Maximizing a Voronoi Region: The Convex Case

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Abstract. Given a set S of s points in the plane, where do we place a new point, p, in order to maximize the area of its region in the Voronoi diagram of S and p? We study the case where the Voronoi neighbors of p are in convex position, and prove that there is at most one local maximum.

Keywords: Computational geometry, locational planning, optimization, Voronoi diagram.

1 Introduction

Suppose that we want to place a new supermarket where it wins over as many customers as possible from the competitors that already exist.

Let us assume that customers are equally distributed and that each customer shops at the market closest to her residence. Our task then amounts to finding a location, p, for the new market amidst the locations p_i of the existing markets, such that the Voronoi region of p, that is, the set of all points in the plane that are closer to p than to any p_i, has a maximum area.

Not much seems to be known about this problem. The area of Voronoi regions has been addressed in the context of games, where players can in turn move their existing sites, or insert new sites, such as to end up with a large total area of their Voronoi regions; see the Hotelling game described in Okabe et al. [6], or recent work by Cheong et al. [3] and Ahn et al. [1]. But none of these papers gives an explicit method for maximizing the region of a new site.

In this paper we take the first, nontrivial step towards a solution of the area maximization problem. Let the Voronoi region of the new point, p, in the Voronoi diagram $V(S \cup \{p\})$ consist of parts of the former regions of certain sites $p_1, \ldots, p_n$ in $V(S)$; these sites form the set N of Voronoi neighbors of p in $V(S \cup \{p\})$. In general, this set N spans a polygon that is star-shaped as seen from p.[1] We show that if the set N is in *convex* position then there can be at most one local maximum for the Voronoi area of p, in the interior of the locus of all positions that have N as their neighbor set. The proof is based on a delicate analysis of certain rational functions; it will be given in Section 3.

[1] A set P is called *star-shaped* as seen from one of its points, p, if any line segment connecting p to a point in P is fully contained in P.

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In Section 4 we describe an overall algorithm for determining the location of p that attains a maximum Voronoi area. Finally, we discuss some directions for future work in Section 5. Section 2 contains some preliminaries, among them a tractable formula for the area of a Voronoi region.

For general properties of Voronoi diagrams see the monograph by Okabe et al. [6] or the surveys by Fortune [4] and Aurenhammer and Klein [2].

2 Preliminaries

First, we restate some basic definitions and facts. Let S be a set of s point sites in the plane that are in general position, that is, no four of them are co-circular, no three of them co-linear. By $V(S)$ we denote the Voronoi diagram of the set S. It consists of Voronoi regions $\mathrm{VR}(q, S)$, one to each point q of S, containing all points in the plane that are closer to q than to any other site in S. The planar dual of $V(S)$ is the Delaunay triangulation, $\mathrm{DT}(S)$, of S. It consists of all triangles with vertices in S whose circum- (or: Delaunay) circle does not contain a site of S in its interior. Both, $V(S)$ and $\mathrm{DT}(S)$, are of complexity $O(s)$ and can be constructed in optimal time $O(s \log s)$.

If we add a new point site, p, to S, it will be connected to a site $q \in S$ by an edge of $\mathrm{DT}(S \cup p)$ if, and only if, there exists a Delaunay triangle with vertex q in $\mathrm{DT}(S)$ whose circumcircle contains p. The set N of such Voronoi or Delaunay neighbors q of p forms a polygon, $P(N)$, that is star-shaped as seen from p. The locus of all placements of p that have N as their neighbor set is denoted by C_N. Its shape will be discussed in Section 4.

In this section we derive some useful formulae for the area of the Voronoi region of a new site p with neighbor set N, assuming that $P(N)$ is convex. It is based on computing the *signed areas* of certain triangles. Let (v_0, v_1, v_2) be the vertices of some triangle, D, where $v_i = (a_i, b_i)$ in Cartesian coordinates. Then,

$$\mathrm{SignedArea}(D) := \frac{1}{2} \sum_{i=0}^{2} (a_i b_{i+1} - a_{i+1} b_i)$$

gives the positive area of D if (v_0, v_1, v_2) appear in counterclockwise order on the boundary of D; otherwise, we obtain the negative value. Here, indices are counted mod 3.

Now let p_i, p_{i+1} be two consecutive vertices on the boundary of $P(N)$, in counterclockwise order. Unless p is co-linear with p_i and p_{i+1}, these three point sites define a Voronoi vertex v_i that may or may not be contained in $P(N)$; see Figure 1.

Let D_i denote the triangle (p_i, v_i, p_{i+1}); its *signed area* is positive if and only if these vertices appear on D_i in counterclockwise order, that is, if and only if v_i lies outside the convex polygon $P(N)$.

Lemma 1. *With the notations from above we have the following identity.*

$$\mathit{Area}(\mathit{VR}(p, S \cup \{p\})) = \frac{1}{2}((\mathit{Area}(P(N)) + \sum_{i=1}^{n} \mathit{SignedArea}(D_i))$$

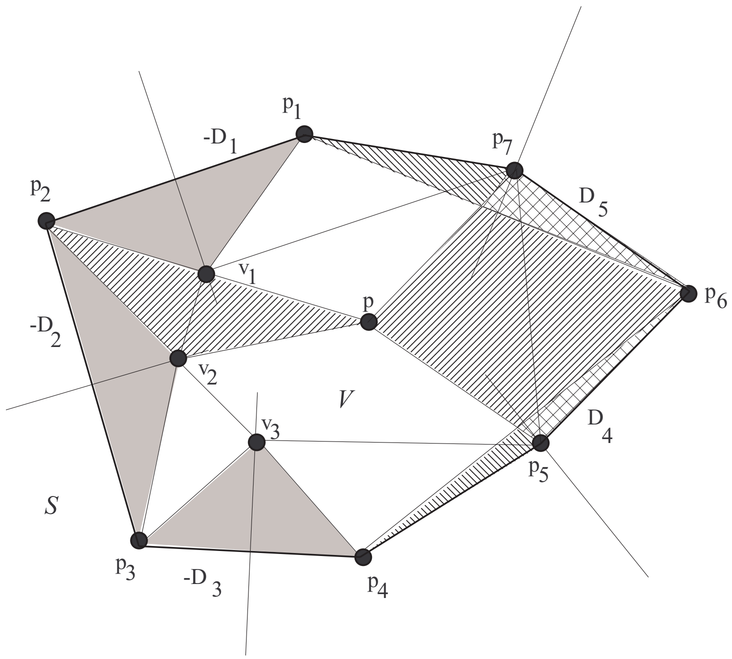

Fig. 1. Decomposing the area of the Voronoi region of site p.

Proof. The area of $VR(p, S \cup \{p\})$ equals the sum of the areas of the triangles (v_{i+1}, p, v_i); each of them is the reflected image of the triangle (v_i, p_{i+1}, v_{i+1}). The union of all these triangles equals $P(N)$ minus those triangles D_j that are contained in $P(N)$, plus those D_i not contained in $P(N)$; see Figure 1.

Lemma 1 reduces the problem of maximizing the area of the Voronoi region of p to maximing the sum of the signed areas of the triangles D_i, assuming N is fixed. Thus, two vertices of D_i are the given points p_i, p_{i+1}, while only the third, v_i, can move, and its movement is constrained to the bisector of p_i, p_{i+1}, depending on the placement of p.

Next, we express the signed area of D_i as a function of p. To this end, let $p_i = (s_i, t_i)$, and let $m_i = (\frac{s_i+s_{i+1}}{2}, \frac{t_i+t_{i+1}}{2})$ be the midpoint of $p_i p_{i+1}$. We put $b_i = |p_i m_i|$ and $l_i = |pm_i|$. Finally, let α_i be the angle at p in the triangle $F_i = (p_i, p_{i+1}, p)$; see Figure 2 for an illustration.

Lemma 2. *Let $p = (X, Y)$ be the new point site. Then the following identities hold.*

$$SignedArea(D_i) = b_i^2 \frac{l_i^2 - b_i^2}{2SignedArea(F_i)} \tag{1}$$

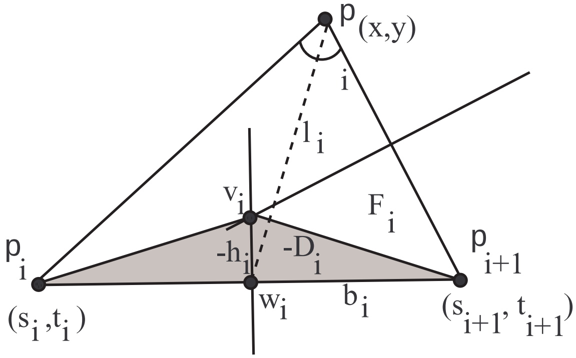

Fig. 2. Computing the signed area of the triangle D_i. In this case, the sign is negative.

$$= b_i^2 \frac{(X - \frac{s_i+s_{i+1}}{2})^2 + (Y - \frac{t_i+t_{i+1}}{2})^2 - b_i^2}{X(t_i - t_{i+1}) + Y(s_{i+1} - s_i) + s_i t_{i+1} - s_{i+1} t_i} \quad (2)$$

Proof. Let h_i denote the signed height of the triangle $D_i = (p_i, v_i, p_{i+1})$, so that SignedArea$(D_i) = b_i h_i$ holds. The Voronoi vertex v_i can be expressed as a vector sum by

$$\mathbf{v_i} = \mathbf{m_i} - h_i \mathbf{e_i},$$

where $\mathbf{e_i} = \frac{1}{2b_i}(t_i - t_{i+1}, s_{i+1} - s_i)$ denotes the unit vector along the bisector of p_i, p_{i+1}. On the other hand, $p = (X, Y)$ lies on a circle of radius $\sqrt{h_i^2 + b_i^2}$ centered at v_i. Plugging the cartesian coordinates of v_i into the equation of this circle, and solving for h_i, leads to formula (2), since the coefficient of h_i reduces to zero. The numerators and denominators in formulae (1) and (2) are identical. We observe that the denominator, that is, the sign of the area of F_i, is positive as long as p stays inside the polygon $P(N)$. It becomes 0 when p hits the line through p_i and p_{i+1}. The numerator of formula (2) is the equation of the circumcircle of the line segment $p_i p_{i+1}$. Thus, if $p \in \{p_i, p_{i+1}\}$ holds then the denominator's zero cancels out, and the area of D_i is zero because the Voronoi vertex v_i equals m_i.

3 Uniqueness of the Local Maximum

In this section we assume that N, the set of Voronoi neighbors of the new site, p, consists of n points in convex position. Now we state our main result.

Theorem 1. *For a convex set, N, of n points, there is at most one interior position in $P(N)$, intersected with the locus C_N of all locations with neighbor set N, where the area of the Voronoi region of p has a local maximum.*

Proof. By Lemma 1 it is sufficient to prove that the sum of the signed areas of the triangles D_i has at most one local maximum in the interior of C_N. It is enough to show that this sum attains at most one maximum along each line through $P(N)$.

If we substitute, in formula (2) of Lemma 2, the variable Y by coordinates $eX + f$ of some line G, and perform partial fraction decomposition, we obtain

$$-\text{SignedArea}(D_i(X)) = \frac{A_i}{X - a_i} + c_i X + d_i.$$

The pole at $X = a_i$ corresponds to the point where the line G intersects the line G_i through p_i, p_{i+1}. Three cases can occur when we hit $P(N)$ from the outside. If the point $G \cap G_i$ lies outside the line segment $p_i p_{i+1}$ then, in formula (1) of Lemma 2, we have $l_i > b_i$, while the sign of the area of F_i changes from $-$ to $+$. Consequently, the sign of $-D_i(X)$ changes from $-$ to $+$. But if G intersects the interior of $p_i p_{i+1}$ then $l_i < b_i$, so that $-D_i(X)$ changes from $+$ to $-$. Finally, if G happens to run through one of p_i, p_{i+1} then there is no pole at a_i, i. e., $A_i = 0$ holds, as we noted at the end of the proof of Lemma 2.

Let us assume that line G equals the X-axis, and let $a_1 \leq a_2 \leq \ldots \leq a_m \leq l < r \leq b_1 \leq \ldots \leq b_k$ denote the n poles that correspond to its intersections with the lines G_i. By the convexity of $P(N)$, the two intersections of the X-axis with the boundary of $P(N)$ must be consecutive in this sequence; they are denoted by l and r.

Figure 3 shows the behavior of

$$f(X) := -\sum_{i=1}^{n} \text{SignedArea}(D_i) =$$
$$= \sum_{i=1}^{m} \frac{A_i}{X - a_i} - \frac{L}{X - l} + \frac{R}{X - r} - \sum_{i=1}^{k} \frac{B_i}{X - b_i} + cX + d$$

as a function of X. By the above discussion, we have $A_i, L, R, B_i \geq 0$.

We want to prove that $f(X)$ has at most one local minimum in the interval (l, r). Since f comes from, and returns to, $-\infty$ at l resp. r it is sufficient to show that its second derivative

$$2f''(X) = \sum_{i-=1}^{m} \frac{A_i}{(X - a_i)^3} - \frac{L}{(X - l)^3} + \frac{R}{(X - r)^3} - \sum_{i=1}^{k} \frac{B_i}{(X - b_i)^3}$$

has at most two zeroes in (l, r). We split function $2f''$ into two constituent parts,

$$g(X) := \sum_{i=1}^{m} \frac{A_i}{(X - a_i)^3} - \frac{L}{(X - l)^3} \text{ and}$$
$$h(X) := \sum_{i=1}^{k} \frac{B_i}{(X - b_i)^3} - \frac{R}{(X - r)^3},$$

such that $2f'' = g - h$ holds, and discuss g and h independently.

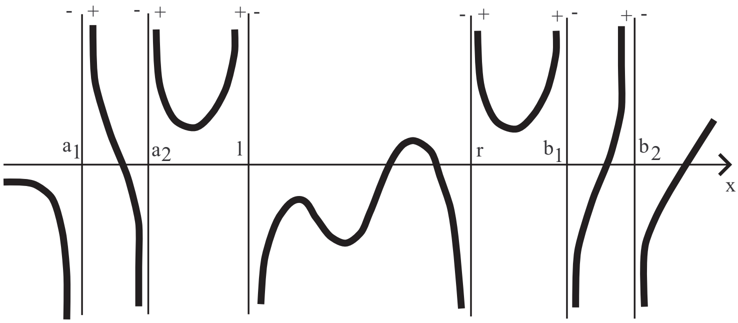

Fig. 3. Discussing the number of minima of $f(X)$ between l and r.

Lemma 3. *Each of the functions g and g'' has at most one zero in (l, ∞), and each of h, h'' has at most one zero in $(-\infty, r)$.*

Proof. Let $x_1 \neq x_0 \in (l, \infty)$ be such that x_0 is a zero of g. Then,

$$0 = g(x_0) = \sum_{i=1}^{m} \frac{A_i}{(x_0 - a_i)^3} - \frac{L}{(x_0 - l)^3} \tag{3}$$

$$= \sum_{i=1}^{m} \frac{A_i}{(x_0 - a_i)^3} \frac{(x_0 - l)^3}{(x_1 - l)^3} - \frac{L}{(x_1 - l)^3} \tag{4}$$

$$= \sum_{i=1}^{m} \frac{A_i}{(x_1 - a_i)^3} \left(\frac{(x_1 - a_i)^3}{(x_0 - a_i)^3} \frac{(x_0 - l)^3}{(x_1 - l)^3} \right) - \frac{L}{(x_1 - l)^3} \tag{5}$$

$$< \sum_{i=1}^{m} \frac{A_i}{(x_1 - a_i)^3} - \frac{L}{(x_1 - l)^3} = g(x_1), \text{ if } x_1 > x_0 \tag{6}$$

$$> g(x_1), \text{ if } x_1 < x_0; \tag{7}$$

observe that formula (4) follows from (3) by multiplying both sides by $\frac{(x_0 - l)^3}{(x_1 - l)^3}$. The alternatives (6) or (7) follow from (5) because $a_i < l < x_0, x_1$ implies that

$$\frac{(x_1 - a_i)^3}{(x_0 - a_i)^3} \frac{(x_0 - l)^3}{(x_1 - l)^3}$$

is of value < 1 if $x_1 > x_0$ holds, and of value > 1, otherwise. Consequently, g has at most one zero in (l, ∞). The other claims are proven analogously.

As a consequence of Lemma 3, the function g has at most one zero and at most one turning point to the right of l. Since g has a negative pole at l and tends to 0 for large values of X, its graph has one of the two possible shapes shown in Figure 4 (i). The possible shapes of the graph of h are shown in (ii).

Our next lemma implies that $2f'' = g - h$ has at most two zeroes in the interval (l, r).

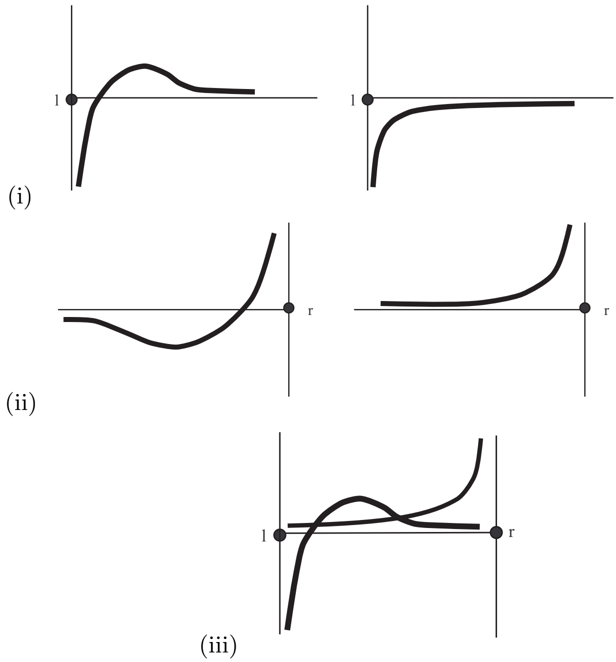

Fig. 4. (i) The possible shapes of the function $g(x)$. (ii) Possible shapes of $h(x)$. (iii) $g(x) = h(x)$ holds for at most two points between l and r.

Lemma 4. *The graphs of the functions g and h have at most two points of intersection over (l, r).*

Proof. If neither g nor h have a zero in (l, r) their graphs do not intersect; see Figure 4. Suppose that h has a zero in (l, r), and assume that p_1 and p_2 are the leftmost points of intersection of the two graphs to the right of l.

We argue that p_2 must be situated to the right of the minimum, m, of h. Indeed, m lies below the X-axis, where g is increasing, and h is decreasing to the left of m, so that only p_1 could lie to the left of m. If p_2 lies to the left of the maximum, M, of function g, or if g does not have a maximum, then the two graphs are separated by their tangents at p_2. If p_2 lies to the right of M then, to the right of p_2, function g is decreasing while function h is increasing. In either case, no third point of intersection can exist.

Now we have shown that the function f takes on at most one local minimum for all points p on L inside the polygon $P(N)$. In the interior of $C_N \subset P(N)$, we have, by Lemma 1,

$$\text{Area}(\text{VR}(p, S \cup \{p\})) = \frac{1}{2}(\text{Area}(P(N)) - f(X)).$$

This completes the proof of Theorem 1.

To give an example, let us assume that n points are evenly placed on the boundary of the unit circle. For $n \leq 4$ there is no local maximum of the Voronoi area. In fact, there is a unique local minimum at the center for $n = 3$; for $n = 4$, the cross formed by the four point sites consists of minimal positions. But for $n \geq 5$ we have a unique local maximum at the center of the circle.

4 Computing the Maximum

In this section we describe a general algorithm for computing, for a new site p, a location of maximum Voronoi area amidst s existing sites. As p is moved over the plane, three events may happen. First, the set N of p's Voronoi neighbors can change.

Let C_N denote a maximal connected region in the plane such that all placements of p inside C_N have N as their set of Voronoi neighbors; we call such a set C_N a *neighborship cell* of N with respect to S. The nature of these cells is quite simple; the proof of the following lemma follows from standard facts on the Delaunay triangulation. Observe that for two neighboring sites, q and r, on the convex hull of S we define, as their Delaunay triangle and circumcircle, the halfplane defined by the line through q, r that does not contain a site of S.

Lemma 5. *Let S be a set of s point sites in the plane.*

1. *The neighborship cells with respect to S are the cells of the arrangement of the Delaunay circles of S. Each cell C has, as its neighbor set, all sites that span a Delaunay circle containing C. The total complexity of all neighborship cells is in $O(s^2)$.*
2. *Let $N \subset S$ be such that $P(N)$ is star-shaped. Then the neighborship cells C_N can be obtained as the intersection of the circumcircles of those Delaunay triangles that are contained in, and share an edge with the boundary of, $P(N)$, minus the union of all Delaunay circles passing through points of $S \setminus N$.*

Lemma 5 is illustrated by Figure 5. Figure 6 shows an example where many neighborship cells are associated with a set, N, of sites.

The arrangement of $O(s)$ many circles can be constructed in time $O(s\lambda_4(s))$[2] by a deterministic algorithm, or in expected time $O(s \log s + k)$, where k denotes the complexity of the arrangement; see Sharir and Agarwal [8].

Another event happens when p hits the boundary of the convex hull of the site set S. At this point, the region of p becomes unbounded. To exclude this phenomenon[3] we assume that a certain feasability domain, F, is given, that consists of neighborship cells contained in the interior of the convex hull of S, and that the placement of p is restricted to F.

Finally, the position of the new site, p, could coincide with one of the existing sites, $p_i \in S$. At these points the area function fails to be continuous; in fact, the

[2] As usual, $\lambda_t(s)$ denotes the maximum length of a Davenport-Schinzel sequence of order t over s characters.

[3] Far out of town there are no customers to win.

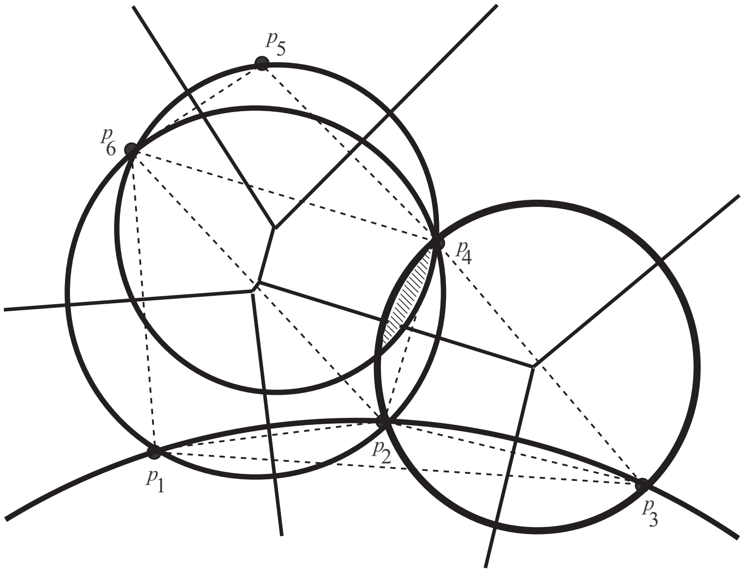

Fig. 5. The shaded area is the neighborship cell of $p_1, \ldots, p_6$. Only the circumcircles of those Delaunay triangles contribute to its boundary that are contained in the star-shaped polygon and share one of its edges.

former region of p_i is split among p and p_i by a bisector through $p = p_i$ whose slope is perpendicular to the direction in which p has approached p_i. But apart from these points, the area function is smooth, as was shown independently by Okabe and Aoyagi [5] and by Piper [7] who generalized work by Sibson [9].

In order to find the optimum placement of p within the whole feasibility domain F, we inspect each cell C of F in turn, and compute the optimal placement of p within the closure of C. Within the interior of C we apply some Newton-based approximation algorithm, which is possible thanks to the smoothness of the area function. If the neighbor set N is convex, we even know that there is at most one local maximum, by Theorem 1, so that following the gradient leads straight to the maximum (or to the boundary of C). Next, we have to check for maxima the boundary of C, which consists of circular arcs, by Lemma 5. This includes checking all placements of p on top of some site p_i; for each of them it takes time proportional to its Delaunay degree to find the optimum slope of the bisector. The solution to our problem is then the maximum of these $O(s^2)$ many cell maxima, together with the corresponding placement of p.

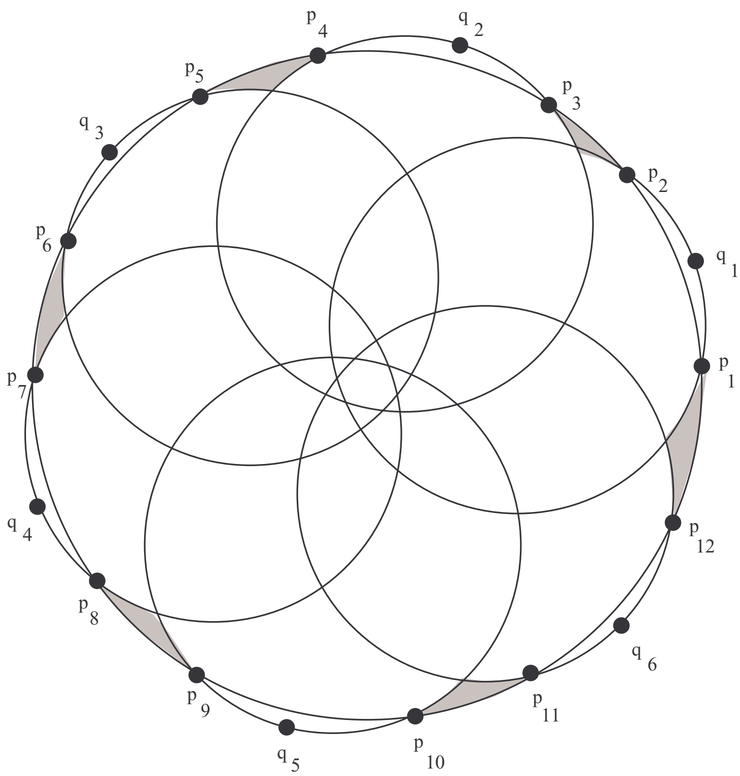

Fig. 6. Each of the shaded cells has $p_1, \ldots, p_{12}$ as neighbor set.

5 Conclusions

In this paper we have shown that the Voronoi area of a new site has at most one local maximum in the interior of each neighborship cell, if the Voronoi neighbors are in convex position. This result gives rise to many further questions.

The obvious open problem is if the maximum is still unique if the neighbors are in star-shaped position. The main difference to the convex case is the following. The line G, along which the new site p was supposed to move in the proof of Theorem 1, can now intersect edge extensions of the neighbor polygon $P(N)$ *inside* $P(N)$, too. Consequently, the functions g and h in the proof of Lemma 3 become more complicated. We expect that considerably more (mathematical) effort will be necessary in order to settle this problem.

Other questions concern the customer model. One could specify bounded populated areas, together with population densities, instead of the uniform distribution, with or without defining a feasibility domain F. Also, it would be interesting to study metrics different from the Euclidean, that are frequently used in location planning. From a theoretical point of view, it would also be interesting to minimize the area of a Voronoi region, and to investigate higher dimensions.

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