as described in the definition of relation $\diamond_r$ for all $r>0$. Thus, $\min(s,s')=\min\{d(s,x)/x\in v\}$. Since v consists of at most two segments of parabolas and three segments of bisectors (Lee and Drysdale 1981) $\min(s,s')$ can be computed in constant time.
Thus, we get

Lemma 3. *The clustering graph $CG(S)$ of a set of N disjoint line segments in the Euclidean plane can be computed in time $O(N\log^2 N)$.*

Given the clustering graph $CG(S)$ and a real value $r>0$ we can compute the optical clusters with respect to separation parameter r (and the number $m(S,r)$ of such clusters) as follows:
Delete all edges (s,s') of $CG(S)$ with $r<\min(s,s')$ and compute the connected components with resp. to the remaining edges (Aho et al. 1974, Chap. 5).
Since linear time suffices to compute the connected components of a graph, we state

Lemma 4. *Given the clustering graph $CG(S)$ and a real number $r>0$ then the optical clusters with respect to separation parameter r and their number $m(S,r)$ can be computed in time $O(N)$.*

However, who knows a suitable r?
Consider Fig. 1: If r is too small, then each line segment might become a cluster of its own and if r is too large then all line segments might become one cluster. In fact, the knowledge of a suitable r already includes nontrivial knowledge about the structure of the picture. We might know, however, that Fig. 1 contains about 4 objects or that the number of objects is between say 3 and 6.
Can we compute a suitable r with such knowledge?
Let $[a,b]\subseteq\{1,\ldots,N\}$ be an interval denoting the desired range of $m(S,r)$. Given the task to find the set $R(a,b)\subset R$ with $\{m(S,r)/r\in R(a,b)\}=[a,b]$ we proceed as follows:
Since $m(S,r)$ is monotone and decreasing with respect to r (see Lemma 1) it is clear, that $R(a,b)$ is a closed interval $[R(b),R(a)]$.
Let $CG(S)$ have k edges and $\min_1,\ldots,\min_k$ be the values of their labels $\min(s,s')$ in increasing sorted order then we have

$$m(S,\min_1)\geqq m(S,\min_2)\geqq\ldots\geqq m(S,\min_k)$$

and

$$(\forall 1\leqq i<k, r\in[\min_i,\min_{i+1})): m(S,r)=m(S,\min_i).$$

Thus, $R(a)$ and $R(b)$ can be calculated using a binary search strategy. Since $m(S,\ldots)$ has to be computed $O(\log k)$ times and $k\in O(N)$ we get an $O(N\log N)$ time complexity for this step, which is dominated by the time complexity for the computation of $CG(S)$.
Summarizing this we get

Theorem 2. *Given a set S of N disjoint line segments in the Euclidean plane.*
(a) *The optical clusters with respect to a given separation parameter $r\in R$ can be computed in time $O(N\log^2 N)$.*
(b) *Given an interval $[a,b]$ for the number of optical clusters which we want to compute, then time $O(N\log^2 N)$ $[O(N\log^2 N+CN)]$ suffices to compute the interval $[R(b),R(a)]=\{r\in R/m(S,r)\in[a,b]\}$ [all C optical clusterings with $R(b)\leqq r\leqq R(a)$].*

Clustering other classes of objects (in R^2)

Since the construction of Voronoi diagrams is crucial for Theorem 2 it can easily be generalized to sets of disjoint polygonial chains (with a total number of N edges) and set of circles in E^2 (Lee and Drysdale 1981). The results do also hold for a class of different metrices in R^2 which is characterized in Lee and Drysdale (1981).
For planar point sets and a large class of convex distance functions (Chew and Drysdale 1985) Theorem 2 holds, too, but with all terms '$N\log^2 N$' replaced by '$N\log N$', respectively.

Conclusion

The definition of optical clusters as given in this paper has three advantages:
(1) It is an analytical definition (not an algorithmic i.e hierarchical specification) of a clustering strategy which is of considerable

interest in picture processing and related applicational fields and which allows efficient computation.

(2) The clustering method can be efficiently applied to various classes of geometric objects in the plane.

(3) Depending on the a priori knowledge about the picture it is possible to chose a local (distance between objects) as well as a global (number of clusters) input parameter for the control of the clustering process.

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