

Coarse Grained Parallel Next Element Search

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Abstract

We present a parallel algorithm for solving the next element search problem on a set of line segments, using a BSP like model referred to as the Coarse Grained Multicomputer (CGM). The algorithm requires $O(1)$ communication rounds (h -relations with $h=O(n/p)$), $O((n/p) \log n)$ local computation, and $O((n/p) \log n)$ storage per processor. Our result implies solutions to the point location, trapezoidal decomposition and polygon triangulation problems. A simplified version for axis parallel segments requires only $O(n/p)$ storage per processor, and we discuss an implementation of this version.

As in a previous paper by Develliers and Fabri[11], our algorithm is based on a distributed implementation of segment trees which are of size $O(n \log n)$. This paper improves on [11] which presented a CGM algorithm for the special case of trapezoidal decomposition only and requires $O((n/p) * \log p * \log n)$ local computation.

1. Introduction

The next element search problem is a well known problem in computational geometry with many applications[1]. Given a set of n non-intersecting line segments s_1, \dots, s_n and a direction D_{next} (without loss of generality we can assume that D_{next} is the direction of the positive Y -axis), the next element search problem consists of finding for each query point q_i of a set of m query points q_1, \dots, q_m the line segment s_j first intersected by the ray starting at q_i in direction D_{next} ($m=O(n)$); see Figure 1. A sequential solution requires $O(n \log n)$ time and $O(n)$ space[17].

In this paper, we present a parallel algorithm for solving the next element search problem on a coarse grained multicomputer, CGM (see Section 2 for a discussion of the model). The algorithm requires $O(1)$

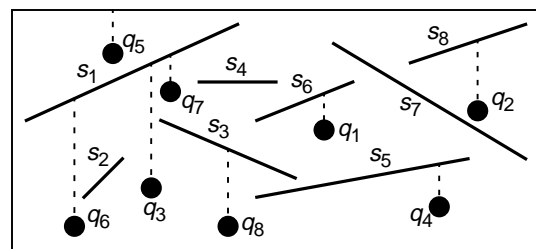


Figure 1: Illustration of Next Element Search

communication rounds, $O((n/p) \log n)$ local computation and needs $O((n/p) \log n)$ storage per processor. A simplified version for axis parallel segments requires only $O(n/p)$ storage per processor.

The next element search algorithm presented here implies immediately solutions for the point location, trapezoidal decomposition and triangulation problems.

As in a previous paper by Develliers and Fabri [11], our algorithm is based on a distributed implementation of segment trees which are of size $O(n \log n)$. This paper improves on [11] which presented a CGM algorithm for the special case of trapezoidal decomposition only and requires $O((n/p) * \log p * \log n)$ local computation.

The organization of this paper is as follows: Section 2 and Section 3 define the coarse grained multicomputer model and segment tree, respectively. The algorithm is presented in Section 4 and Section 5. In Section 6 we outline a simplified version for axis parallel segments and discuss an implementation of this version. Section 7 concludes the paper and outlines some important applications.

2. The coarse grained multicomputer model

Speedup results for theoretical PRAM algorithms do not necessarily match the speedups observed on real machines [2][8]. Given sufficient slackness in the number of processors, Valiant's BSP approach [20] simulates

PRAM algorithms optimally on distributed memory parallel systems. Valiant points out, however, that one may want to design algorithms that utilize local computations and minimize global operations [19][20]. The BSP approach requires that g (= local computation speed / router bandwidth) is low, or fixed, even for increasing number of processors. Gerbessiotis and Valiant [13] describe circumstances where PRAM simulations cannot be performed efficiently, among others if the factor g is high. Unfortunately, this is true for most currently available multiprocessors. The algorithms presented here consider this case for the next element search problem. Furthermore, as pointed out in [20], the cost of a message also contains a constant overhead cost s . The value of s can be fairly large and the total message overhead cost can have a considerable impact on the speedup observed (see e.g. [6]).

We are therefore using a variation of the BSP model, referred to as *Coarse Grained Multicomputer*, CGM. It is comprised of a set of p processors P_1, \dots, P_p with $O(m/p)$ local memory per processor and an arbitrary communication network (or shared memory). The term “coarse grained” refers to the fact that we assume that the size $O(m/p)$ of each local memory is “considerably larger” than $O(1)$. Our definition of “considerably larger” will be that $m/p \geq p$.

All algorithms consist of alternating local computation and global communication rounds. Each communication round consists of routing a single h -relation with $h=O(m/p)$, i.e. each processor sends $O(m/p)$ data and receives $O(m/p)$ data. We require that all information sent from a given processor to another processor in one communication round is packed into one message. In the BSP model, a computation/communication round is equivalent to a superstep with $L = (m/p)g$ (plus the above “packing” and “coarse grained” requirement).

Finding an optimal algorithm in the coarse grained multicomputer model is equivalent to minimizing the number of communication rounds as well as the total local computation time. This considers all parameters discussed above that are affecting the final observed speedup, and it requires no assumption on g . Furthermore, it has been shown that minimizing the number of supersteps also leads to improved portability across different parallel architectures [9][19][20]. The above model has been used (explicitly or implicitly) in parallel algorithm design for various problems ([4][5][6][7][8][10][12][15]) and shown very good practical timing results.

We now list the basic operations required by our algorithms. Each of these operations requires $O(1)$ communication rounds, $O((n/p) \log n)$ local computation, and requires $n/p \geq p$.

Global sort: Sort $O(n)$ data items stored on a CGM, n/p

data items per processor, with respect to the CGM’s processor numbering[14].

All-to-all broadcast: Every processor sends one message to all other processors[6].

Personalized all-to-all broadcast: Every processor (in parallel) sends a different message to every other processor[6].

Partial sum (Scan): Every processor stores one value, and all processors compute the partial sums of these values with respect to some associative operator[6].

3. Segment tree definition

A well known method for solving the next element search problem is to apply a plane sweep in direction D_{next} using a segment tree [3][16][17]. Let $s_i^{(x)}[q_i^{(x)}]$ be the projection of the line segment s_i [query point q_i , respectively] onto the x -axis, and let $(x_1, x_2, \dots, x_{2n})$ be the sorted sequence of the projections of the $2n$ endpoints of s_1, \dots, s_n onto the x -axis. The segment tree $T(S) = (V_s, E_s)$ is a complete binary tree with leaves x_1, x_2, \dots, x_{2n} . For every node v of $T(S)$ an interval $xrange(v)$ is defined as follows:

- if v is a leaf x_i , then $xrange(v) = [x_i, x_{i+1})$, assuming that $[x_{2n}, x_{2n+1}) = [x_{2n}, x_{2n}]$.
- if v is an internal node, then $xrange(v)$ is the union of all intervals $xrange(v')$ such that v' is a leaf of the subtree of $T(S)$ rooted at v .

With every node v of the segment tree $T(S)$ there is associated a catalog $C(v) \in S$ defined as follows:

- $C(v) = \{s \in S \mid xrange(v) \subseteq s(x) \text{ and } \text{not } (xrange(\text{father of } v) \subseteq s(x))\}$.

Note that each line segment can occur in $O(\log n)$ catalogs. The size of the segment tree $T(S)$, denoted $|T(S)|$, is equal to the number of nodes and edges of $T(S)$ plus the total size of all catalogs. Therefore $|T(S)| = O(n \log n)$. Hence, storing the segment tree with all of its catalogs requires $N = O(n \log n)$ space. Also note that the sum of the lengths of all catalogs of all nodes with the same level (height) is $O(n)$ [16]. For the remainder, define $xrange(T(S)) = xrange(r)$ where r is the root of $T(S)$. Also define $xrange(s)$ and $xrange(q)$ to be $s_i^{(x)}[q_i^{(x)}]$ respectively.

4. Parallel segment tree construction

In this section we will show how to construct a distributed representation of a segment tree $T(S)$, called a *parallel segment tree*, for a set of n line segments on a CGM such that the resulting data structure can be efficiently used to process next element search queries in parallel. The approach will be to partition the segment tree (without associated catalogs) into substructures of size

$O(n/p)$ such that no processor stores more than $O(1)$ such substructures; see Figure 2. The catalogs for nodes in

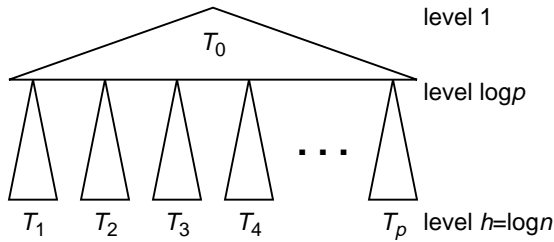


Figure 2: Decomposition of the Segment Tree

subtrees T_1, \dots, T_p will be stored with their associated subtrees, while the catalogs for nodes in tree T_0 will be partitioned, when necessary, into lists of size $((n \log p) / p)$ and distributed such that no processor stores more than $O(1)$ such lists; see Figure 3. We first describe our

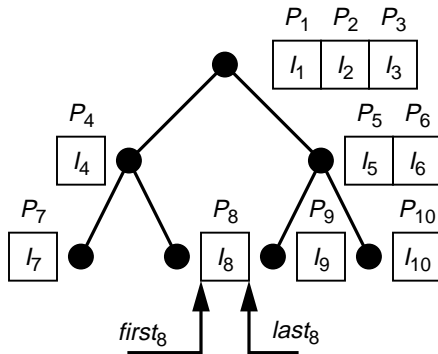


Figure 3: The tree T_0 with catalogs partitioned into lists l_i , where list l_i is of size $|l_i| \leq n/p$

distributed segment tree representation and then give an algorithm that efficiently constructs such a segment tree on a CGM.

A parallel segment tree $T(S)$ (without catalogs) is a complete tree with n leaves which consists of $k = \log n$ levels where the level of a node v is defined recursively as follows: $level(v) = 1$, if v is the root, and $level(v) = level(parent(v)) + 1$, otherwise. Let $rank(v)$ denote the rank of the vertex v in the left to right ordering of the level for v . For $v \in V$, let $subtree(v) \in 1 \dots p$ be defined as follows: $subtree(v) = p$, if $1 \leq level(v) \leq \log p - 1$ and $subtree(v) = rank(v')$ otherwise, where v' is the ancestor of v such that $level(v') = \log p$. Let $V_i = \{v \in V \mid subtree(v) = i\}$. Let T_i denote the subtree of T with vertex set V_i and edge set $E_i = \{(v, v') \in E \mid subtree(v) = subtree(v') = i\}$. Let $S(T_i)$ be the set of segments of S whose endpoint is in $xrange(T_i)$.

In our distributed representation of a segment tree $T(S)$, every subtree T_i ($1 \leq i \leq p$) will be stored on processor P_i . Each processor P_i will also store the catalogs associated

with nodes in V_i .

Observation 1 If a line segment contains $xrange(T_i)$ then it is not contained in any catalog of T_i (except for possibly the root).

Observation 2 $S(T_i)$ is of size $O(n/p)$.

A consequence of Observation 2 is that each subset $S(T_i)$ ($1 \leq i \leq p$) with its catalogs can be stored in the memory of a single processor. The tree T_0 consists of $O(p)$ nodes and catalogs whose combined size is $O(n \log p)$. Therefore T_0 is too big to be stored on a single processor. Instead, each processor will store a copy of T_0 (without catalogs) and a list l_i which is a portion, or all, of the catalog of a node v of T_0 . Let L denote the list formed by concatenating the catalogs associated with nodes of T_0 , where catalogs are ordered by level and then rank in level and all catalogs are padded to be of a length evenly divisible by $((n \log p) / p)$. The list l_i consisting of elements $in/p, \dots, (i+1)n/p$ from L will be stored on processor P_i ; see Figure 3.

Observation 3 Since T_0 has height $\log p$ and a line segment can appear in at most 2 catalogs of T_0 at the same level, the total size of list L is $O(n \log p)$.

Algorithm 1: "Parallel Segment Tree Construction".

Input: Processor P_i ($1 \leq i \leq p$) stores a subset S_i of n/p elements of S . *Output:* Processor P_i ($1 \leq i \leq p$) stores $S(T_i)$, the tree T_0 without catalogs but with the values $xrange(v)$ for each $v \in V_0$, and the list l_i which is a portion, or all, of the catalog of a node v of T_0 .

- (1) Create for each $s \in S$ two copies, one for each endpoint. Refer to the new set as S' . Sort S' by x -coordinate, such that each processor P_i contains a subset S_i of size $O(n/p)$. Now, processor P_i stores $S(T_i)$ and $xrange(T_i)$. From Observation 2, $|S(T_i)| = O(n/p)$.
- (2) Use an all-to-all broadcast to distribute all $xrange(T_i)$ ($1 \leq i \leq p$) to all processors. Each processor computes T_0 without any catalogues but with $xrange(v)$ values for all $v \in V_0$.
- (3) Processor P_i computes the catalogs of T_0 with respect to S_i only. We refer to this reduced version of T_0 as $T_{0,i}$. Note that $|T_{0,i}| = O((n/p) \log p)$.
- (4) Assume that all nodes of T_0 have a unique index. Consider a line segment s in the catalog of the node v in T_0 with index j . Let l be the left vertical boundary of the vertical slab defined by $xrange(v)$; see Figure 4. Let $y(s)$ denote the y -coordinate of the intersection of s and l . We define j and $y(s)$ as the primary and secondary key for s , respectively. Using a global sort, all line segments in the catalogs of all $T_{0,i}$, ($1 \leq i \leq p$), are sorted with respect to their primary and secondary key in such a way that no processor stores two line segments with different primary keys. The latter can be achieved by using $2p$ virtual processors.

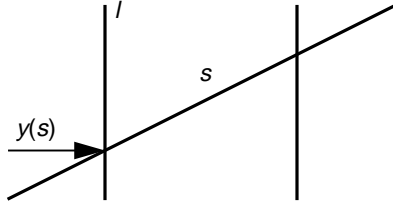


Figure 4: The vertical slab defined by $xrange(v)$

Theorem 1 Algorithm 1 constructs a parallel segment tree on a CGM using $O((n/p) \log n)$ memory per processor, $O(1)$ communication rounds and $O((n/p) \log n)$ local computation.

Proof. The memory bound follows from Observation 2 and Observation 3. The algorithm uses a constant number of the basic communication operations of Section 2. The local computation time is bounded by the local time for sorting (Steps 1 and 4). \square

5. Parallel query processing

Given a segment tree $T(S)$ and a query point $q \in Q$, the next element of q in S can be determined by a simple search in $T(S)$ from the root of $T(S)$ to the leaf v whose $xrange$ contains q (see e.g. [17] for details).

Recall that, at the end of Algorithm 1, processor P_i ($1 \leq i \leq p$) stores $S(T_i)$, the tree T_0 without catalogs but with the values $xrange(v)$ for each $v \in V_0$, and a list l_i which is a portion, or all, of the catalog of a node v of T_0 . Let $first_i$ and $last_i$ refer to the first and last element of l_i , respectively (see Figure 3).

The following algorithm uses the parallel segment tree to answer all queries in parallel. Each individual query is first “routed” through T_0 and then through the respective T_i . In T_0 , the tree structure is used to schedule the computation. However, the catalog lookups are reduced to sequential next element search problems. For the subtrees T_i , a load balancing scheme is used to ensure equal distribution of work. In each T_i , all search processes are reduced to a single sequential next element search problem.

Algorithm 2: “Parallel Query Processing”. *Input:* A parallel segment tree $T(S)$ as produced by Algorithm 1 and a set Q of n queries, where each processor P_i stores a subset Q_i of size n/p . *Output:* Each Processor P_i ($1 \leq i \leq p$) stores for each $q \in Q_i$ its next element s .

- (1) Using an all-to-all broadcast, send all $first_i$ and $last_i$ to all processors P_i . Recall that every processor P_i ($1 \leq i \leq p$) stores T_0 without catalogs but with values $xrange(\cdot)$.
- (2) Using the $first_i$ and $last_i$ values of all processors, processor P_i computes for each $q \in Q_i$ the sublists l_j ($1 \leq i \leq p$) that have to be searched in order to process

q on T_0 . The problem reduces to solving for each node v of T_0 a next element search problem for a certain number X_v of line segments and a certain number Y_v of queries. Note that, each such problem requires $O((X_v + Y_v) \log(X_v + Y_v))$ computation. Since $\sum_v X_v = n/p$ and $\sum_v Y_v = p$, it follows that the total local time is $O((n \log n) / p)$.

- (3) Using global sort and partial sum operations, determine for each sublist l_i the number of queries, $g(l_i)$, that have to be searched in l_i . Let $k(l_i) = \lceil g(l_i) p / n \rceil$.
- (4) Create $k(l_i)$ copies of l_i ($1 \leq i \leq p$). Note that, this requires $2p$ virtual processors. Broadcast the new addresses of the sublists l_i .
- (5) Each processor P_i makes $\log p$ copies of its query set Q_i and routes the queries to the respective sublists using sort.
- (6) The queries are processed on the sublists to which they were sent in Step 5, and the $\log p$ results for each query are collected in a single processor by using a global sort operation. (Note that, $\log p \leq n/p$)
- (7) Determine for each T_i the number, $a(T_i)$, of queries whose search path includes the root of T_i ($1 \leq i \leq p$). This can be computed by using global sort and partial sum operations. Let $b(T_i) = \lceil a(T_i) / (n/p) \rceil$.
- (8) Create $b(T_i)$ copies of $S(T_i)$. Note that, this requires $2p$ virtual processors. Route n/p queries to each processor such that a processor storing $S(T_i)$ receives n/p queries whose search path contains the root of T_i .
- (9) Each processor processes the queries for its subtree T_i ($1 \leq i \leq p$) by applying the standard sequential next element search algorithm [17] for $S(T_i)$ and its query set.
- (10) Combine the results of Step 9 with those obtained in Step 6, using sort.

Theorem 2 Algorithm 2 solves the next element search problem for n line segments on a CGM with $O((n/p) \log n)$ memory per processor using $O(1)$ communication rounds and $O((n/p) \log n)$ local computation.

Proof. The memory bound follows from Theorem 1. The algorithm uses a constant number of the basic communication operations of Section 2. The local computation time is bounded by the local time for sorting and sequential next element search. \square

6. A simplified algorithm for axis parallel line segments

If we limit the segments to be axis parallel (i.e. they are all horizontal), we can reduce the space requirement to $O(n/p)$ per processor by applying the lower envelope algorithm presented in [6].

Algorithm 3: “Next Element Search for Axis

Parallel Line Segments². *Input:* A set S of n axis parallel line segments and a query set Q , where each processor P_i stores n/p line segments and queries, respectively. *Output:* The next element in S for each query point $q \in Q$, where each processor stores n/p next element results. Note: During the algorithm, queries will be handled like line segments of zero length.

- (1) Sort $S \cup Q$ by increasing y -coordinate. Each processor P_i solves, both, the next element search problem and lower envelope problem sequentially for the data $S_i \cup Q_i$ it has received. Let Q' be all queries whose next element has not yet been found, and let S' be the union of all lower envelopes.
- (2) Sort $S' \cup Q'$ by x -coordinate of the right endpoints and the x -coordinates of the query points, respectively.
- (3) Let l_1, \dots, l_{p-1} be the vertical lines that separate the sorted segments in the p different processors. Perform an all-to-all broadcast where processor P_i sends l_i to all other processors.
- (4) Perform a personalized all-to-all broadcast, where processor P_i sends segment $s \in S'$ to processor P_j if and only if s intersects the vertical line l_j .
- (5) Each processor P_i solves locally the next element search problem for its subset of Q' and the line segments of S' received in Steps 2 and 4.

Theorem 3 *Algorithm 3 solves the next element search problem for n axis parallel line segments on a CGM with $O(n/p)$ memory per processor using $O(1)$ communication rounds and $O((n/p) \log n)$ local computation.*

Proof. The correctness follows from the fact that for each $q \in Q_i$, its next element is either in S_i or in the lower envelope of an S_j with $j > i$. The algorithm uses a constant number of the basic communication operations of Section 2 and duplicates no data. \square

Algorithm 3 was implemented on an Intel iPSC/860 hypercube and tested for $p = 2, 4$ and 8 processors. For each value of p , we ran tests for $n = 100, 200, 500, 1000, 2000, 5000$ and 10000. We used 10 sets of data on each combination of p and n . In 5 of them the line segments were evenly distributed in a unit square and in the other 5 the line segments were evenly distributed in a unit circle. The average length of the segments are 1/10 unit length.

The result is summarized in the Figure 5. Observe the close to linear speedup obtained.

7. Applications and conclusions

In this paper, we presented a BSP like coarse grained parallel algorithm for the next element search problem which requires $O(1)$ h -relations ($h = O(n/p)$), $O((n \log n) / p)$ memory per processor and $O((n/p) \log n)$ local computation. An important advantage of our model is that

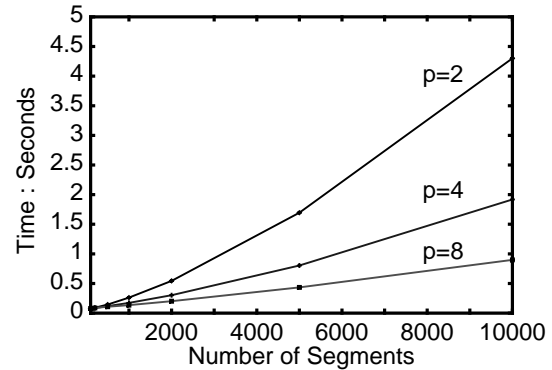


Figure 5: Running time of Algorithm 3 on an Intel iPSC/860

it gives a very good indication of the running time observed in an actual implementation. Our implementation on an Intel iPSC/860 hypercube obtained a very close to linear speedup. As demonstrated in [5][6][7], coarse grained parallel algorithms with $O(1)$ communication rounds are also portable across very different parallel platforms. Therefore, we expect that our algorithm presented here will also run well on other parallel machines.

Next element search can be used to solve many other geometric problems. Some of the more important examples include the following:

1. **Planar subdivision search problem.** Given a plane graph $G=(V,E)$ with vertex coordinates, and a set of n query points q_i ($1 \leq i \leq n$), find for each query point q_i , the face of G containing q_i .
2. **Trapezoidal map problem.** Given a set of segments in the plane, decompose the plane into a set of trapezoids, based on the arrangement of the segments.
3. **Triangulation problem for a simple polygon.** Partition the interior of a simple polygon into a set of triangles.

The above three problems can be reduced to $O(1)$ next element search problems (obvious for 1 and 2; see [21] for 3). Hence, Theorem 1 applies to these problems as well and we obtain

Corollary 1 *The planar subdivision search problem, trapezoidal map problem, and triangulation problem for a simple polygon can be solved on a CGM with $O((n/p) \log n)$ memory per processor in $O(1)$ communication rounds and $O((n/p) \log n)$ local computation.*

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